Chapter 2

Local analysis I: Linear differential equations

Perturbation and asymptotic methods can be divided into two main categories: ***local*** and ***global*** analysis. In local analysis one approximates a function in a neighborhood of some point, whereas in global analysis one approximates a function throughout the domain. IVPS can be treated by either approach whereas BVPS require inherently a global approach. Local analysis is easier and is therefore the first approach we learn. This chapter is devoted to the local analysis of solutions of linear differential equations. In cases where the equation is solvable we can explicitly assess the accuracy of the approximation by comparing the exact and approximate solutions.

Example: The fourth-order differential equation

$$\frac{d^4y}{dx^4}(x) = (x^4 + \sin x)\, y(x),$$

cannot be solved in terms of elementary functions. Yet, we will be able to determine very easily (Bender and Orszag claim that on the back of a stamp) that as $x \to \infty$ the solution is well-approximated by a linear combination of the functions

$$x^{-3/2} e^{\pm x^2/2}, \quad x^{-3/2} \sin(x^2/2) \quad \text{and} \quad x^{-3/2} \cos(x^2/2).$$

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2.1 Classification of singular points

We are concerned with homogeneous linear equations of the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0. \qquad (2.1)$$

In local analysis we approximate its solution near a point x_0.

Definition 2.1 A point x_0 is called an **ordinary point** *of* (2.1) *if the coefficient functions $a_i(x)$ are (real) analytic in a neighborhood of x_0, that is, the Taylor series at x_0 converges to the function in a neighborhood of x_0. (Recall that real-analyticity implies the complex analyticity of its analytic continuation.)*

Example: Consider the equation

$$y''(x) = e^x y(x).$$

Every point $x_0 \in \mathbb{R}$ is an ordinary point because the function e^z is entire. ▲▲▲

It was proved in 1866 (Fuchs) that all n independent solutions of (2.1) are analytic in the neighborhood of an ordinary point. Moreover, if these solutions are Taylor expanded about x_0 then the radius of convergence is at least as the distance of the nearest singularity of the coefficient functions to x_0. This is not surprising. We know that an analytic equation has analytic solutions. In the case of linear equations, this solution can be continued indefinitely as long as no singularity has been encountered.

Example: Consider the equation

$$y'(x) + \frac{2x}{1+x^2} y(x) = 0.$$

The point $x = 0$ is an ordinary point. The complexified coefficient function

$$\frac{2z}{1+z^2}$$

has singularities at $z = \pm\imath$, i.e., at a distance of 1 from the origin. The general solution is

$$y(x) = \frac{c}{1+x^2} = c \sum_{n=0}^{\infty} (\imath x)^{2n}$$

and the Taylor series has a radius of convergence of 1. Note that the solution $y(x) = 1/(1+x^2)$ is analytic on the whole of $\mathbb{R}$, yet the radius of convergence of the Taylor series is bounded. ▲▲▲

Definition 2.2 A point x_0 is called a ***regular singular point*** *of* (2.1) *if it is not an ordinary point, but the functions*

$$(x - x_0)^n a_0(x),\ (x - x_0)^{n-1} a_1(x), \dots, (x - x_0) a_{n-1}(x)$$

are analytic in a (complex) neighborhood of x_0. Alternatively, x_0 is a regular singular point if the equation is of the form

$$y^{(n)}(x) + \frac{b_{n-1}(x)}{x - x_0} y^{(n-1)}(x) + \cdots + \frac{b_1(x)}{(x - x_0)^{n-1}} y'(x) + \frac{b_0(x)}{(x - x_0)^n} y(x) = 0,$$

and the coefficient functions $b(x)$ are analytic at x_0.

Example: Consider the equation

$$y'(x) = \frac{y(x)}{x - 1}.$$

The point $x = 1$ is a regular singular point. However, the point $x = 0$ is *not* a regular singular point of the equation

$$y'(x) = \frac{x + 1}{x^3} y(x).$$

▲▲▲

It was proved (still Fuchs) that a solution *may be* analytic at a regular singular point. If it is not, then its singularity can only be either a pole, or a branch point (algebraic or logarithmic). Moreover, there always exists at least one solution of the form

$$(x - x_0)^{\alpha} g(x),$$

where α is called the ***indical exponent*** and $g(x)$ is analytic at x_0. For equations of order two and higher, there exists another independent solution either of the form

$$(x - x_0)^{\beta} h(x),$$

or of the form

$$(x - x_0)^{\alpha} g(x) \log(x - x_0) + (x - x_0)^{\beta} h(x),$$

where $h(x)$ is analytic at x_0. This process can be continued.

Example: For the case where the coefficients $b_i(x)$ are constant the equation is of Euler type, and we know that the solutions are indeed of this type, with $g(x)$ and $h(x)$ constant. ▲▲▲

Example: Consider the equation

$$y'(x) = \frac{y(x)}{\sinh x}.$$

It has a regular singular point at $x = 0$. The general solution is

$$y(x) = c \tanh \frac{x}{2},$$

which is analytic at $x = 0$. It's Taylor series at zero, which involves the ***Bernoulli numbers***, has a radius of π, which is the distance of the nearest singularity of the coefficient function (at $z = \pm\imath\pi$). ▲▲▲

Definition 2.3 A point x_0 is called an ***irregular singular point*** *of* (2.1) *if it is neither an ordinary point nor a regular singular point.*

There are no general properties of solutions near such point.

Finally, we consider also the point $x = \infty$ by changing the dependent variable into $t = 1/x$ and looking at $t = 0$. The point $x = \infty$ inherits the classification of the point $t = 0$.

Examples:

1. Consider the three equations

$$\begin{aligned} y'(x) - \frac{1}{2}y(x) &= 0 \\ y'(x) - \frac{1}{2x}y(x) &= 0 \\ y'(x) - \frac{1}{2x^2}y(x) &= 0. \end{aligned}$$

   Changing variables $t = 1/x$ we have

$$\begin{aligned} -y'(t) - \frac{1}{2t^2}y(t) &= 0 \\ -y'(t) - \frac{1}{2t}y(t) &= 0 \\ -y'(t) - \frac{1}{2}y(t) &= 0. \end{aligned}$$

Thus, every point $x \neq 0$ is an ordinary point of the first equation, the point $x = 0$ is a regular singular point of the second equation and an irregular singular point of the third equation. At $x = \infty$ it is the exact opposite. The solutions are

$$y(x) = c\, e^{x/2}, \qquad y(x) = c\sqrt{x}, \quad \text{and} \quad y(x) = c\, e^{-1/2x},$$

respectively. Note that the second solution has branch cuts at $x = 0$ and $x = \infty$, whereas the third solution has an essential singularity at $x = 0$.

✎ *Exercise 2.1* Classify all the singular points (finite and infinite) of the following equations

1. $y'' = xy$ (Airy equation).
2. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (Bessel equation).
3. $y'' + (h - 2\theta \cos 2x)y = 0$ (Mathieu equation).

2.2 Local solution near an ordinary point

In the vicinity of an ordinary point, the solution to a linear differential equation can be sought by explicitly constructing its Taylor series. The latter is guaranteed to converge in a ball whose radius is a property of the coefficient functions; that is, it can be determined directly from the problem, without need to solve it. While this method can (in principle) provide the full solution, we are interested in it as a ***perturbation series***, i.e., as an ***approximation*** to the exact solution as a power series of the "small" parameter $x - x_0$. We will discuss perturbation series in more generality later on.

Example: Consider the IVP

$$y'(x) = 2x\, y(x) \qquad y(0) = 1.$$

The exact solution is $y(x) = ce^{x^2}$, but we will ignore it, and obtain it via a power series expansion.

Note first that the coefficient function $2z$ is entire, hence the Taylor series of the solution has an infinite radius of convergence. We now seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting into the equation we get

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = 2 \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Since the Taylor series is unique the two sides must have equal coefficients, hence we resort to a term by term identification. Equating the x^0 terms we get that $a_1 = 0$. Then,

$$\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+2} = 2 \sum_{n=0}^{\infty} a_n x^{n+1},$$

and

$$a_{n+2} = \frac{2a_n}{n+2}, \qquad n = 0, 1, \dots.$$

The first coefficients a_0 is determined by the initial data $a_0 = 1$. All the odd coefficients vanish. For the even coefficients

$$a_2 = \frac{2}{2}, \quad a_4 = \frac{2^2}{2 \cdot 4}, \qquad a_6 = \frac{2^3}{2 \cdot 4 \cdot 6},$$

and thus,

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!},$$

which is the right solution.

Suppose that we interested in $y(1)$ within an error of 10^{-8}. If we retain n terms, then the error is

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots \right) \\ &\le \frac{1}{(n+1)!} \left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots \right) = \frac{1}{(n+1)!} \frac{n}{n-1} \le \frac{2}{(n+1)!}.\end{aligned}$$

Taking $n = 11$ will do. ▲▲▲

The Gamma function Before going to the next example, we remind ourselves the properties of the ***Gamma function***. It is a complex-valued function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

This function is analytic in the upper half-plane. If has the property that

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z}\, dt = z\,\Gamma(z).$$

In particular,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

from which follows that $\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$, and more generally, for n integer,

$$\Gamma(n) = (n-1)!.$$

In fact, the Gamma-function can be viewed as the analytic extension of the factorial. The Gamma function can be used to shorten notations as

$$x(x+1)(x+2)\dots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Example: Consider the ***Airy equation***[1]

$$y''(x) = x\,y(x).$$

Our goal is to analyze the solutions near the ordinary point $x = 0$. Here again, the solutions are guaranteed to be entire.

Again, we seek for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

[1]The Airy function is a special function named after the British astronomer George Biddell Airy (1838). The Airy equation is the simplest second-order linear differential equation with a turning point (a point where the character of the solutions changes from oscillatory to exponential). The Airy function describes the appearance of a star–a point source of light–as it appears in a telescope. The ideal point image becomes a series of concentric ripples because of the limited aperture and the wave nature of light.

Substituting we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

The coefficient a_0, a_1 remain undetermined, and are in fact the two integration constants. Equating the x^0 terms we get $a_2 = 0$. Then

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \qquad n = 0, 1, 2, \dots .$$

This recursion relation can be solved: first the multiple of three,

$$a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},$$

and more generally,

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\dots(3n-1)3n} = \frac{a_0}{3^{2n}\frac{2}{3}\cdot 1\cdot(1+\frac{2}{3})\cdot 2\dots(n-1+\frac{2}{3})n} = \frac{a_0\,\Gamma(\frac{2}{3})}{3^{2n}n!\Gamma(n+\frac{2}{3})}.$$

Similarly,

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_1}{3\cdot 4\cdot 6\cdot 7},$$

hence

$$a_{3n+1} = \frac{a_0}{3\cdot 4\cdot 6\cdot 7\dots 3n(3n+1)} = \frac{a_1}{3^{2n}1\cdot(1+\frac{1}{3})\cdot 2\cdot(2+\frac{1}{3})\dots n(n+\frac{1}{3})} = \frac{a_0\,\Gamma(\frac{4}{3})}{3^{2n}n!\Gamma(n+\frac{4}{3})}.$$

Thus, the general solution is

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} + a_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})},$$

where we absorbed the constants into the coefficients.

These are very rapidly converging series, and their radius of convergence is infinite. An approximate solution is obtained by truncating this series. To solve an initial value problem one has to determine the coefficients a_0, a_1 first.

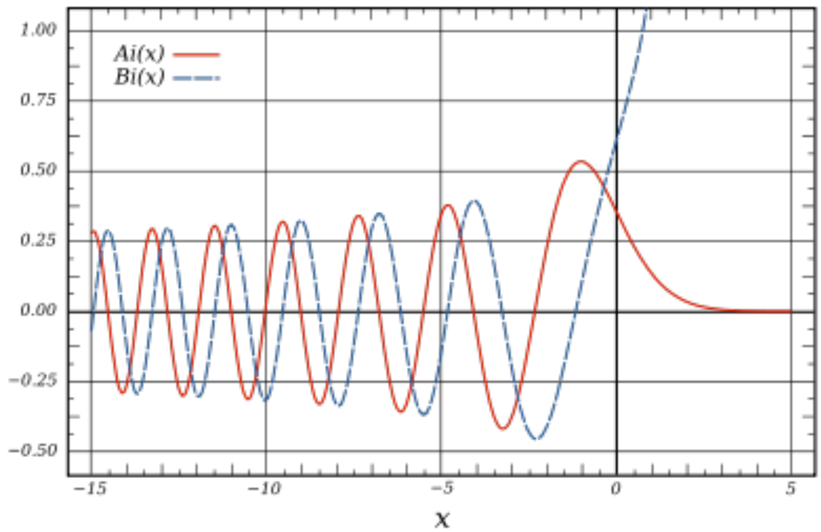

Figure 2.1: The Airy functions

The two terms we obtained are independent solutions. There is arbitrariness in the choice of independent solutions. It is customary to the refer to ***Airy functions*** as the two special (independent) choices of

$$\operatorname{Ai}(x) = 3^{-2/3} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})}$$

$$\operatorname{Bi}(x) = 3^{-1/6} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma(n+\frac{4}{3})}.$$

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✎ *Exercise 2.2* Find the Taylor expansion about $x = 0$ of the solution to the initial value problem

$$(x-1)(x-2)y''(x) + (4x-6)y'(x) + 2y(x) = 0, \qquad y(0) = 2, \; y'(0) = 1.$$

For which values of x we should expect the series to converge? What is its actual radius of convergence?

✎ *Exercise 2.3* Estimate the number of terms in the Taylor series need to estimate the Airy functions $\operatorname{Ai}(x)$ and $\operatorname{Bi}(x)$ to three decimal digits at $x = \pm 1$, $x = \pm 100$ and $x = \pm 10,000$.

2.3 Local solution near a regular singular point

Let us first see what may happen if we Taylor expand the solution about a regular singular point:

Example: Consider the Euler equation,

$$y''(x) + \frac{y(x)}{4x^2} = 0, \qquad x_0 = 0.$$

Substituting a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n-2} = 0,$$

i.e.,

$$(n - \tfrac{1}{2})^2 a_n = 0.$$

This gives $a_n = 0$ for all n, i.e., we only find the trivial solution. The general solution, however, is of the form $y(x) = c_1 \sqrt{x} + c_2 \sqrt{x} \log x$. ▲▲▲

The problem is that Taylor series are not general enough for this kind of problems. Yet, we know from Fuchs' theory that there exists at least one solution of the form

$$y(x) = (x - x_0)^{\alpha} g(x),$$

where $g(x)$ is analytic at x_0. This suggest to expand the solution in a series known as a ***Frobenius series***,

$$y(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To remove indeterminacy we require $a_0 \neq 0$.

Example: Going back to the previous example, we search a solution of the form

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n.$$

Substituting we get

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+\alpha-2} = 0,$$

i.e.,

$$\left[(n+\alpha)(n+\alpha-1) + \tfrac{1}{4}\right] a_n = 0.$$

Since we require $a_0 \neq 0$, the indical exponent α satisfies the quadratic equation

$$P(\alpha) = (\alpha - \tfrac{1}{2})^2 = 0.$$

This equation has a double root at $\alpha = 1/2$. For $n = 1, 2, \dots$ we have $a_n = 0$, hence we found an exact solution, $y(x) = \sqrt{x}$. On the other hand, this method does not allow us, for the moment, to find a second independent solution. ▲▲▲

We will discuss now, in generality, local expansions about regular singular points of second-order equations,

$$y''(x) + \frac{p(x)}{x - x_0} + \frac{q(x)}{(x-x_0)^2} = 0.$$

We assume that the functions $p(x), q(x)$ are analytic at x_0, i.e., they can be locally expanded as

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n.$$

We then substitute into the equation a Frobenius series,

$$y(x) = (x-x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

This gives

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n(x-x_0)^{n+\alpha-2} \\ &\quad + \left(\sum_{k=0}^{\infty} p_k (x-x_0)^k\right)\sum_{n=0}^{\infty}(n+\alpha)a_n(x-x_0)^{n+\alpha-2} \\ &\quad + \left(\sum_{k=0}^{\infty} q_k (x-x_0)^k\right)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+\alpha-2} = 0. \end{aligned}$$

Equating same powers of $(x - x_0)$ we get

$$(n+\alpha)(n+\alpha-1)a_n + \sum_{k=0}^{n} \left[p_k(n-k+\alpha)+q_k\right] a_{n-k} = 0.$$

Separating the $k = 0$ term we get

$$P(n+\alpha)\, a_n = -\sum_{k=1}^{n} \left[p_k(n-k+\alpha)+q_k\right] a_{n-k},$$

where

$$P(z) = z^2 + (p_0 - 1)z + q_0.$$

We write the left-hand side as $P(n + \alpha)\, a_n$. The requirement that $a_0 \neq 0$ implies that $P(\alpha) = 0$, i.e., α is a solution of a quadratic equation. a_0 is indeterminate (integration constant), whereas the other a_n are then given by a recursion relation,

$$a_n = -\frac{1}{P(\alpha+n)} \sum_{k=1}^{n} \left[p_k(n-k+\alpha)+q_k\right] a_{n-k}, \qquad n = 1, 2, \ldots.$$

A number of problems arise right away: (1) α may be a double root in which case we're lacking a solution. (2) The recursion relation may break down if for some $n \in \mathbb{N}$, $P(\alpha + n) = 0$. Yet, if α_1, α_2 are the two roots of the indical equation, and $\Re\alpha_1 \geq \Re\alpha_2$, then it is guaranteed that $P(\alpha_1 + n) \neq 0$, and the recursion relation can be continued indefinitely. This is why there is always at least one solution in the form of a Frobenius series. More generally, we have the following possible scenarios:

1. $\alpha_1 \neq \alpha_2$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$. In this case there are two solutions in the form of Frobenius series.

2. (a) $\alpha_1 = \alpha_2$. There is one solution in the form of a Frobenius series and we will see how to construct a second independent solution.

   (b) $\alpha_1 - \alpha_2 = N$, $N \in \mathbb{N}$:

   i. If $\sum_{k=1}^{N} \left[p_k(\alpha_1 - k) + q_k\right] a_{N-k} = 0$ then $a_N = 0$ and the series can be continued past the "bad" index.

   ii. Otherwise, there is only one solution in the form of a Frobenius series. We will see how to construct another independent solution.

✎ *Exercise 2.4* Find series expansions about $x = 0$ for the following differential equations. Try to sum (if possible) the infinite series.

① $2xy''(x) - y'(x) + x^2 y(x) = 0$.
② $x(x+2)y''(x) + (x+1)y'(x) - 4y(x) = 0$.
③ $x(1-x)y''(x) - 3xy'(x) - y(x) = 0$.
④ $\sin x\, y''(x) - 2\cos x\, y'(x) - \sin x\, y(x) = 0$.

Example: We start with an example of type 1. Consider the modified Bessel equation[2]

$$y''(x) + \frac{1}{x} y(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0,$$

where ν is a parameter. In this case

$$p(x) = 1 \qquad \text{and} \qquad q(x) = -x^2 - \nu^2,$$

and therefore

$$P(z) = z^2 - \nu^2.$$

The point $x = 0$ is a regular singular point, hence we substitute the Frobenius series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

This gives

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} + \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-2} - \sum_{n=0}^{\infty} a_n x^{n+\alpha} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+\alpha-2} = 0.$$

Equating powers of x we get

$$\left[(n+\alpha)^2 - \nu^2\right] a_n = a_{n-2}.$$

For $n = 0$ we get the indical equation

$$P(\alpha) = \alpha^2 - \nu^2 = 0,$$

[2]The Bessel equation arises in the solution of numerous partial differential equations in cylindrical and spherical coordinates.

i.e., $\alpha = \pm\nu$. Take first $\alpha = \nu > 0$, in which case $P(\alpha + n) > 0$ for all n.

For $n = 1$, since $P(\nu + 1) \neq 0$ we have $a_1 = 0$. For $n \geq 2$,

$$a_n = \frac{a_{n-2}}{(n+\nu)^2 - \nu^2} = \frac{a_{n-2}}{n(n+2\nu)}.$$

That is,

$$a_{2n} = \frac{a_0}{2n(2n-2)\dots 2 \cdot (2n+2\nu)(2n+2\nu-2)\dots(4+2\nu)(2+2\nu)}.$$

We can express this using the Γ function,

$$a_{2n} = \frac{\Gamma(\nu+1)}{4^n n!\,\Gamma(n+\nu+1)} a_0.$$

Thus, a first solution is

$$y(x) = \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{x^{2n+\nu}}{4^n n!\,\Gamma(n+\nu+1)}.$$

It is conventional to define the ***modified Bessel function***

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}.$$

This series has an infinite radius of convergence, as expected from the analyticity of the coefficients.

A second solution can be found by setting $\alpha = -\nu$. In order for $P(-\nu + n) \neq 0$ we need 2ν not to be an integer. Note however that $I_{-\nu}(x)$ given by the above power series is well defined as long as 2ν is not an even integer, i.e., $I_{-1/2}(x)$, $I_{-3/2}(x)$ and so on are well-defined and form a second independent solution.

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✎ *Exercise 2.5* Show that all the solutions of the modified Bessel equation

$$y''(x) + \frac{y'(x)}{x} - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0.$$

with $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, can be expanded in Frobenius series.

Case 2b(i) This is the simplest "bad" case, where nevertheless as second solution in the form of a Frobenius series can be constructed.

Example: You will be asked as homework to examine the half-integer modified Bessel equation. ▲▲▲

Case 2a This is that case where α is a double root, $\alpha_1 = \alpha_2 = \alpha$. Recall that when we substitute in the equation a Frobenius series, we get

$$P(n+\alpha)a_n = -\sum_{k=1}^{n}[p_k(n-k+\alpha)+q_k]a_{n-k},$$

where

$$P(z) = z^2 + (p_0 - 1)z + p_0.$$

In the present case we have

$$P(z) = (z-\alpha)^2.$$

One solution can be obtained by this procedure. We solve iteratively for the a_n and have

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+\alpha}.$$

We may generalize this type of solutions by replacing α by an arbitrary β, i.e., form a function

$$y(x;\beta) = \sum_{n=0}^{\infty} a_n(\beta)(x-x_0)^{n+\beta},$$

where the coefficients $a_n(\beta)$ satisfy the recursion relations,

$$P(n+\beta)a_n(\beta) = -\sum_{k=1}^{n}[p_k(n-k+\beta)+q_k]a_{n-k}(\beta).$$

Of course, this is a solution only for $\beta = \alpha$.

Let's see now what happens if we substitute $y(x;\beta)$ into the differential equation,

$$\mathcal{L}[y](x) = y''(x) + \frac{p(x)}{x-x_0}y'(x) + \frac{q(x)}{(x-x_0)^2}y(x) = 0.$$

We get

$$\begin{aligned}\mathcal{L}[y(\cdot;\beta)] &= \sum_{n=0}^{\infty}(n+\beta)(n+\beta-1)a_n(\beta)(x-x_0)^{n+\beta-2} \\ &+ p(x)\sum_{n=0}^{\infty}(n+\beta)a_n(\beta)(x-x_0)^{n+\beta-2} \\ &+ q(x)\sum_{n=0}^{\infty}a_n(\beta)(x-x_0)^{n+\beta-2}.\end{aligned}$$

If we substitute the series expansions for $p(x)$, $q(x)$, we find that almost all the terms vanish, because the $a_n(\beta)$ satisfy the correct recursion relations. The only terms that do not vanish are those proportional to $(x-x_0)^{\beta-2}$,

$$\mathcal{L}[y(\cdot;\beta)](x) = a_0\left[\beta^2 + (p_0-1)\beta + q_0\right](x-x_0)^{\beta-2} = a_0 P(\beta)(x-x_0)^{\beta-2}.$$

Indeed, this vanishes if and only if $\beta = \alpha$. If we now differentiate both sides with respect to β and set $\beta = \alpha$ the right-hand side vanishes because α is a double root of P. Thus,

$$\mathcal{L}\left[\frac{\partial}{\partial\beta}y(\cdot;\beta)\bigg|_{\beta=\alpha}\right] = 0,$$

that is we found another independent solution,

$$\frac{\partial}{\partial\beta}y(\cdot;\beta)\bigg|_{\beta=\alpha} = \sum_{n=0}^{\infty}\frac{da_n}{d\beta}\bigg|_{\beta=\alpha}(x-x_0)^{n+\alpha} + \log(x-x_0)\sum_{n=0}^{\infty}a_n(\alpha)(x-x_0)^{n+\alpha},$$

where we have used the fact that

$$\frac{\partial}{\partial\beta}x^\beta = x^\beta \log x.$$

We write it in the more compact form,

$$y_2(x) = \sum_{n=0}^{\infty}b_n(x-x_0)^{n+\alpha} + \log(x-x_0)\,y_1(x), \qquad b_n = \frac{da_n}{d\beta}\bigg|_{\beta=\alpha}.$$

Example: Consider the modified Bessel for $\nu = 0$. Recall that we get the recursion relation

$$(n+\alpha)^2 a_n = a_{n-2},$$

and therefore conclude that $\alpha = 0$ and that

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2}$$

is a first solution. We then define the coefficients $a_n(\beta)$ by the recursion relation

$$(n + \beta)^2 a_n(\beta) = a_{n-2}(\beta),$$

i.e.,

$$a_{2n}(\beta) = \frac{a_{2n-2}(\beta)}{(2n + \beta)^2} = \frac{a_0}{(2n + \beta)^2 (2n - 2 + \beta)^2 \dots (2 + \beta)^2}.$$

Differentiating with respect to β and setting $\beta = \alpha = 0$ we get

$$b_{2n} = -a_{2n}(0) \left(\frac{1}{n} + \frac{1}{n-1} + \dots 1 \right) = -\frac{a_0}{4^n (n!)^2} \left(\frac{1}{n} + \frac{1}{n-1} + \dots + 1 \right).$$

Thus we have found another (independent solution)

$$y_2(x) = a_0 \log x \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2} - a_0 \sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \frac{(\frac{1}{2}x)^{2n}}{(n!)^2}.$$

It is conventional to choose for other independent function a linear combination of $y_2(x)$ and $I_0(x)$ (it is called $K_0(x)$). ▲▲▲

Case 2b(ii) We are left with the case where

$$P(z) = (z - \alpha_1)(z - \alpha_2),$$

with $\alpha_1 - \alpha_2 = N \in \mathbb{N}$, and no "miracle" occurs. As before, using $y(x;\beta)$ we have

$$\mathcal{L}[y(\cdot;\beta) = a_0 P(\beta)(x - x_0)^{\beta-2}.$$

If we try to do again as before, differentiating both sides with respect to β and setting $\beta = \alpha_1$ we find

$$\mathcal{L}\left[\frac{\partial}{\partial \beta} y(\cdot, \beta) \Big|_{\beta=\alpha_1} \right] = a_0 N (x - x_0)^{\alpha_1 - 2} = a_0 N (x - x_0)^{\alpha_2 + N - 2}.$$

In other words,

$$\frac{\partial}{\partial \beta} y(\cdot, \beta) \Big|_{\beta=\alpha_1}$$

satisfies the linear inhomogeneous equation

$$\mathcal{L}[y](x) = a_0 N (x - x_0)^{\alpha_2 + N - 2}.$$

A way to obtain a solution to the homogeneous equation is to subtract any particular solution to this inhomogeneous equation. Can we find such? It turns out that we can find a solution in the form of a Frobenius series.

Setting

$$z(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\alpha_2},$$

and substituting into the inhomogeneous equation, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} (n + \alpha_2)(n + \alpha_2 - 1) c_n (x - x_0)^{n+\alpha_2-2} \\ &\qquad + \left(\sum_{k=0}^{\infty} p_n (x - x_0)^k \right) \sum_{n=0}^{\infty} (n + \alpha_2) c_n (x - x_0)^{n+\alpha_2-2} \\ &\qquad + \left(\sum_{k=0}^{\infty} q_n (x - x_0)^k \right) \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\alpha_2-2} = a_0 N (x -_0)^{\alpha_2+N-2}. \end{aligned}$$

Equating the coefficients of $(x - x_0)^{\alpha_2 - 2}$ we get

$$\left[\alpha_2^2 + (p_0 - 1)\alpha_2 + q_0 \right] c_0 = 0,$$

which is indeed satisfies since the pre-factor is $P(\alpha_2)$. In particular, c_0 is not (yet) determined. For all powers of $(x - x_0)$ other than $\alpha_2 + N - 2$ we have the usual recursion relation,

$$P(n + \alpha_2) c_n = - \sum_{k=1}^{n} [p_k (n - k + \alpha_2) + q_k] c_{n-k}.$$

Since $n \neq N$ there is no problem. Remain the terms proportional to $(x - x_0)$ to the power $\alpha_2 + N - 2$, which give

$$P(N + \alpha_2) c_N = - \sum_{k=1}^{N} [p_k (N - k + \alpha_2) + q_k] c_{N-k} + a_0 N.$$

While the left hand side is zero, we can view this equation as determining c_0 (i.e., relating it to a_0). Then, c_N is left arbitrary, but that is not a problem. Particular solutions are not unique. Thus, we have constructed a second (independent) solution which is

$$y_2(x) = \frac{\partial}{\partial\beta} y(\cdot,\beta)\Big|_{\beta=\alpha_1} - z(x),$$

or,

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+\alpha_1} + \log(x-x_0) y_1(x) - \sum_{n=0}^{\infty} c_n (x-x_0)^{n+\alpha_2}.$$

✎ *Exercise 2.6* Consider the modified Bessel equation with $\nu = 1$,

$$y''(x) + \frac{1}{x} y(x) - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0,$$

① Calculate explicitly the solution in the form of a Frobenius series with index $\alpha_1 = 1$; we denote this solution $y(x;\alpha_1)$.

② Show that one cannot construct a second such series $y(x;\alpha_2)$ with index $\alpha_2 = -1$.

③ Show that

$$z = \frac{d}{d\beta} y(\cdot;\beta)\Big|_{\beta=\alpha_1}$$

is *not* a solution by calculating $\mathscr{L}[z]$.

④ Find a second independent solution by subtracting from z a series of the form

$$\sum_{n=0}^{\infty} c_n (x-x_0)^{n+\alpha_2},$$

and determining the coefficients c_n.

2.4 Local solution near irregular singular points

So far everything was very straightforward (though sometimes tedious). Rigorous methods to find local solutions, always guaranteed to work, reflecting the fact that the theory of local solutions near ordinary and regular singular points is complete. In the presence of irregular singular points, no such theory exists, and one has to build up approximation methods that are often based on heuristics and intuition.

In the same way as we examined the breakdown of Taylor series near regular singular points, let's examine the breakdown of Frobenius series near irregular singular points.

Example: Let's start with a non-dramatic example,

$$y'(x) = x^{1/2} y(x).$$

The point $x = 0$ is an irregular singular point (note that nothing diverges). The general solution is obtained by separation of variables,

$$\frac{d}{dx} \log y(x) = \frac{2}{3} \frac{d}{dx} x^{3/2},$$

i.e.,

$$y(x) = c\, e^{\frac{2}{3} x^{3/2}}.$$

This can be written as a series,

$$y(x) = c \sum_{n=0}^{\infty} \frac{(\frac{2}{3} x^{3/2})^n}{n!},$$

that has nothing of a Frobenius series, where all powers must be of the form $n + \alpha$.
▲▲▲

Example: We next consider the equation

$$x^3 y''(x) = y(x),$$

where $x = 0$ is clearly an irregular singular point. If we attempt a Frobenius series,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha},$$

we get

$$\sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1) a_n x^{n+\alpha+1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

The first equation is $a_0 = 0$, which is a contradiction. ▲▲▲

Example: This third example is much more interesting,

$$x^2 y''(x) + (1 + 3x)y'(x) + y(x) = 0.$$

Substituting a Frobenius series gives

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha} + \sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1} + 3\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0.$$

Equating the coefficients of $\alpha - 1$ we get $\alpha a_n = 0$, i.e., $\alpha = 0$, which means that the Frobenius series is in fact a power series. Then,

$$n a_n = -\left[(n-1)(n-2) + 3(n-1) + 1\right] a_{n-1} = -n^2 a_{n-1},$$

from which we get that $a_n = (-1)^n n!\, a_0$, *i.e.*, the solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n n! x^n.$$

This is a series whose radius of convergence is zero! Thus, it does not look as a solution at all. This is indeed a divergent series, on the other hand, it is a perfectly good ***asymptotic series***, something we are going to explore in depth. If we truncate this series at some n, it provides a very good approximation for small x. ▲▲▲

Some definitions We will address the notion of asymptotic series later on, and at this stage work "mechanically" in a way we will learn.

*Definition 2.4 We say that $f(x)$ is **much smaller** than $g(x)$ as $x \to x_0$,*

$$f(x) \ll g(x) \qquad \text{as } x \to x_0,$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

*We say that $f(x)$ is **asymptotic** to $g(x)$ as $x \to x_0$,*

$$f(x) \sim g(x) \qquad \text{as } x \to x_0,$$

if

$$f(x) - g(x) \ll g(x).$$

Note that asymptoticity is symmetric as $f(x) \sim g(x)$ implies that

$$\lim_{x \to x_0} \frac{f(x) - g(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)} - 1 = 0,$$

i.e., that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Examples:

1. $x \ll 1/x$ as $x \to 0$.
2. $x^{1/2} \ll x^{1/3}$ as $x \to 0^+$.
3. $(\log x)^5 \ll x^{1/4}$ as $x \to \infty$.
4. $e^x + x \sim e^x$ as $x \to \infty$.
5. $x^2 \nsim x$ as $x \to 0$.
6. A function can never be asymptotic to zero!
7. $x \ll -1$ as $x \to 0^+$ even though $x > -1$ for all $x > 0$.

In the following, until we do it systematically, we will assume that asymptotic relations can be added, multiplied, integrated and differentiated. Don't worry about justifications at this point.

Example: Let's return to the example

$$x^3 y''(x) = y(x),$$

for which we were unable to construct a Frobenius series. It turns out that as $x \to 0$, the two independent solutions have the asymptotic behavior,

$$y_1(x) \sim c_1 x^{3/4} e^{2x^{-1/2}}$$
$$y_2(x) \sim c_2 x^{3/4} e^{-2x^{-1/2}}.$$

▲▲▲

Example: Recall the other example,

$$x^2y''(x) + (1 + 3x)y'(x) + y(x) = 0,$$

for which we were able to construct only one series, which was divergent everywhere. It turns out that the second solution has the asymptotic behavior,

$$y_2(x) \sim c^2 \frac{1}{x} e^{1/x} \qquad \text{as } x \to 0^+.$$

▲▲▲

All these solutions exhibit an exponential of a function that diverges at the singular point. This turns out to be typical. These asymptotic expressions turn out to be the most significant terms in an infinite series expansion of the solution. We call these terms the ***leading terms*** (it is not clear how well-defined this concept is). Each of these leading terms is itself a product of functions among which we can identify one which is the "most significant"—the ***controlling factor***. Identifying the controlling factor is the first step in finding the leading term of the solution.

Comment: Note that if $z(x)$ is a leading term for $y(x)$ it does not mean that their difference is small; only that their ratio tends to one.

Consider now a linear second-order equation,

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where x_0 is an irregular singular point. The first step in approximating the solution near x_0 is to substitute,

$$y(x) = \exp S(x),$$

which gives,

$$S''(x) + [S'(x)]^2 + p(x)S'(x) + q(x) = 0.$$

This substitution goes back to Carlini (1817), Liouville (1837) and Green (1837).

The resulting equation is of course as complicated as the original one. It turns out, however, that it is typical for

$$S''(x) \ll [S'(x)]^2 \qquad \text{as } x \to x_0.$$

(Check for all above examples.) Then, it implies that

$$[S'(x)]^2 \sim -p(x)S'(x) + q(x) \qquad \text{as } x \to x_0.$$

Note that we moved two terms to the right-hand side since no function can be asymptotic to zero. We then proceed to integrate this relation (ignoring again any justification) to find $S(x)$.

Example: Let us explore the example

$$x^3 y''(x) = y(x)$$

in depth. The substitution $y(x) = \exp S(x)$ yields,

$$x^3 S''(x) + x^3 [S'(x)]^2 = 1,$$

which is not more solvable than the original equation (even less as it is nonlinear). Assuming that $S''(x) \ll [S'(x)]^2$, an assumption that we need to justify a posteriori, we get

$$[S'(x)]^2 \sim x^{-3},$$

hence

$$S'(x) \sim \pm x^{-3/2}.$$

If this were an equation we would have $S(x) = \mp 2x^{-1/2} + c$. Here this constant could depend on x, i.e.,

$$S(x) = \mp 2x^{-1/2} + c_\pm(x),$$

as long as

$$S'(x) = \pm x^{-3/2} + c'_\pm(x) \sim \pm x^{-3/2},$$

i.e., $c'_\pm(x) \ll x^{-3/2}$ as $x \to 0^+$. Note that this is consistent with the assumption that

$$S''(x) \sim - \mp \frac{3}{2} x^{-5/2} \ll [S'(x)]^2 \sim x^{-3}.$$

Let us focus on the positive solution and see if this result can be refined. Make the ansatz

$$S(x) = 2x^{-1/2} + c(x), \qquad c'(x) \ll x^{-3/2},$$

which substituted into the (full) equation for $S(x)$ gives

$$\frac{3}{2} x^3 x^{-5/2} + x^3 c''(x) + x^3 (-x^{3/2} + c'(x))^2 = 1,$$

or,

$$\frac{3}{2} x^{1/2} + x^3 c''(x) - 2x^{3/2} c'(x) + x^3 [c'(x)]^2 = 0.$$

Since $c'(x) \ll x^{-3/2}$ the last term is much smaller than the third. Moreover, assuming that we can differentiate the asymptotic relation,

$$c''(x) \ll -\frac{3}{2} x^{-5/2},$$

we remain with $\frac{3}{2}x^{1/2} \sim 2x^{3/2}c'(x)$, or,

$$c'(x) \sim \frac{3}{4x}.$$

Again, if this were an equality we would get $c(x) = \frac{3}{4}\log x$. Here we have

$$c(x) = \frac{3}{4}\log x + d(x),$$

where $d'(x) \ll \frac{3}{4x}$.

Once again, we restart, this time with the ansatz

$$S(x) = 2x^{-1/2} + \frac{3}{4}\log x + d(x).$$

Substituting in the equation for $S(x)$,

$$\frac{3}{2}x^{-5/2} - \frac{3}{4x^2} + d''(x) + \left(-x^{-3/2} + \frac{3}{4x} + d'(x)\right)^2 = \frac{1}{x^3},$$

which simplifies into

$$-\frac{3}{16x^2} + d''(x) + [d'(x)]^2 - 2x^{-3/2}d'(x) + \frac{3}{2x}d'(x) = 0.$$

Since $x^{-1} \ll x^{-3/2}$ and $d'(x) \ll 3/4x$ (from which we conclude that $d''(x) \ll x^{-2}$), we remain with the asymptotic relation,

$$-\frac{3}{16x^2} \sim 2x^{-3/2}d'(x),$$

i.e.,

$$d'(x) \sim -\frac{3}{32}x^{-1/2}.$$

From this we deduce that

$$d(x) \sim \frac{3}{16}x^{1/2} + \delta(x),$$

where $\delta'(x) \ll x^{-1/2}$. This time, the leading term vanishes as $x \to 0$. Thus, we conclude that

$$S(x) \sim 2x^{-1/2} + \frac{3}{4}\log x + c,$$

which gives that

$$y(x) \sim c\, x^{3/4} e^{2x^{-1/2}}.$$

This was long, exhausting, and relying on shaky grounds!

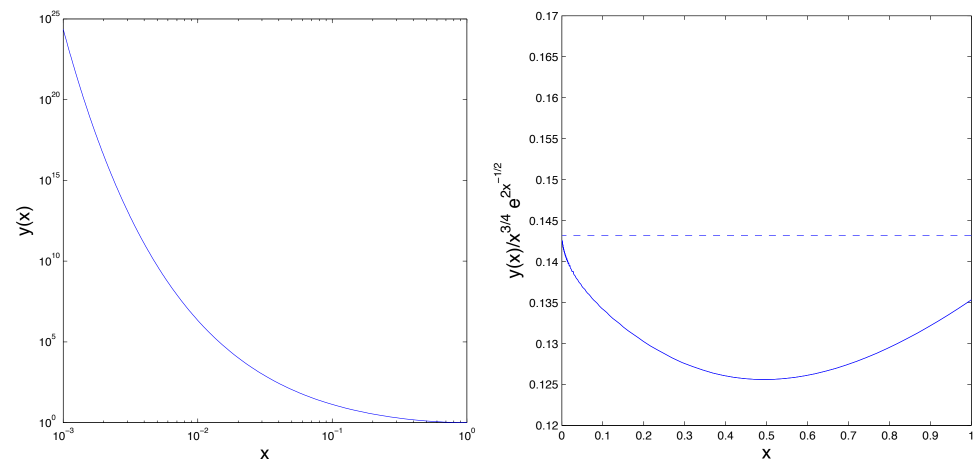

Figure 2.2:

Numerical validation Since we do not have a theoretical justification to the above procedure, let us try to evaluate the quality of the approximation numerically. On the one hand, let us solve the equation $x^3y''(x) = y(x)$ numerically, with the initial date

$$y(1) = 1 \qquad \text{and} \qquad y'(1) = 0.$$

The solution is shown in Figure 2.2*a* on a log-log scale.

We know, on the other hand that the solution is *asymptotically* a linear combination of the form

$$y(x) = c_1 x^{3/4} e^{2x^{-1/2}}(1 + \epsilon_1(x)) + c_2 x^{3/4} e^{-2x^{-1/2}}(1 + \epsilon_2(x)),$$

where $\epsilon_{1,2}(x)$ tend to zero as $x \to 0$. Since one of the two solutions terns to zero as $x \to 0$, we expect that

$$\frac{y(x)}{x^{3/4}e^{2x^{-1/2}}} \sim c_1(1 + \epsilon_2(x)).$$

In Figure 2.2*a* we show $y(x)/x^{3/4}e^{2x^{-1/2}}$ versus x. The deviation from the constant $c_1 \approx 0.1432$ is $c_1\epsilon_1(x)$. ▲▲▲

The technique which we have used above is called the ***method of dominant balance***. It based on (i) identifying the terms that appear to be small, dropping them, thus replacing the equation by an asymptotic relation. (ii) We then replace the asymptotic sign by an equality and solve the differential equation. (iii) We check that the result is consistent and allow for additional weaker variations. (iv) We iterate this procedure.

Example: We go back once more to our running example and try to improve the approximation. At this stage we have

$$y(x) = x^{3/4} e^{2x^{-1/2}} [1 + \epsilon(x)].$$

We will substitute into the equation and try to solve for $\epsilon(x)$ as a power series of x. Setting $w(x) = 1 + \epsilon(x)$, we have

$$y'(x) = x^{3/4} e^{2x^{-1/2}} \left[\frac{3}{4x} w(x) - x^{-3/2} w(x) + w'(x) \right],$$

and

$$\begin{aligned} y''(x) = x^{3/4} e^{2x^{-1/2}} &\left[\frac{3}{4x} - x^{-3/2} \right] \left[\frac{3}{4x} w(x) - x^{-3/2} w(x) + w'(x) \right] \\ &+ x^{3/4} e^{2x^{-1/2}} \left[\frac{3}{4x} w'(x) - x^{-3/2} w'(x) + w''(x) - \frac{3}{4x^2} w(x) + \frac{3}{2} x^{-5/2} w(x) \right]. \end{aligned}$$

This equals $x^{3/4} e^{2x^{-1/2}} w(x) x^{-3}$, which leaves us with the equation,

$$\begin{aligned} x^{-3} w(x) = &\left[\frac{3}{4x} - x^{-3/2} \right] \left[\frac{3}{4x} w(x) - x^{-3/2} w(x) + w'(x) \right] \\ &+ \left[\frac{3}{4x} w'(x) - x^{-3/2} w'(x) + w''(x) - \frac{3}{4x^2} w(x) + \frac{3}{2} x^{-5/2} w(x) \right]. \end{aligned}$$

This further simplifies into

$$w''(x) + \left(\frac{3}{2x} - 2x^{-3/2} \right) w'(x) - \frac{3}{16x^2} w(x) = 0.$$

This is a linear equation. Since we have extracted the singular parts of the solution, there is hope that this remaining equation can be dealt with by simpler means. This does not mean that the resulting equation no longer has an irregular singularity at zero. The only gain is that $w(x)$ does not diverge at the origin.

We proceed to solve this equation by the method of dominant balance. The equation for $\epsilon(x)$ is

$$\epsilon''(x) + \left(\frac{3}{2x} - 2x^{-3/2}\right)\epsilon'(x) - \frac{3}{16x^2}(1 + \epsilon(x)) = 0.$$

Since $\epsilon(x) \ll 1$ as $x \to 0$ we remain with

$$\epsilon''(x) - 2x^{-3/2}\epsilon'(x) \sim \frac{3}{16x^2},$$

subject to the constraint that $\epsilon(x) \to 0$. We do not know whether among the three remaining terms there are some greater than the others. We therefore need to investigate four possible scenarios:

① $\epsilon'' \sim 3/16x^2$ and $x^{-3/2}\epsilon' \ll \epsilon''$. In this case we get that

$$\epsilon(x) \sim -\frac{3}{16}\log x + ax + b,$$

which does not vanish at the origin.

② $\epsilon'' \sim 2x^{-3/2}\epsilon'$ and $3/16x^2 \ll \epsilon''$. In this case,

$$(\log \epsilon(x))' \sim 2x^{-3/2},$$

i.e.,

$$\log \epsilon(x) \sim -4x^{-1/2} + c,$$

which again violates the condition at the origin.

③ All three terms are of equal importance. This means that

$$(e^{-4x^{-1/2}}\epsilon'(x))' \sim \frac{3}{16x^2}e^{-4x^{-1/2}}.$$

Integrating we get again divergence at the origin.

④ This leaves for unique possibility that $-2x^{-3/2}\epsilon' \sim \frac{3}{16x^2}$ and $\epsilon'' \ll -2x^{-3/2}\epsilon'$. Then

$$\epsilon'(x) \sim -\frac{3}{32}x^{-1/2},$$

i.e.,

$$\epsilon(x) \sim -\frac{3}{16}x^{1/2} + \epsilon_1(x),$$

where $\epsilon_1'(x) \ll x^{-1/2}$.

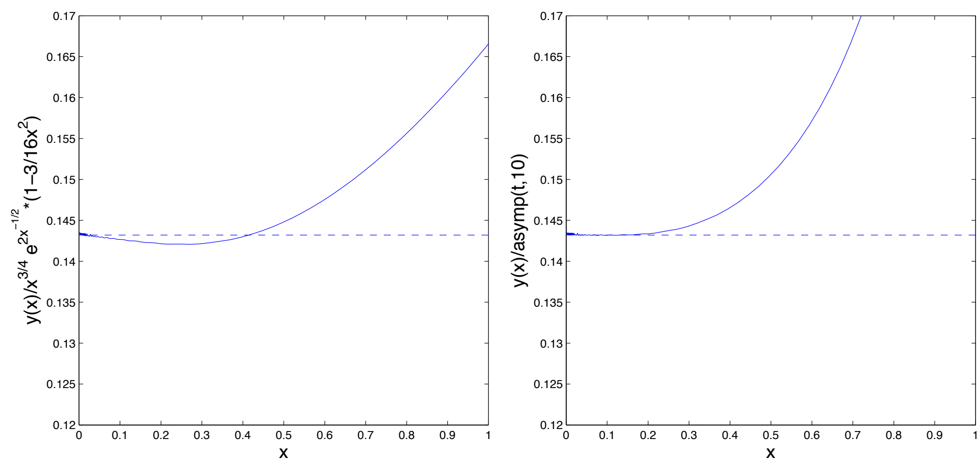

Figure 2.3:

Thus, we already have for approximation

$$y(x) = x^{3/4} e^{2x^{-1/2}} \left[1 - \frac{3}{16} x^{1/2} + \epsilon_1(x) \right].$$

See Figure 2.3*a*.

We can proceed further, discovering doing so that each correction is of power of $x^{1/2}$ greater than its predecessor. At this stage we may well substitute the asymptotic series,

$$w(x) = \sum_{n=0}^{\infty} a_n x^{n/2},$$

and $a_0 = 1$. This yields after some manipulations,

$$y(x) \sim x^{3/4} e^{2x^{-1/2}} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})\,\Gamma(n + \frac{3}{2})}{\pi 4^n \, n!} x^{n/2}.$$

In Figure 2.3*b* we show the ratio between the exact solution and the approximate solution truncated at $n = 10$. Note how the solution becomes more accurate near the origin, although it becomes less accurate further away, reflecting the fact that the series has a vanishing radius of convergence. ▲▲▲

✎ *Exercise 2.7* Using the method of dominant balance, investigate the second solution to the equation

$$x^2 y''(x) + (1 + 3x)y'(x) + y(x) = 0.$$

Try to imitate all the steps followed in class. You should actually end up with an exact solution!

✎ *Exercise 2.8* Find the leading behavior, as $x \to 0^+$, of the following equations:

① $y''(x) = \sqrt{x}\, y(x)$.
② $y''(x) = e^{-3/x} y(x)$.

2.5 Asymptotic series

Definition 2.5 The power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to be ***asymptotic to the function*** *$y(x)$ as $x \to x_0$, denoted*

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0,$$

if for every $N \in \mathbb{N}$

$$y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \ll (x - x_0)^N, \qquad x \to x_0.$$

This does not require the series to be convergent.

Comment: The asymptotic series does not need to be in integer powers of $x - x_0$. For example,

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^{\alpha n}, \qquad x \to x_0$$

where $\alpha > 0$, if for every N

$$y(x) - \sum_{n=0}^{N} a_n (x - x_0)^{\alpha n} \ll (x - x_0)^{\alpha N}, \qquad x \to x_0.$$

Comment: For $x_0 = \infty$ the definition of the relation

$$y(x) \sim \sum_{n=0}^{\infty} a_n x^{-\alpha n}, \qquad x \to \infty,$$

is that for every $N \in \mathbb{N}$

$$y(x) - \sum_{n=0}^{N} a_n x^{-\alpha n} \ll x^{-\alpha N}, \qquad x \to \infty.$$

Comment: In particular, for $N = 0$,

$$\lim_{x \to x_0} [y(x) - a_0] = 0,$$

i.e., for a function to have an asymptotic series at x_0 it must have a finite limit at this point.

Example: Not all functions have asymptotic series expansions. The function $1/x$ does not have a asymptotic series expansion at $x_0 = 0$ because it diverges. Similarly, the function e^x does not have an asymptotic series expansion at $x_0 = \infty$. ▲▲▲

The difference between a convergent series and an asymptotic series is worth stressing. Recall that a series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is convergent if

$$\lim_{N \to \infty} \sum_{n=N+1}^{\infty} a_n (x - x_0)^n = 0, \qquad \text{for } x \text{ fixed}.$$

Convergence is an absolute property. A series is either convergent or not, and convergence can be determined regardless of whether we know what the limit is (we even have criteria for that). In contrast, a series is asymptotic to a function $f(x)$ if

$$f(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \ll (x - x_0)^N, \qquad \text{for } N \text{ fixed}.$$

Asymptoticity is relative to a function. It makes no sense to ask whether a series is asymptotic. In fact, every power series is asymptotic to some function at x_0.

Proposition 2.1 *Let (a_n) be a sequence of numbers. Then there exists a function $y(x)$ such that*

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0.$$

Proof: Without loss of generality, let us take $x_0 = 0$. We define the following continuous function,

$$\phi(x;\alpha) = \begin{cases} 1 & |x| \le \frac{1}{2}\alpha \\ 2\left(1 - \frac{|x|}{\alpha}\right) & \frac{1}{2}\alpha < |x| < \alpha \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $\phi(x;\alpha)$ has suppose in $[-\alpha, \alpha]$. Then we set

$$\alpha_n = \min(1/|a_n|^2, 2^{-n}),$$

and

$$y(x) = \sum_{n=0}^{\infty} a_n \phi(x;\alpha_n) x^n.$$

Note that $\phi(x;\alpha_n)$ is non-zero only for $|x| < 1/|a_n|^2$ and $x < 1/2^n$.

For every x this series is finite and continuous because it truncates after a finite number of terms. We will now show that

$$y(x) \sim \sum_{n=0}^{\infty} a_n x^n.$$

Fixing N we can find a $\delta > 0$ sufficiently small such that

$$\phi(x;\alpha_n) = 1, \qquad \text{for all } n = 0, 1, \dots, N \text{ for all } |x| < \delta.$$

Thus,

$$\frac{y(x) - \sum_{n=0}^{N} a_n x^n}{x^N} = \sum_{n=N+1}^{\infty} a_n \phi(x;\alpha_n) x^{n-N}.$$

It remains to show that the right hand side tends to zero as $x \to 0$. For $|x| \le \delta$ we only get contributions from n's such that

$$|x| < 2^{-n} \qquad \text{and} \qquad |x| < 1/a_n^2,$$

i.e.,

$$n < \frac{\log x}{\log 2} \qquad \text{and} \qquad |a_n| < \frac{1}{\sqrt{x}}.$$

Hence,

$$\left| \sum_{n=N+1}^{\infty} a_n \phi(x; \alpha_n) x^{n-N} \right| \le \frac{\log x}{\log 2} \sqrt{x} \to 0.$$

■

Before demonstrating the properties of asymptotic series, let us show that solutions to differential equations can indeed be represented by asymptotic series:

Example: Recall that we "found" that a solution to the differential equation

$$x^2 y''(x) + (1 + 3x) y'(x) + y(x) = 0.$$

is a (diverging) power series

$$y(x) = \sum_{n=0}^{\infty} (-x)^n n!.$$

This is of course meaningless. We will now *prove* that this series is indeed asymptotic to the solution.

We start by noting that

$$n! = \int_0^\infty e^{-t} t^n \, dt$$

(recall the definition of the Γ function). We then do formal manipulations, which we do not justify,

$$y(x) = \sum_{n=0}^{\infty} (-x)^n \int_0^\infty e^{-t} t^n \, dt = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} (-x)^n t^n \, dt = \int_0^\infty \frac{e^{-t}}{1 + xt} \, dt.$$

This integral exists, and in fact defines an analytic function (it is called a ***Stieltjes integral***). Moreover, we can check directly that this integral solves the differential equation.

We will now show that this solution has the above asymptotic expansion. Integrating by parts we have

$$\begin{aligned} y(x) = \int_0^\infty \frac{e^{-t}}{1 + xt} \, dt &= - \left. (1 + xt)^{-1} \right|_0^\infty - x \int_0^\infty \frac{e^{-t}}{(1 + xt)^2} \, dt \\ &= 1 - x \int_0^\infty \frac{e^{-t}}{(1 + xt)^2} \, dt. \end{aligned}$$

We may proceed integrating by parts to get

$$y(x) = 1 - x - 2x^2 \int_0^\infty \frac{e^{-t}}{(1+xt)^3}\, dt,$$

and after N steps,

$$y(x) = \sum_{n=1}^{N} n!(-x)^n + (N+1)!\,(-x)^{N+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{N+1}}\, dt.$$

Since the integral is bounded by 1, we get that

$$y(x) - \sum_{n=1}^{N} n!(-x)^n \le (N+1)!\,(-x)^{N+1} \ll x^N, \qquad x \to 0.$$

A more interesting question is how many terms we need to take for the approximation to be optimal. *It is not true that the more the better!* We may rewrite the error as follows

$$\begin{aligned}\epsilon_n &= y(x) - \sum_{n=0}^{N} n!(-x)^n = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt - \sum_{n=0}^{N} \int_0^\infty e^{-t}(-xt)^n\, dt \\ &= \int_0^\infty e^{-t}\left(\frac{1}{1+xt} - \sum_{n=0}^{N}(-xt)^n\right) \\ &= \int_0^\infty e^{-t}\frac{(-xt)^N}{1+xt}\, dt.\end{aligned}$$

What is the optimal N? Note that the coefficients of the series are alternating in sign and their ratio is $(-nx)$. As long as this ratio is less that 1 in absolute value, the error decreases, otherwise it increases. The optimal N is therefore the largest integer less than $1/x$. An evaluation of the error at the optimal N gives

$$\epsilon_N \sim \frac{\pi}{2x} e^{-1/x}.$$

▲▲▲

We are now in measure to prove the properties of asymptotic series:

Proposition 2.2 (Non-uniqueness) *Let*

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0.$$

Then there exists a function $g(x) \ne f(x)$ asymptotic to the same series.

Proof: Take

$$g(x) = f(x) + e^{-1/(x-x_0)^2}.$$

This follows from the fact that

$$e^{-1/x^2} \ll x^n, \qquad x \to 0$$

for every n. The function e^{-1/x^2} is said to be ***subdominant***. ■

Proposition 2.3 (Uniqueness) If a function $y(x)$ has an asymptotic series expansion at x_0 then the series is unique.

Proof: By definition,

$$y(x) - \sum_{n=0}^{N-1} a_n(x-x_0)^{\alpha n} - a_N(x-x_0)^{\alpha N} \ll (x-x_0)^{\alpha N},$$

hence,

$$a_N = \lim_{x \to x_0} \frac{y(x) - \sum_{n=0}^{N-1} a_n(x-x_0)^{\alpha n}}{(x-x_0)^{\alpha N}},$$

which is a constructive definition of the coefficients. ■

Comment: It follows that if two sides of an equation are have asymptotic series expansions we can equate the coefficients term by term.

Proposition 2.4 (Arithmetic operations) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad \textit{and} \qquad g(x) \sim \sum_{n=0}^{\infty} b_n(x-x_0)^n,$$

then

$$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n,$$

and

$$f(x)g(x) \sim \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proof: This follows directly from the definitions. Take the product rule for example. Define $h(x) = f(x)g(x)$. Then,

$$h(x) - a_0 b_0 = f(x)g(x) - a_0 b_0 = (f(x) - a_0)g(x) + a_0(g(x) - b_0),$$

and therefore,

$$\lim_{x \to x_0} [h(x) - a_0 b_0] = 0.$$

Likewise,

$$\begin{aligned} h(x) - a_0 b_0 - (a_0 b_1 + a_1 b_0)(x - x_0) = (f(x) - a_0 - a_1(x - x_0))g(x) \\ + (g(x) - b_0 - b_1(x - x_0))a_0 \\ + a_1(x - x_0)(g(x) - b_0), \end{aligned}$$

hence

$$\begin{aligned} \frac{h(x) - a_0 b_0 - (a_0 b_1 + a_1 b_0)(x - x_0)}{x - x_0} &= \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0} g(x) \\ &+ \frac{g(x) - b_0 - b_1(x - x_0)}{x - x_0} a_0 \\ &+ a_1(g(x) - b_0), \end{aligned}$$

which tends to zero as $x \to x_0$. ■

✎ *Exercise 2.9* Show that if

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad g(x) \sim \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

then

$$f(x)g(x) \sim \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition 2.5 (Integration) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If f is integrable near x_0 then

$$\int_{x_0}^{x} f(t)\,dt \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}.$$

Proof: Set N. By the asymptotic property of f it follows that for every ϵ there exists a δ such that

$$\left| f(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \right| \leq \epsilon (x-x_0)^N, \qquad |x| \leq \delta.$$

Thus,

$$\left| \int_{x_0}^{x} f(t)\,dt - \sum_{n=0}^{N} \frac{a_n}{n+1}(x-x_0)^{n+1} \right| \leq \frac{\epsilon (x-x_0)^{N+1}}{N+1},$$

which proves the claim. ■

Proposition 2.6 (Differentiation 1) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

Then it is not necessarily true that

$$f'(x) \sim \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}.$$

Proof: The problem is tightly related to the presence of subdominant functions. Defining

$$g(x) = f(x) + e^{-1/x^2} \sin(e^{1/x^2}),$$

the two functions have the same asymptotic expansion at zero, but not their derivatives. ■

Proposition 2.7 (Differentiation 2) If $f'(x)$ has an asymptotic expansion and is integrable near x_0 then

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

implies that

$$f'(x) \sim \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}.$$

Proof: Set

$$f'(x) \sim \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

Using the integration Proposition and the uniqueness of the expansion we get the desired result. ■

We come now to the ultimate goal of this section. Suppose we have a differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Suppose that $p(x)$ and $q(x)$ have asymptotic expansions at x_0. Does it imply that $y(x)$ has an asymptotic expansion as well, and that its coefficient can be identified by term-by-term formal manipulations? In general this is true.

First we need to assume that $p'(x)$ also has an asymptotic expansion. Then we usually proceed in two steps. First we *assume* that $y(x)$ has an asymptotic expansion. Then, since

$$y''(x) + [p(x)y(x)]' + [q(x) - p'(x)]y(x) = 0,$$

it follows that

$$y'(x) - y'(x_0) + p(x)y(x) - p(x_0)y(x_0) + \int_{x_0}^{x} [q(t) - p'(t)]y(t)\, dt.$$

Hence $y'(x)$ has an asymptotic expansion and so does $y''(x)$ (by the arithmetic properties). We are then allowed to use the arithmetic properties and the uniqueness of the expansion to identify the coefficients.

It remains however to show that $y(x)$ can indeed to expanded in an asymptotic series. In the next section we will demonstrate the standard approach to do so.

2.6 Irregular singular points at infinity

Irregular singular points at infinity are ubiquitous in equations that arise in physical applications (*e.g.*, Bessel, Airy), and the asymptotic behavior at infinity is of major importance in such applications. In principle, the investigation of irregular singular points at infinity can be dealt with by the change of variables $t = 1/x$, yet, we can use the method of dominant balance to study the asymptotic behavior in the original variables.

Example: Consider the function

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},$$

reminiscent of the Bessel function

$$I_0(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2},$$

i.e., $y(x) = I_0(\sqrt{2x})$. This series is convergent everywhere, yet to evaluate it at, say, $x = 10000$ to ten significant digits requires at least,

$$\frac{10000^n}{(n!)^2} < 10^{-10},$$

and using Stirling's formula,

$$n \log 10000 - 2n \log n + 2n < -10 \log 10,$$

this roughly gives $n > 284$. It would be useful to obtain an approximation that does not require the addition of hundreds of numbers.

Consider the following alternative. First, note that

$$y'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!},$$

hence

$$(xy'(x))' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{[(n-1)!]^2} = y(x),$$

i.e., $y(x)$ is a solution of the differential equation

$$x\,y''(x) + y'(x) = y(x).$$

We are looking for a solution as $x \to \infty$, in the form $y(x) = \exp S(x)$, yielding

$$xS''(x) + x[S'(x)]^2 + S'(x) = 1.$$

As before, we assume that $S''(x) \ll [S'(x)]^2$. We remain with

$$x[S'(x)]^2 + S'(x) \sim 1.$$

This is a quadratic equation, whose solution is

$$S'(x) \sim \frac{-1 \pm \sqrt{1+4x}}{2x} \sim \pm x^{-1/2}, \qquad x \to \infty.$$

Thus,

$$S(x) \sim \pm 2x^{1/2},$$

or

$$S(x) \pm 2x^{1/2} + C(x),$$

where $C'(x) \ll x^{-1/2}$.

Since all the coefficients in the power series are positive, $y(x)$ is an increasing function of x, and the leading behavior must be dominated by the positive sign. We then go to the next equation,

$$x[2x^{1/2} + C(x)]'' + x([2x^{1/2} + C(x)])'^2 + [2x^{1/2} + C(x)]' = 1.$$

Expanding we get

$$x\left[-\frac{1}{2}x^{-3/2} + C''(x)\right] + x\left[x^{-1/2} + C'(x)\right]^2 + \left[x^{-1/2} + C'(x)\right] = 1,$$

and

$$\frac{1}{2}x^{-1/2} + xC''(x) + 2x^{1/2}C'(x) + x[C'(x)]^2 + C'(x) = 0,$$

Recall that $C'(x) \ll x^{-1/2}$ hence $C''(x) \ll x^{-3/2}$, and so we remain with

$$2x^{1/2}C'(x) \sim -\frac{1}{2}x^{-1/2},$$

or

$$C(x) \sim -\frac{1}{4}\log x.$$

The next correction is asymptotic to a constant.

The leading solution is then

$$y(x) \sim c\, x^{-1/4} e^{2x^{1/2}}.$$

We cannot (at this point) evaluate the constant (the equation is homogeneous!), which turns out to be $\frac{1}{2}\pi^{-1/2}$. In Figure 2.4 we show the ratio of this asymptotic

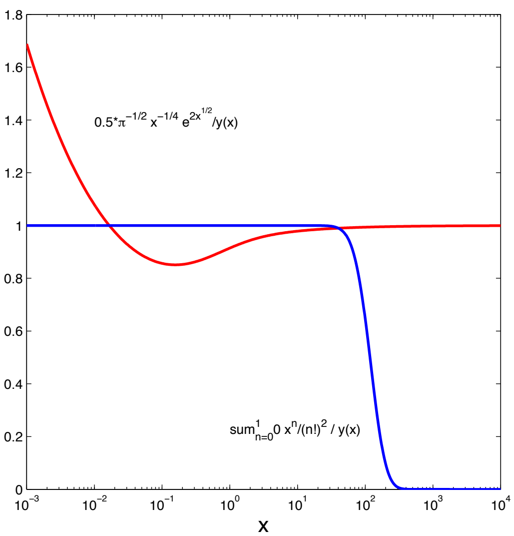

Figure 2.4:

solution and $y(x)$ versus x. Interestingly, the approximation is (relatively) excellent for $x > 100$, whereas the power series truncated at $n = 10$ is very accurate up to that point. Together, the two approximations yield a "uniformly accurate" approximation.

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✎ *Exercise 2.10* Show that the asymptotic behavior at infinity of the solutions to the modified Bessel equation,

$$x^2 y''(x) + x y'(x) - (x^2 + \nu^2) y(x) = 0$$

is

$$y_1(x) \sim c_1 x^{-1/2} e^{x}$$
$$y_2(x) \sim c_2 x^{-1/2} e^{-x}.$$

Example: The modified Bessel equation

$$x^2 y''(x) + x y'(x) - (x^2 + \nu^2) y(x) = 0$$

has an irregular singular point at $x = \infty$. There are two independent solutions, one which decays at infinity and one which diverges. We will study the behavior of the converging one.

By using the method of dominant balance (see above exercise) we find that

$$y(x) \sim cx^{-1/2}e^{-x}.$$

We "peel off" the leading behavior by setting

$$y(x) = x^{-1/2}e^{-x}w(x).$$

Then,

$$y'(x) = x^{-1/2}e^{-x}\left(-\frac{1}{2}x^{-1}w(x) - w(x) + w'(x)\right)$$

$$y''(x) = x^{-1/2}e^{-x}\left(-\frac{1}{2}x^{-1} - 1\right)\left(-\frac{1}{2}x^{-1}w(x) - w(x) + w'(x)\right)$$
$$+ x^{-1/2}e^{-x}\left(\frac{1}{2}x^{-2}w(x) - \frac{1}{2}x^{-1}w'(x) - w'(x) + w''(x)\right).$$

Substituting we get

$$x^2w''(x) - 2x^2w'(x) + \left(\tfrac{1}{4} - \nu^2\right)w(x) = 0.$$

At this point we construct an asymptotic series for $w(x)$.

$$w(x) \sim \sum_{n=0}^{\infty} a_n x^{-n},$$

and proceed formally. Substituting we get

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{-n} - 2\sum_{n=0}^{\infty} na_n x^{-n+1} + \left(\tfrac{1}{4} - \nu^2\right)a_n x^{-n} = 0.$$

Equating the power of x^{-n} we get

$$n(n+1)a_n - 2(n+1)a_{n+1} + \left(\tfrac{1}{4} - \nu^2\right)a_n = 0,$$

or

$$a_{n+1} = \frac{(n+\frac{1}{2})^2 - \nu^2}{2(n+1)}a_n.$$

Recall that we proved in the previous section that if $w(x)$ assumes a power series expansion, then it is given by the above procedure. We will now prove that this is indeed the case. Setting $\lambda = \frac{1}{4} - \nu^2$ we have

$$w''(x) - 2w'(x) + \frac{\lambda}{x^2}w(x) = 0,$$

and we want to show that there exists a solution that can be expanded about infinity. We first write this equation as an integral equation. First,

$$(e^{-2x}w'(x))' + \frac{\lambda e^{-2x}}{x^2}w(x) = 0,$$

from which we deduce that

$$w'(x) = \lambda \int_x^\infty \frac{e^{-2(s-x)}}{s^2} w(s)\, ds.$$

Note that we chose the integration constant such that $w'(x) \to 0$ at infinity. One more integration yields

$$w(x) = 1 + \lambda \int_x^\infty \int_t^\infty \frac{e^{-2(s-t)}}{s^2} w(s)\, dsdt.$$

Exchanging the order of integration we end up with

$$\begin{aligned} w(x) &= 1 + \lambda \int_x^\infty \int_x^s \frac{e^{-2(s-t)}}{s^2} w(s)\, dtds \\ &= 1 + \frac{\lambda}{2} \int_x^\infty \frac{e^{-2(s-x)} - 1}{s^2} w(s)\, ds \end{aligned}$$

We now claim that the solution to this integral equation is bounded for sufficiently large x. That is, there exist $a, B > 0$ such that $|w(x)| \le B$ for $x \ge a$. To show that we proceed formally and iterate this integral,

$$w(x) = 1 + \frac{\lambda}{2} \int_x^\infty \frac{K(x,s)}{s^2}\, ds + \left(\frac{\lambda}{2}\right)^2 \int_x^\infty \int_{s_1}^\infty \frac{K(x,s_1)}{s_1^2} \frac{K(s_1,s_s)}{s_s^2}\, ds + \dots,$$

where $K(x, s) = e^{-2(s-x)} - 1$. Since $|K(x, s)| \le 1$ for $x \ge s$, it follows that the k-th term of this series is bounded by

$$|I_k| \le \left(\frac{\lambda}{2}\right)^n \int_x^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \frac{1}{s_1^2} \cdots \frac{1}{s_n^2}\, ds_n \dots ds_1 \le \left(\frac{\lambda}{2}\right)^n \frac{x^{-n}}{n!},$$

i.e., the series converges absolutely and is bounded by $e^{-\lambda/2x}$. Since we constructed an absolutely converging series that satisfies an iterative relation satisfied by $w(x)$, it is indeed the solution.

Having proved the boundedness of $w(x)$, it remains to show that it has an asymptotic expansion. We start with $w(x) = 1 + w_1(x)$, and

$$|w_1(x)| = \left|\frac{\lambda}{2}\int_x^\infty \frac{K(x,s)}{s^2} w(s)\,ds\right| \le \frac{\lambda}{2} B \int_s^\infty \frac{ds}{s^2} = \frac{\lambda}{2x} B,$$

i.e., $w(x) \to 1$. Next,

$$\begin{aligned} w_1(x) &= \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)} - 1}{s^2}\,ds + \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)} - 1}{s^2} w_1(s)\,ds \\ &= -\frac{\lambda}{2x} + \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)} - 1}{s^2} w_1(s)\,ds. \end{aligned}$$

Using the bound, $|w_1(x)| \le \frac{\lambda B}{2x}$, we get that

$$\left|w_1(x) + \frac{\lambda}{2x}\right| \le \frac{\lambda^2 B}{4}\int_x^\infty \frac{ds}{s^3} = \frac{\lambda^2 B}{12x^2},$$

and so on.

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