Preface

Probability theory is concerned with rationalizing about an uncertain future. From today's perspective, trying to assign likelihoods to events that haven't yet occurred is natural. Yet, until several centuries ago, the future was considered totally unpredictable. Not only it was believed that you could not be *certain* about the future, it was believed that you could not *reason* at all about uncertain future events. Quite expectedly, this vision had religious roots, which forbade humans trying to predict the future. The Fate is in the hands of God.

One of the precursors of probability theory was gambling games. Although often banned by the Church, gambling games were gaining enormous popularity, and gamblers were trying to find ways to improve their chances to win (the word *chance* (סיכוי) is in a sense the ancestor of *likelihood* or *probability* (הסתברות)).

In 1494, Luca Pacioli posed a problem known as the *problem of the unfinished game*. Consider two players, Player A and Player B, bidding an equal amount of money. The two players toss repeatedly a fair coin. Suppose that Player A bids on Heads and Player B ids bon Tails. The winner is the first to attain 6 points, taking then the whole pot. Suppose that Player A leads 5:3, when the game has to be stopped. How should they share the bids?

Several important mathematicians (e.g., Girolamo Cardano and Niccoló Tartaglia) tried to solve this problem over two centuries, all having it wrong. The recurring error was looking into the past rather than looking into the future (which was viewed as unpredictable). It is interesting that none of them tried to simulate an experiment, exploring repeatedly possible futures.

The turning point in the problem of the unfinished game was an exchange of letters in 1654 between Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665) in which they analyzed the problem, getting the correct solution (after endless hesitations). More importantly, this letter exchange laid the foundations to what would later become probability theory. It is hard to over-estimate the influence of probability theory on our lives. It paved the way to all aspects of *risk management* (ניהול סיכונים), without which no banking, insurance, and modern engineering, to name a few, are possible.

The work of Pascal and Fermat had to wait for more than half a century before it was taken over by Jacob Bernoulli, who, in 1713, was the first to coin the term *probability*. Bernoulli is responsible for inventing what would later become the common terminology of probability theory. Many of the great mathematicians in following years (e.g., Leonhard Euler, Carl Friedrich Gauß and Pierre-Simon Laplace) further developed the theory of probability, applying it to a growing range of applications.

The next major leap in the development of probability theory occurred in the first half of the 20th century, with its axiomatization by Andrey Nikolaevich Kolmogorov. Modern probability theory hinges on the foundations of *measure theory* (תורת המידה). In this course, which is an introduction to the theory of probability, we will not assume any knowledge in measure theory; we will basically learn 19th century probability, which is sufficiently rich, and applies to many contemporary applications. Whenever measure theoretic issues are relevant, we will state it, explaining at least what the issue is.

What does probability mean? This question may sound strange to you: Were you ever asked what does a number mean? a set? a group? a complex function? Probability is so deeply rooted in applications, that this question is inevitable. In 1900, David Hilbert presented a collection of 23 problems that set the course for much of the mathematical research of the 20th century. Hilbert's sixth problem was:

> Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.

This is to show that even in 1900, probability was viewed as a branch in physics. Hilbert's "problem" was solved in the following 20 years by Kolmogorov and others.

But let's get back to the question: what does probability mean? There are two common answers to it. Consider the two following assertions:

(a) The probability of a fair coin returning Heads is 0.5.
(b) There is a 70% probability of rain tomorrow (or better, there is a 95% probability that the Iliad and the Odyssey were written by the same person).

We tend to think of the first assertion in a *frequentist* (שכיחותני) view. If you toss a coin many times, then as the number of tosses tends to infinity, the *relative* *frequency* (שכיחות יחסית) of Heads tends to 0.5. We tend to think of the second assertion in an *epistemic* (הכרתית) view (*rational belief*). According to the second view, probability is a measure of belief (מידה של ודאות) based on certain knowledge and rational evaluation.

While both views help guiding our intuition, none of them can serve as a mathematical definition, even not the frequentist approach (the limit of many independent repetitions will turn out to be a theorem toward the end of this course). We are going to formulate probability theory axiomatically, starting with the definition of a *probability space* (מרחב הסתברות). Yet, we will constantly use this formal framework to solve practical problems, relying on its meanings. Note that this is a source of confusion for the learner: unlike, say, group theory, where *everything* follows from the axiomatic definition of a group, in probability theory, much of what we will do will depends on implicit knowledge (e.g., how to relate the probabilistic framework to an experiment of drawing balls out of a hat).

Yet, don't hurry to conclude that probability is "only an application of mathematics". Probability is a mathematical branch of its own, linked in many intricate ways to other branches of mathematics, such as functional analysis, discrete mathematics, abstract algebra, geometry and more.