Ordinary Differential Equations

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Contents

Bibliography

1. Theory of ordinary differential equations by E.A. Coddington and N. Levinson.
2. Ordinary differential equations by V.I. Arnold.
3. Solving ordinary differential equations by E. Hairer, N.P Nørsett and G. Wanner.

Chapter 1

Existence and uniqueness

1.1 Introduction

Definition 1.1 A ***differential equation*** *is an equation that relates a function to its derivative(s). The unknown is the function. A differential equation is said to be* **ordinary** *(רגילה) if the function is uni-variate, and more precisely if its domain is a connected subset of* $\mathbb{R}$*. We abbreviate ordinary differential equation into* ODE.

Example: Find a function $y : \mathbb{R} \to \mathbb{R}$ that satisfies the equation

$$y' = 6y,$$

or in a different notation,

$$\forall t \in \mathbb{R} \qquad y'(t) = 6\,y(t).$$

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A number of questions arise right away:

① Does the equation have a solution?
② If it does, is the solution unique?
③ What is the solution?
④ Is there a systematic way to solve such an equation?

In the above example, direct substitution shows that any function of the form $y(t) = a\,e^{6t}$, $a \in \mathbb{R}$ is a solution of the equation. Thus, a solution exists and it is not unique. This is not surprising. We know from elementary calculus that knowing the derivative of a function does not uniquely determine the function; it does only up to an ***integration constant***. We could try to obtain a problem for which there exists a unique solution by imposing an additional condition, like the value of the function at a certain point, say, $y(2) = 8$. In this case, the unique solution that belongs to the above exponential family of solutions is found by setting

$$y(2) = a\,e^{12} = 8,$$

i.e., $a = 8\,e^{-12}$. Still, can we be sure that this is the unique solution to the equation

$$y' = 6y \qquad y(2) = 8 \;?$$

We will soon know the answer to this question.

Definition 1.2 An ordinary differential equation is said to be of **order** *k if the highest derivative of the unknown function that appears in the equation is the k-th derivative.*

Example: The following equation is a first-order ODE:

$$y'(t) = t \sin\sqrt{3 + y^2(t)},$$

which we will often write in the more sloppy form:

$$y' = t \sin\sqrt{3 + y^2}.$$

(It is sloppy because it is not clear in this notation that t is a point in the domain of y; suppose we wanted to denote the independent variable by x, as we often do.)
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First-order ODE The most general first-order ODE is of the form

$$\Phi(y'(t), y(t), t) = 0.$$

Such an equation is said to be ***implicit*** (הצגה סתומה). We will always assume that the equation has been brought into ***explicit form*** (הצגה מפורשת):

$$y'(t) = f(t, y(t)).$$

Second-order ODE The most general second-order ODE is of the form

$$\Phi(t, y''(t), y'(t), y(t)) = 0,$$

and in explicit form

$$y''(t) = f(t, y'(t), y(t)).$$

Example: Consider the following equation:

$$y''(t) + y(t)y'(t) + y(t)\,\sin t = 0.$$

As in the first example, we ask ourselves whether this equation has a solution and whether it is unique. At this point it is clear that there won't be a unique solution unless we specify some extra information about the function at certain points. The question is how much extra information is needed in order to obtain an equation that has a unique solution (assuming that such exists). ▲▲▲

Systems of differential equations In the same way as we go from univariate algebraic equations into multivariate ones, so we can extend the idea of a differential equation into a system of differential equations in which more than one function is unknown.

Example: Find functions y, z, such that

$$\begin{aligned} y'(t) &= 6tz \\ z'(t) &= z^3 y' + 4. \end{aligned}$$

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More generally, in a first-order system of n (explicit) ODEs the unknown is a vector-valued function, $y : \mathbb{R} \to \mathbb{R}^n$, that satisfies an equation of the form

$$y'(t) = f(t, y(t)),$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. In components,

$$y_i(t) = f_i(y_1(t), \ldots, y_n(t), t), \qquad i = 1, \ldots, n.$$

High-order equations and first-order systems Every n-th order (scalar) equation can be turned into a first-order system of n ODEs by a simple procedure. Suppose we have an equation of the form

$$y^{(n)}(t) = f(t, y^{n-1}(t), \dots, y'(t), y(t)).$$

We then define the following vector-valued function,

$$Y = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

It satisfies the following system of equations,

$$Y_1' = Y_2 \qquad Y_2' = Y_3 \qquad \cdots \qquad Y_{n-1}' = Y_n \qquad Y_n' = f(t, Y_1, \dots, Y_n),$$

which we can rewrite in vector form,

$$Y'(t) = F(t, Y(t)).$$

What is ordinary in an ODE? A differential equation is called ordinary when the unknown function depends on a single real-valued variable. This is in contrast to ***partial differential equations*** (משוואות חלקיות), in which the unknown function depends on more than one variable.

Example: An example of a PDE is the ***heat equation*** (also known as the ***diffusion equation***): the unknown function depends on two variables (position and time), $y : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and satisfies the equation,

$$\frac{\partial y}{\partial t}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t).$$

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The theory of PDEs is infinitely harder than the theory of ODEs, and will not be considered in this course (we will eventually explain why there is such a fundamental difference between the two).

Why do we care about ODES? Differential and integral calculus were developed in the 17th century motivated by the need to describe the laws of nature. Classical mechanics is "one big system of ODEs"; if the instantaneous position of all the particles in the world at time t can be viewed as a huge vector $y(t)$, then according to classical mechanics, the second derivative of this vector (the acceleration) is a god-given function of the current positions and velocities of all the particles. That is,

$$y''(t) = f(t, y'(t), y(t)).$$

As we will see, such a system may have a unique solution if we specify both y (positions) and y' (velocities) at an initial time (the big bang?). By Newton's laws, the evolution of the universe is determined forever from its initial conditions.

Example: Consider a single "point particle" that can only move along a single axis. The particle is attached to a spring that is connected to the origin. Newton's law is

$$my''(t) = -k\,y(t),$$

where m is the mass of the particle and k is the spring constant (the force law is that the force on the particle is propositional to its distance from the origin and inverse in sign). As we will learn, the most general solution to this equation is

$$y(t) = a\,\cos\sqrt{\tfrac{k}{m}}t + b\,\sin\sqrt{\tfrac{k}{m}}t,$$

where a, b are two "integration constants". They can be determined by specifying, say, $y(0)$ and $y'(0)$. ▲▲▲

ODEs are important not only in physics. Any (deterministic) process that evolves continuously can be described by an ODE. ODEs are used generically to describe evolving systems in chemistry, economics, ecology, demography, and much more.

The oldest ODE The first documented instance of an ODE is an equation studied by Newton (1671), while developing differential calculus:

$$y'(t) = 1 - 3t + y(t) + t^2 + ty(t) \qquad y(0) = 0.$$

Newton's approach to solve this equation was based on a method that we call today an ***asymptotic expansion***. He looked for a solution that can be expressed as a power series (טור חזקות),

$$y(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots.$$

Substituting into the equation,

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 = 1 - 3t + \left(a_1 t + a_2 t^2 + a_3 t^3\right) + t^2 + t\left(a_1 t + a_2 t^2 + a_3 t^3\right).$$

He then proceeded to identify equal powers of t,

$$a_1 = 1 \qquad 2a_2 = -3 + a_1 \qquad 3a_3 = a_2 + 1 + a_1 \qquad 4a_4 = a_3 + a_2,$$

etc. This yields, $a_1 = 1$, $a_2 = -1$, $a_3 = 1/3$, $a_4 = -1/6$, so that

$$y(t) = t - t^2 + \frac{t^3}{3} - \frac{t^4}{6} + \cdots$$

For fixed t, as more terms as added, the closer we are to the true solution, however, for every fixed number of terms the approximation deteriorates for large t.

A problem studied by Leibniz In 1674 Leibniz (Newton's competitor on the development of calculus) studied a geometrical problem that had been studied earlier by Fermat. He tried to solve the following differential equation,

$$y'(t) = -\frac{y(t)}{\sqrt{a^2 - y^2(t)}}.$$

Remember that at that time, derivatives were not defined as rigorously as you learned it (it took 200 more years until calculus took the form you know). For Leibniz, the derivative was roughly,

$$y'(t) \approx \frac{\Delta y}{\Delta t} = -\frac{y(t)}{\sqrt{a^2 - y^2(t)}}.$$

If we consider Δy and Δt as finite numbers, then we may write

$$\frac{\sqrt{a^2 - y^2}}{y}\Delta y = -\Delta t.$$

Suppose that $y(0) = y_0$. Then summing over many small steps Δt and Δy,

$$\int_{y_0}^{y} \frac{\sqrt{a^2 - \xi^2}}{\xi} d\xi = -t.$$

This integral can actually be calculated analytically (the primitive function is known) and so Leibniz could get an implicit representation of the solution.

Initial and boundary value problems Recall that any n-th order equation can be represented as a first-order system, thus, until further notice we will assume that we deal with first order systems,

$$y'(t) = f(t, y(t)).$$

As we saw, even if such an equation has a solution, it will fail to be unique unless additional conditions on y are prescribed. In general, a system of n first-order equations requires n additional data on y. If all the data are prescribed at a single point we call the resulting problem an ***initial value problem***. The origin of this name is realizations where t is time, and the data are prescribed as some initial time. If the data are distributed between two points of more then we call the resulting problem a ***boundary value problem***. The origin of the name is situations where the independent variable is space (and then often denoted by x), and $y(x)$ is some function of space that is prescribed at the boundary of the region of interest.

While the distinction between initial and boundary value problems may seem immaterial it is fundamental. The theories of initial and boundary value problems are very different. The first part of this course will be devoted to initial value problems.

Integral formulation Consider a first order system,

$$y'(s) = f(s, y(s)) \qquad y(t_0) = y_0.$$

Integrating both sides from t_0 to t,

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

This is an ***integral equation*** ($y(t)$ is still unknown), which shares a lot in common with the differential equation. We will later see in which sense the differential and integral systems are equivalent, and in what sense they differ.

1.2 Euler's approximation scheme

Consider an initial value problem,

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

where y and f are vector-valued, and suppose that we want to find the function in some interval $[t_0, T]$ (we will often refer to t as "time" as a suggestive interpretation).

In 1768 Euler proposed a method to approximate the solution. He considered partitions of the interval:

$$t_0 < t_1 < t_2 < \cdots < t_n = T,$$

and approximated $y(t_j)$ by y_j given by

$$\frac{y_{j+1} - y_j}{t_{j+1} - t_j} = f(t_j, y_j),$$

or equivalently,

$$y_{j+1} = y_j + f(t_j, y_j)(t_{j+1} - t_j), \qquad j = 0, \dots, n-1.$$

Thus, the first-order differential equation is approximated by a ***one-step difference equation*** (משוואת הפרשים).

Euler's belief was that if we refine the partition sufficiently, then we will get arbitrarily close to the true solution (whose existence was not questioned). Euler's method (usually variants of it) is used until today to numerically approximate the solution of ordinary differential systems.

In 1820 Cauchy proved that Euler's approximation does indeed converge, as the partition is refined, to a solution of the differential equation, and in fact Cauchy's proof is the first proof that the differential system has a solution and that this solution is unique.

✎ *Exercise 1.1* Approximate the solution to the equation

$$y'(t) = 6y(t) \qquad y(0) = 1$$

at the point $t = 1$ using Euler's scheme for a uniform partition $t_{j+1} - t_j = 1/n$. (i) Obtain an explicit expression for the approximate solution. (ii) What is the true solution? Does the approximate solution converge to the true solution as $n \to \infty$?

TA material 1.1 Learn to solve a large class of equations.

1.3 The Cauchy-Peano existence proof

In order to be able to analyze the differential system, we must have some a priori assumptions about the function f. Throughout, we will assume that f is continuous in its two arguments in some open and connected domain $D \subset \mathbb{R} \times \mathbb{R}^n$.

$\mathbb{R}^n$ can be endowed with a norm. As we know from advanced calculus, all norms on $\mathbb{R}^n$ are equivalent[1], and without loss of generality we will use the ***Euclidean norm***, which we denote by $\|\cdot\|$.

(2 hrs, (תשעב))

Definition 1.3 Let $D \subset \mathbb{R}^n \times \mathbb{R}$ be a domain in which f is continuous. Let I be an open connected interval. A function $\varphi : I \to \mathbb{R}^n$ is called an **ε-approximation** *to the differential system in D if:*

① *φ is continuous.*
② *φ is continuously differentiable, except perhaps at a finite set of points S, where φ' has one-sided limits.*
③ *For every $t \in I$, $(t, \varphi(t)) \in D$.*
④ *For every $t \in I \setminus S$,*

$$\|\varphi'(t) - f(t, \varphi(t))\| \le \varepsilon.$$

Let $(t_0, y_0) \in D$ and consider a rectangle of the form

$$R = [t_0, t_0 + a] \times \{y : \|y - y_0\| \le b\} \subset D.$$

Since f is continuous and R is compact, it follows that f is bounded in R; define

$$M = \max_{(t,y)\in R} \|f(t, y)\|.$$

Then, we define

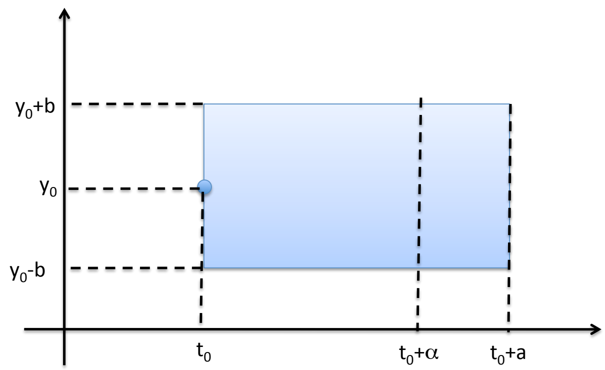
$$\alpha = \min(a, b/M),$$

and replace R by another rectangle,

$$R_0 = [t_0, t_0 + \alpha] \times \{y : \|y - y_0\| \le b\} \subset D.$$

[1]Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent if there exist constants $c, C > 0$ such that

$$(\forall x) \qquad c\|x\|_1 \le \|x\|_2 \le C\|x\|_1.$$



Theorem 1.1 There exists for every $\varepsilon > 0$ an ε-approximation in R_0 to the equation $y'(t) = f(t, y(t))$ on the interval $[t_0, t_0 + \alpha]$ satisfying the initial condition $y(t_0) = y_0$.

Comment: A similar theorem can be formulated for an interval on the left of t_0. Mathematically, there is no difference between "past" and "future".

Proof: Since f is continuous on R_0, it is uniformly continuous (רציפה במידה שווה). Thus, given $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$|t - s| < \delta \qquad \text{and} \qquad \|y - z\| < \delta \quad \text{then} \quad \|f(t, y) - f(s, z)\| < \varepsilon.$$

Take the interval $[t_0, t_0 + \alpha]$ and partition it,

$$t_0 < t_1 < \dots t_n = t_0 + \alpha,$$

such that

$$\max_j (t_{j+1} - t_j) < \min(\delta, \delta/M).$$

We then construct the Euler approximation to this solution, and connect the point by straight segments, so that Euler's approximation is a polygon, which we denote by $\varphi(t)$. It is a piecewise continuously differentiable function.

As long as the polygon is in R_0 its derivative, which is equal to some $f(t_j, y(t_j))$, has norm less than M. Thus, over an interval of length α, $\varphi(t)$ can differ from y_0 (in norm) by at most $\alpha M \le b$. This means that all Euler's polygons remain in R_0 on the entire interval $[t_0, t_0 + \alpha]$, no mater how the partition of the segment is done.

Let t be a point in the j-th interval. Then,

$$\varphi'(t) = f(t_j, \varphi(t_j)),$$

and further,

$$\|\varphi'(t) - f(t, \varphi(t))\| = \|f(t_j, \varphi(t_j)) - f(t, \varphi(t))\|.$$

By our choice of the partition,

$$|t - t_j| < \delta.$$

Because the derivative of φ is bounded (in norm) by M,

$$\|\varphi(t) - \varphi(t_j)\| \le M|t - t_j| \le M(\delta/M) = \delta.$$

By our choice of δ,

$$\|f(t_j, \varphi(t_j)) - f(t, \varphi(t))\| < \varepsilon,$$

which proves that φ is an ε-approximation to our system. ■

Definition 1.4 A family $\mathscr{F}$ of functions $I \to \mathbb{R}^n$ is said to be **equicontinuous** *(רציפים במידה אחידה) if there exists for every $\varepsilon > 0$ a $\delta > 0$, such that for every $|t_1 - t_2| < \delta$ and every $g \in \mathscr{F}$,*

$$\|g(t_1) - g(t_2)\| < \varepsilon.$$

(Equicontinuity is uniform continuity with the same δ valid for all $g \in \mathscr{F}$.)

Theorem 1.2 (Arzela-Ascoli) Let $\mathscr{F}$ be an infinite family of functions $I \to \mathbb{R}^n$ that are uniformly bounded, namely,

$$\sup_{f \in \mathscr{F}} \max_{t \in I} \|f(t)\| < \infty,$$

and equicontinuous. Then there exists a sequence $f_n \in \mathscr{F}$ that converges uniformly on I.

(3 hrs, (תשעב))

Proof: Let (r_k) be a sequence that contains all the rational numbers in I. Consider the set

$$\{f(r_1) : f \in \mathscr{F}\} \subset \mathbb{R}^n.$$

Since $\mathscr{F}$ is uniformly bounded, this set is bounded, and therefore has an accumulation point: there exists a sequence $f_j^{(1)} \in \mathscr{F}$ for which $f_j^{(1)}(r_1)$ converges.

Consider then the set

$$\{f_j^{(1)}(r_2) : j \in \mathbb{N}\}.$$

This set is bounded, therefore there exists a subsequence $f_j^{(2)} \in (f_k^{(1)})$ for which $f_j^{(2)}(r_2)$ converges (and also $f_j^{(2)}(r_1)$ converges).

We proceed inductively, and for every k derive a sequence $f_j^{(k)}$ such that $f_j^{(k)}(r_i)$ converges for every $i \le j$.

Consider now the diagonal sequence $F_k = f_k^{(k)}$. It converges pointwise on all the rationals in I. Why? Because for every ℓ the diagonal sequence is eventually a subsequence of $f_j^{(\ell)}$. This means that

$$(\forall r \in I \cap \mathbb{Q})(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) : (\forall m, n > N) \; (\|F_n(r) - F_m(r)\|) < \varepsilon.$$

Note that since we don't know what the limit is, we use instead the Cauchy criterion for convergence.

Fix $\varepsilon > 0$. Since the family of functions $\mathscr{F}$ is equicontinuous,

$$(\exists \delta > 0) : (\forall t_1, t_2, |t_1 - t_2| < \delta)(\forall f \in \mathscr{F}) \; (\|f(t_1) - f(t_2)\| < \varepsilon).$$

Partition the interval I into subintervals,

$$I = I_1 \cup I_2 \cup \cdots \cup I_K,$$

such that each subinterval if shorter than δ. In every subinterval I_k select a rational number r_k. Since there is a finite number of such r_ks,

$$(\exists N \in \mathbb{N}) : (\forall m, n > N)(\forall k = 1, \dots, K) \; (\|F_n(r_k) - F_m(r_k)\|) < \varepsilon$$

Then for every $t \in I_k$, and $m, n > N$,

$$\|F_n(t) - F_m(t)\| \le \underbrace{\|F_n(t) - F_n(r_k)\|}_{\text{equicontinuity}} + \underbrace{\|F_n(r_k) - F_m(r_k)\|}_{\text{convergence}} + \underbrace{\|F_m(r_k) - F_m(t)\|}_{\text{equicontinuity}} \le 3\varepsilon.$$

Thus, the sequence F_n satisfies Cauchy's criterion for uniform convergence, which concludes the proof. ■

With this, we can prove Cauchy's theorem:

Theorem 1.3 (Cauchy-Peano) Let $f(t,y)$ and the rectangle R_0 be defined as above. Then the differential equation $y'(t) = f(t,y(t))$ has a solution on the interval $[t_0, t_0+\alpha]$ satisfying the initial condition $y(t_0) = y_0$.

Comments:

① Note that our only conditions on f is that it be continuous.

② At this stage not a word about uniqueness.

③ This theorem is known as a ***local existence theorem***. It only guarantees the existence of a solution on some interval. We have no a priori knowledge on how long this interval is.

Proof: Let (ε_k) be a sequence of positive numbers that tends to zero. We proved that for every k there exists a function $\varphi_k : I \to \mathbb{R}^n$ that is an ε_k-approximation on $I = [t_0, t_0+\alpha]$ to the differential equation, and satisfies the initial conditions.

The sequence (φ_k) is uniformly bounded on I because

$$(\forall t \in I)(\forall k \in \mathbb{N}) \ (\|\varphi_k(t) - y_0\| < b).$$

It is also equicontinuous because

$$(\forall t_1, t_2 \in I)(\forall k \in \mathbb{N}) \ (\|\varphi_k(t_2) - \varphi_k(t_1)\| \le M\,|t_2 - t_1|),$$

and therefore, given $\varepsilon > 0$ we set $\delta = \varepsilon/M$, and

$$(\forall |t_1 - t_2| < \delta)(\forall k \in \mathbb{N}) \ (\|\varphi_k(t_2) - \varphi_k(t_1)\| \le \varepsilon).$$

It follows from the Arzela-Ascoli theorem that (φ_k) has a subsequence (which we do not relabel) that converges uniformly on I; denote the limit by y. It remains to show that y is a solution to the differential equation.

For each $k \in \mathbb{N}$ it follows from the fundamental theorem of calculus that[2]

$$\begin{aligned}\varphi_k(t) &= y_0 + \int_{t_0}^{t} \varphi_k'(s)\,ds \\ &= y_0 + \int_{t_0}^{t} f(s,\varphi_k(s))\,ds + \int_{t_0}^{t} [\varphi_k'(s) - f(s,\varphi_k(s))]\,ds.\end{aligned}$$

The fact that φ_k' is not defined at a finite number of points is immaterial.
Using the fact that φ_k is an ε_k-approximation to the differential equation,

$$\left\| \int_{t_0}^{t} [\varphi_k'(s) - f(s,\varphi_k(s))]\,ds \right\| \le \varepsilon_k |t - t_0|.$$

Letting $k \to \infty$ and using the fact that the convergence of φ_k to y is uniform, we get that

$$y(t) = y_0 + \int_{t_0}^{t} f(s,y(s))\,ds.$$

Since $y \in C^1(I)$, we can differentiate this equation and obtain $y'(t) = f(t,y(t))$. ■

✎ *Exercise 1.2* Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open domain, $f \in C(D)$, and $(t_0, y_0) \in D$. Show that the equation

$$y'(t) = f(t,y(t)), \qquad y(t_0) = y_0$$

has a solution on some open interval I that contains t_0.

1.4 Uniqueness

Example: Consider the initial value problem,

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0.$$

[2] The functions φ_k are ***absolutely continuous*** (רציפות בהחלט), which means that there exists for every $\varepsilon > 0$ a $\delta > 0$ such that for every finite sequence of disjoint intervals (x_k, y_k) that satisfies $\sum_k (y_k - x_k) < \delta$,

$$\sum_k \|f(y_k) - f(x_k)\| < \varepsilon.$$

A function is absolutely continuous if and only if it is almost everywhere differentiable, and its derivative satisfies the fundamental theorem of calculus.

By the Cauchy-Peano theorem there exists an $\alpha > 0$ such that this equation has a solution in the interval $[0, \alpha]$. The Cauchy-Peano theorem, however, does not guarantee the uniqueness of the solution. A trivial solution to this equation is

$$y(t) = 0.$$

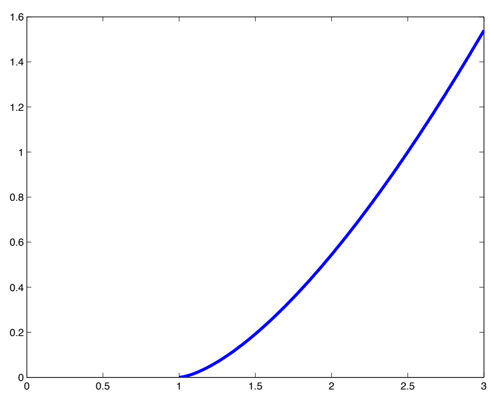
Another solution is

$$y(t) = \left(\frac{2t}{3}\right)^{3/2}.$$

In fact, there are infinitely many solutions of the form

$$y(t) = \begin{cases} 0 & 0 \le t \le c \\ \left(\frac{2(t-c)}{3}\right)^{3/2} & t > c, \end{cases}$$

for every $c > 0$.



(What would have happened had we approximated the solution with Euler's method?)
▲ ▲ ▲

The above example shows that existence of solutions does not necessarily go hand in hand with uniqueness. It turns out that in order to guarantee uniqueness we need more stringent requirements on the function f.

Proposition 1.1 *Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by continuous in t and Lipschitz continuous in y with constant L in the domain D. Let φ_1 and φ_2 be ε_1- and ε_2-approximations to the equation $y'(t) = f(t, y(t))$ contained in D on the interval I, and moreover for a certain $t_0 \in I$,*

$$\|\varphi_1(t_0) - \varphi_2(t_0)\| < \delta.$$

Then, for all $t \in I$,

$$\|\varphi_1(t) - \varphi_2(t)\| < \delta e^{L|t-t_0|} + \frac{\varepsilon_1 + \varepsilon_2}{L}\left(e^{L|t-t_0|} - 1\right).$$

Comment: By Lipschitz continuity in y we mean that for every (t, y) and (t, z) in D,

$$\|f(t, y) - f(t, z)\| \le L\|y - z\|.$$

Comment: This proposition states that the deviation between two approximate solutions can be split into two: (i) an initial deviation that may grow exponentially fast, and (ii) a deviation due to the fact that both functions are only approximate solutions to the same equation.

Proof: Suppose, without loss of generality that $t > t_0$. By the definition of ε-approximations, for every $t_0 \le s \le t$,

$$\begin{aligned}
\|\varphi_1'(s) - f(s, \varphi_1(s))\| &\le \varepsilon_1 \\
\|\varphi_2'(s) - f(s, \varphi_2(s))\| &\le \varepsilon_2.
\end{aligned}$$

For $i = 1, 2$,

$$\varphi_i(t) = \varphi_i(t_0) + \int_{t_0}^t f(s, \varphi_i(s))\, ds + \int_{t_0}^t [\varphi_i'(s) - f(s, \varphi_i(s))]\, ds,$$

hence

$$\left\|\varphi_i(t) - \varphi_i(t_0) - \int_{t_0}^t f(s, \varphi_i(s))\, ds\right\| \le \varepsilon_i(t - t_0).$$

Using the triangle inequality, $\|a - b\| \le \|a\| + \|b\|$,

$$\left\|[\varphi_1(t) - \varphi_2(t)] - [\varphi_1(t_0) - \varphi_2(t_0)] - \int_{t_0}^t [f(s, \varphi_1(s)) - f(s, \varphi_2(s))]\, ds\right\| \le (\varepsilon_1 + \varepsilon_2)(t - t_0).$$

It further follows that

$$\begin{aligned}
\|\varphi_1(t) - \varphi_2(t)\| \le{}& \|\varphi_1(t_0) - \varphi_2(t_0)\| \\
&+ \int_{t_0}^t \|f(s, \varphi_1(s)) - f(s, \varphi_2(s))\|\, ds + (\varepsilon_1 + \varepsilon_2)(t - t_0).
\end{aligned}$$

Define now $r(t) = \|\varphi_1(t) - \varphi_2(t)\|$. Then, using the Lipschitz continuity of f and the bound on the initial deviation,

$$r(t) \le \delta + L \int_{t_0}^{t} r(s)\, ds + (\varepsilon_1 + \varepsilon_2)(t - t_0). \tag{1.1}$$

This is an ***integral inequality***. Our goal is to get an explicit bound for $r(t)$. We proceed as follows: define

$$R(t) = \int_{t_0}^{t} r(s)\, ds,$$

so that

$$R'(t) \le \delta + LR(t) + (\varepsilon_1 + \varepsilon_2)(t - t_0), \qquad R(t_0) = 0,$$

which is a ***differential inequality***.

Next, for every $t_0 \le s \le t$:

$$\left(e^{L(s-t_0)} R(s)\right)' = e^{L(s-t_0)}[R'(s) - LR(s)] \le e^{L(s-t_0)} \left(\delta + (\varepsilon_1 + \varepsilon_2)(s - t_0)\right),$$

Since integration preserves order, we may integrate this inequality from t_0 to t and get

$$e^{L(t-t_0)} R(t) - \underbrace{R(0)}_{0} \le \int_{t_0}^{t} e^{L(s-t_0)} \left(\delta + (\varepsilon_1 + \varepsilon_2)(s - t_0)\right) ds,$$

and it only remains to integrate the right hand side explicitly[3] and substitute back into (1.1). ■

TA material 1.2 Various Gronwall inequalities.

Theorem 1.4 (Uniqueness) Let φ_1 and φ_2 be two solutions of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

where f is continuous in t and Lipschitz continuous in y. Then $\varphi_1 = \varphi_2$.

Proof: This is an immediate corollary of the previous proposition as $\delta = \varepsilon_1 = \varepsilon_2 = 0$. ■

[3]

$$\int_0^t s e^{-Ls}\, ds = -\frac{t}{L} e^{-Lt} - \frac{1}{L^2}\left(e^{-Lt} - 1\right).$$

Comment: We have obtained also an estimate for the error of Cauchy's approximation. It y is the solution and φ is an ε-approximation with exact initial data, then

$$\|\varphi(t) - y(t)\| < \frac{\varepsilon}{L}\left(e^{L|t-t_0|} - 1\right).$$

(5 hrs, (תשעב))

1.5 The Picard-Lindlöf existence proof

There is another standard way to prove the existence of solutions, and it is of sufficient interest to justify a second proof. Recall that a solution to the initial value problem $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ is also a solution to the integral equation,

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Define the function $y^{(0)}(t) \equiv y_0$, and further define

$$y^{(1)}(t) = y_0 + \int_{t_0}^{t} f(s, y^{(0)}(s))\, ds.$$

For what values of t is it defined? For the same $[t_0, t_0 + \alpha]$ as in the previous sections. We can show that $(t, y^{(1)}(t))$ remains in R_0 as follows: let

$$\tau = \sup\{t \in I : \|y^{(1)}(t) - y_0\| \le b\}.$$

If $\tau < t_0 + \alpha$ then

$$\|y^{(1)}(\tau) - y_0\| \le M(t - t_0) < b,$$

contradicting the fact that τ is the supremum.

We then define inductively for every $n \in \mathbb{N}$:

$$y^{(n+1)}(t) = y_0 + \int_{t_0}^{t} f(s, y^{(n)}(s))\, ds,$$

and show as above that $(t, y^{(n+1)}(t))$ remains in R_0 for all $t \in I$.

We are now going to show that the sequence $y^{(n)}$ converges uniformly to a solution to the initial value problem. The difference with the proof based on Euler's method is that we need to assume the Lipschitz continuity of f in R_0. Define,

$$\Delta^{(n)} = y^{(n+1)} - y^{(n)},$$

then,

$$\Delta^{(n+1)}(t) = \int_{t_0}^{t} \left[f(s, y^{(n)}(s)) - f(s, y^{(n-1)}(s)) \right] ds,$$

and

$$\|\Delta^{(n+1)}(t)\| \le L \int_{t_0}^{t} \|\Delta^{(n)}(s)\| \, ds,$$

with

$$\|\Delta^{(1)}(t)\| \le \int_{t_0}^{t} \|f(s, y_0)\| \, ds \le M \, |t - t_0|.$$

Once again, we have an integral inequality, but we may proceed differently:

$$\|\Delta^{(2)}(t)\| \le L \int_{t_0}^{t} \|\Delta^{(1)}(s)\|| \, ds \le LM \int_{t_0}^{t} (s - t_0) \, ds \le \frac{LM}{2} \, |t - t_0|^2,$$

$$\|\Delta^{(3)}(t)\| \le L \int_{t_0}^{t} \|\Delta^{(2)}(s)\|| \, ds \le \frac{L^2 M}{2} \int_{t_0}^{t} (s - t_0)^2 \, ds \le \frac{L^2 M}{3!} \, |t - t_0|^3,$$

and generally,

$$\|\Delta^{(n)}(t)\| \le \frac{M}{L} \frac{(L|t - t_0|)^n}{n!}.$$

Now we write the n-th function as follows:

$$y^{(n)} = \left(y^{(n)} - y^{(n-1)} \right) + \left(y^{(n-1)} - y^{(n-2)} \right) + \cdots + y_0 = y_0 + \sum_{k=0}^{n-1} \Delta^{(k)}.$$

By the Weierstaß M-test this series converges uniformly to a limit y. All that remains is to let $n \to \infty$ in the equation

$$y^{(n+1)}(t) = y_0 + \int_{t_0}^{t} f(s, y^{(n)}(s)) \, ds,$$

to obtain

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds.$$

Comment: Usually one puts this proof in the context of the Picard's ***contractive mapping theorem***. I purposely did it here "by hand" to show how straightforward the proof is. This method is known as the method of ***successive approximations***.

Note also that we can estimate the error of the successive approximations. We have

$$y^{(n)} - y^{(m)} = \sum_{k=m}^{n-1} \Delta^{(k)},$$

hence

$$
\begin{aligned}
\|y^{(n)}(t) - y^{(m)}(t)\| &\le \sum_{k=m}^{n-1} \|\Delta^{(k)}(t)\| \le \sum_{k=m}^{n-1} \frac{L^{k-1}M}{k!}(t-t_0)^k \\
&\le \sum_{k=m}^{\infty} \frac{L^{k-1}M}{k!}(t-t_0)^k = \sum_{k=0}^{\infty} \frac{L^{m+k-1}M}{(m+k)!}(t-t_0)^{m+k} \\
&\le \sum_{k=0}^{\infty} \frac{L^{m+k-1}M}{m!k!}(t-t_0)^{m+k} = \frac{L^{m-1}M(t-t_0)^m}{m!} \sum_{k=0}^{\infty} \frac{L^k}{k!}(t-t_0)^k \\
&= \frac{M[L(t-t_0)]^m}{Lm!} e^{L(t-t_0)}.
\end{aligned}
$$

The right hand side does not depend on n, so we can let $n \to \infty$ to get an estimate for $y - y^{(m)}$.

1.6 Continuation of solutions

The existence and uniqueness proof shows that there exists an α such that a unique solution exists in the interval $[t_0, t_0 + \alpha]$. Does it really mean that the domain of existence is limited? What prevents the existence of solution for all $t \in \mathbb{R}$?

Example: Consider the initial value problem:

$$
y'(t) = y^2(t), \qquad y(0) = 1.
$$

It is easy to check that

$$
y(t) = \frac{1}{1-t},
$$

is a solution on the interval $[0,1)$, and we know therefore that it is unique. This solution however diverges as $t \to 1$, indicating that it can't be further continued. ▲▲▲

This situation is typical to equations that do not have ***global solutions***. Suppose that $f(t,y)$ is continuous in t and Lipschitz continuous in y for all $(t,y) \in \mathbb{R} \times R^n$ (as is $f(t,y) = y^2$), and define

$$
a = \sup\{\tau > t_0 : \text{there exists a solution on } [t_0, \tau]\}.
$$

If a is finite and

$$
\lim_{t \to a} y(t) = b < \infty,
$$

then consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(a) = b.$$

By the existence and uniqueness theorems there exists a γ such that a unique solution exists on $[a, a+\gamma]$, i.e., a unique solution exists on $[t_0, a+\gamma]$, violating the definition of a as a supremum. Thus, if a is finite then

$$\lim_{t \to a} y(t) = \infty,$$

namely, the only way a solution can fail to exist indefinitely is if it diverges.

This observation has an important implication: whenever you want to show that a certain equation has global solutions, you can try to show that as long as a solution exists it is bounded uniformly.

(6 hrs, (תשעב))

1.7 Generalized solutions

1.8 Continuity of solutions with respect to initial conditions

The solution of an IVP is of course a function of "time" t, but it can also be viewed as a function of the initial data, namely, we can viewed the solution of

$$y'(t) = f(t, y(t)), \qquad y(\tau) = \xi$$

as a function of t,τ, and ξ; for now we denote this solution as $\varphi(t, \tau, \xi)$. This is a different point of view in which we have a function that represents the solution not to a single IVP, but to all IVPs together.

We know that φ is differentiable in t, as it satisfies the equations,

$$\frac{\partial \varphi}{\partial t}(t, \tau, \xi) = f(t, \varphi(t, \tau, \xi)), \qquad \varphi(\tau, \tau, \xi) = \xi. \tag{1.2}$$

The question is how regular it is with respect to the initial data τ and ξ. This is an important question, as if it were, for example, discontinuous with respect to the initial data, it would not be an appropriate model for any real life problem. In this context, one speaks of the ***well-posedness*** of an equation, which includes existence, uniqueness, and continuity on initial data (and possibly other parameters in the equation).

Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a domain in which f is continuous in t and Lipschitz in y. Let $I = [a, b]$ and let $\psi : I \to \mathbb{R}^n$ be a solution to the differential equation, such that

$$\{(t, \psi(t)) : t \in I\} \subset D.$$

Theorem 1.5 *There exists a* $\delta > 0$ *such that for every*

$$(\tau, \xi) \in U_\delta = \{(\tau, \xi) : \tau \in I, \|\xi - \psi(\tau)\| < \delta\}$$

There exists a solution φ *to* (1.2) *on* I *with* $\varphi(\tau, \tau, \xi) = \xi$. *Moreover,* φ *is continuous on* $I \times U_\delta$.

Chapter 2

Linear equations

2.1 Preliminaries

Recall that for every normed space $(\mathscr{X}, \|\cdot\|)$ we can endow the vector space of linear operators $L(\mathscr{X}, \mathscr{X})$ with an ***operator norm***,

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

It can be shown that this is indeed a norm. This norm has the following (additional) properties:

1. For every $x \in \mathscr{X}$ and $A \in L(\mathscr{X}, \mathscr{X})$,
$$\|Ax\| \le \|A\| \|x\|.$$
2. For every $A, B \in L(\mathscr{X}, \mathscr{X})$,
$$\|AB\| \le \|A\| \|B\|.$$
3. It is always the case that $\|\mathrm{id}\| = 1$.

We will deal with the vector space $\mathbb{R}^n$ endowed with the Euclidean norm. One can ask then what is the explicit form of the corresponding operator norm for matrices $A : \mathbb{R}^{n \times n}$.

✎ *Exercise 2.1* Find the explicit form of the operator norm corresponding to the Euclidean norm on R^n.

2.2 Linear homogeneous systems

2.2.1 The space of solutions

Let $I \subset \mathbb{R}$ be a closed interval and let $A : I \to \mathbb{R}^{n\times n}$ be a continuous n-by-n matrix-valued function of I. We denote its entries, which are real-valued functions, by a_{ij}. The first-order system of differential equations,

$$y'(t) = A(t)y(t) \tag{2.1}$$

is called a ***linear homogeneous system***. In component notation,

$$y_j'(t) = \sum_{k=1}^{n} a_{jk}(t)y_k(t).$$

Example: A linear homogeneous system for $n = 2$:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'(t) = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{2}\sin t & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t).$$

▲▲▲

Proposition 2.1 For every $\tau \in I$ and $\xi \in \mathbb{R}^n$, there exists a unique function $y : I \to \mathbb{R}^n$ such that

$$y'(t) = A(t)y(t), \qquad y(\tau) = \xi.$$

Comment: The main claim is that a (unique) solution exists on the whole of I; in fact this remains true also if $I = \mathbb{R}$.

Proof: The function $f(t, y) = A(t)y$ is continuous in t and Lipschitz in y, hence a local solution exists for some interval containing τ. We need to show that the solution can be continued on the entire of I, i.e., that it cannot diverge. This is done as follows. We first write the corresponding integral equation,

$$y(t) = \xi + \int_\tau^t A(s)y(s)\, ds,$$

which holds as long as a solution exists.

Taking norms,

$$\|y(t)\| \le \|\xi\| + \int_\tau^t \|A(s)y(s)\|\, ds \le \|\xi\| + \int_\tau^t \|A(s)\| \|y(s)\|\, ds.$$

Because A is continuous on I, so is $\|A\|$, which means that there exists a uniform bound

$$\|A(s)\| \le M, \qquad \forall t \in I.$$

This implies that

$$\|y(t)\| \le \|\xi\| + M \int_\tau^t \|y(s)\|\, ds.$$

This is an integral inequality that we can solve. Defining $Y(t) = \int_\tau^t \|y(s)\|\, ds$ we get

$$Y'(t) \le \|\xi\| + MY(t), \qquad Y(\tau) = 0.$$

Then,

$$e^{-M(t-\tau)}[Y'(t) - MY(t)] \le \|\xi\| e^{-M(t-\tau)}.$$

Integrating from τ to t,

$$e^{-M(t-\tau)} Y(t) \le \frac{\|\xi\|}{-M} \left(e^{-M(t-\tau)} - 1\right),$$

and further

$$Y(t) \le \frac{\|\xi\|}{M} \left(e^{M(t-\tau)} - 1\right).$$

Substituting back into the inequality for $\|y(t)\|$,

$$\|y(t)\| \le \|\xi\| + M \frac{\|\xi\|}{M} \left(e^{M(t-\tau)} - 1\right) = \|\xi\| e^{M(t-\tau)}.$$

This can never diverge in I, which proves that a solution to the IVP exists on I. ■

The trivial solution One characteristic of linear homogeneous systems is that $y \equiv 0$ is always a solution. This solution is called the ***trivial solution***.

Proposition 2.2 Let $\mathscr{X}$ be the space of functions $y : I \to \mathbb{R}^n$ satisfying the linear homogeneous system (2.1). *Then $\mathscr{X}$ is a vector space over the complex field $\mathbb{C}$ with respect to pointwise addition and scalar multiplication.*

Proof: Let $y, z \in \mathscr{X}$, i.e.,

$$y'(t) = A(t)y(t) \qquad \text{and} \qquad z'(t) = A(t)z(t).$$

For every $\alpha, \beta \in \mathbb{C}$,

$$(\alpha y + \beta z)'(t) = \alpha A(t)y(t) + \beta A(t)z(t) = A(t)(\alpha y + \beta z)(t),$$

which proves that $\alpha y + \beta z \in \mathscr{X}$. ■

Comment: The elements of $\mathscr{X}$ are functions $I \to \mathbb{R}^n$.

Comment: This is what physicists call the ***principle of superposition***. Any linear combination of solutions is also a solution.

Recall that the dimension of a vector space is defined as the maximal number n for which there exists a set of n linearly independent vectors, but every $(n+1)$ vectors are linearly dependent.

Proposition 2.3 Let $\mathscr{X}$ be defined as above, then

$$\dim \mathscr{X} = n,$$

that is there exists a set of functions $\varphi_1, \dots, \varphi_n \in \mathscr{X}$, such that

$$\sum_{k=1}^{n} c_k \varphi_k = 0$$

implies that $c_k = 0$ for all k. Moreover, for every set of $n+1$ function $\varphi_k \in \mathscr{X}$ there exist scalars c_k, not all of them zero, such that

$$\sum_{k=1}^{n+1} c_k \varphi_k = 0.$$

Proof: Take $\tau \in I$, and select n independent vectors $\xi_1, \dots, \xi_n \in \mathbb{R}^n$; we denote the k-th component of ξ_j by ξ_{kj}. For each ξ_j, the IVP

$$\varphi_j'(t) = A(t)\varphi_j(t), \qquad \varphi_j(\tau) = \xi_j$$

has a unique solution. We claim that these solution are independent. Indeed, suppose that

$$\sum_{k=1}^{n} c_k \varphi_k \equiv 0,$$

then in particular for $t = \tau$,

$$\sum_{k=1}^{n} c_k \xi_k = 0,$$

but because the ξ_k are independent, the c_k are all zero.

Let $\varphi_1, \dots, \varphi_n$ be linear independent solutions. Take any $\tau \in I$ and let

$$\xi_j = \varphi(\tau).$$

The ξ_n are linearly independent, for if they were linearly dependent, we would have

$$\xi_n = \sum_{k=1}^{n-1} c_k \xi_k,$$

but then

$$\psi = \sum_{k=1}^{n-1} c_k \varphi_k$$

is a solution to the differential system satisfying $\psi(\tau) = \xi_n$, and by uniqueness $\psi = \varphi_n$, which contradicts the linear independence of the φ_k.

Suppose now that φ is another solution of the differential system. Set

$$\xi = \varphi(\tau).$$

Because the ξ_j span $\mathbb{R}^n$, there exist coefficients c_k such that

$$\xi = \sum_{k=1}^{n} c_k \xi_k.$$

By the same argument as above,

$$\varphi = \sum_{k=1}^{n} c_k \varphi_k,$$

which proves that the φ_k form a basis. ■

2.2.2 Fundamental matrices

Let now $\varphi_1, \dots, \varphi_n$ be a basis. Let $\Phi : I \to \mathbb{R}^{n\times n}$ be the matrix-valued function whose columns are the vectors φ_j,

$$\Phi(t) = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix} (t),$$

that is, φ_{ij} is the i-th component of the j-th basis function. This matrix satisfies the matrix-valued differential equation,

$$\Phi'(t) = A(t)\Phi(t),$$

which in components reads

$$\varphi'_{ij}(t) = a_{ik}(t)\varphi_{kj}(t).$$

$\Phi(t)$ is called a ***fundamental matrix*** (מטריצה יסודית) for the linear homogeneous system (2.1); we call a fundamental matrix for (2.1) any solution $X : I \to \mathbb{R}^{n\times n}$ of the matrix equation

$$X'(t) = A(t)X(t), \tag{2.2}$$

whose columns are linearly independent.

Because the columns of a fundamental matrix span the space of solutions to (2.1), any solution $y : I \to \mathbb{R}^n$ can be represented as

$$y_i = \sum_{j=1}^{n} c_j \varphi_{ij},$$

where the c_j's are constant, and in vector form,

$$y = \Phi c.$$

Proposition 2.4 For a matrix-valued function to be a fundamental matrix of (2.1)*, it has to satisfy the matrix equation* (2.2) *and its columns have to be independent for some $\tau \in I$. In particular, if this holds then its columns are independent for all $t \in I$.*

Proof: Let $\Phi : I \to \mathbb{R}^{n\times n}$ be a fundamental matrix; we know that it is a solution of (2.2) and that any solution $y : I \to \mathbb{R}^n$ of (2.1) can be represented as

$$y = \Phi c,$$

Fix a $\tau \in I$. The vector c is determined by the equation

$$y(\tau) = \Phi(\tau)c.$$

This is a linear equation for c; we know that it has a unique solution (because the columns of Φ span the space of solutions), hence from basic linear algebra, $\det\Phi(\tau) \neq 0$. Since the same considerations hold for every $t \in I$, $\det\Phi(t) \neq 0$ for all $t \in I$. ■

(8 hrs, (תשעב))

Here is another proof that the columns of the matrix remain independent, but it comes with a quantitative estimate:

Proposition 2.5 *Let $\Phi : I \to \mathbb{R}^{n\times n}$ be a fundamental matrix. Then*

$$\det\Phi(t) = \det\Phi(t_0)\exp\left(\int_{t_0}^{t} \operatorname{Tr} A(s)\,ds\right).$$

In particular, if $\det\Phi(t_0) \neq 0$ then $\det\Phi(t) \neq 0$ for all $t \in I$.

Proof: Since

$$\Phi = \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix},$$

we differentiate to obtain

$$\begin{aligned} \Phi' &= \begin{vmatrix} \varphi'_{11} & \cdots & \varphi'_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi'_{n1} & \cdots & \varphi'_{nn} \end{vmatrix} \\ &= \begin{vmatrix} \sum_j a_{1j}\varphi_{j1} & \cdots & \sum_j a_{1j}\varphi_{jn} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \sum_j a_{nj}\varphi_{j1} & \cdots & \sum_j a_{nj}\varphi'_{jn} \end{vmatrix}. \end{aligned}$$

Consider the first determinant. We subtract from the first line a_{12} times the second line, a_{13} times the third line, etc. Then we remain with $a_{11} \det \Phi$. We do the same with the second determinant and remains with $a_{22} \det \Phi$, and so on. Thus,

$$\Phi' = \left(\operatorname{Tr} A\right) \Phi.$$

This is a scalar linear ODE, and it remains to integrate it.

■

Example: Consider the matrix

$$\Phi(t) = \begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix}.$$

This matrix is singular, but its columns are linearly independent. What does it imply? That this matrix-valued function cannot be a fundamental matrix of a linear homogeneous system; there exists no matrix $A(t)$ such that

$$\begin{pmatrix} t \\ 0 \end{pmatrix}' = A(t) \begin{pmatrix} t \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} t^2 \\ 0 \end{pmatrix}' = A(t) \begin{pmatrix} t^2 \\ 0 \end{pmatrix}.$$

▲▲▲

Theorem 2.1 Let Φ be a fundamental matrix for (2.1) *and let $C \in \mathbb{C}^{n \times n}$ be a nonsingular complex (constant) matrix. Then ΦC is also a fundamental matrix. Moreover, if Ψ is a fundamental matrix then $\Psi = \Phi C$ for some nonsingular complex matrix C.*

Proof: By definition $\Phi' = A\Phi$ and we know that $\det \Phi(t) \neq 0$ for all t. Then for every nonsingular complex matrix C,

$$(\Phi C)' = A\Phi C$$

and $\det \Phi(t) C = \det \Phi(t)\, \det C \neq 0$ for all $t \in I$.

Let Ψ be a fundamental matrix. Because it is nonsingular for all t, Ψ^{-1} exists, and

$$0 = (\Psi\Psi^{-1})' = \Psi(\Psi^{-1})' + \Psi'\Psi^{-1},$$

from which we get that

$$(\Psi^{-1})' = -\Psi^{-1}\Psi'\Psi^{-1} = -\Psi^{-1}A\Psi\Psi^{-1} = -\Psi^{-1}A.$$

Now,

$$(\Psi^{-1}\Phi)' = (\Psi^{-1})'\Phi + \Psi^{-1}\Phi' = -\Psi^{-1}A\Phi + \Psi^{-1}A\Phi = 0.$$

It follows that $\Psi^{-1}\Phi = C$, or $\Phi = \Psi C$. ■

Comment: The matrix $(\Psi^*)^{-1}$ satisfies the equation,

$$[(\Psi^*)^{-1}]' = -A^*(\Psi^*)^{-1}.$$

This equation is known as the equation ***adjoint*** (משוואה צמודה) to (2.1).

Proposition 2.6 If Φ is a fundamental matrix for (2.1) *then Ψ is a fundamental matrix for the adjoint system if and only if*

$$\Psi^*\Phi = C,$$

where C is a nonsingular constant matrix.

(9 hrs, (תשעב))

Proof: Let Ψ satisfy the above equation then

$$(\Psi^*)'\Phi + \Psi^*\Phi' = 0,$$

i.e.,

$$(\Psi^*)'\Phi = -\Psi^*A\Phi,$$

and since Φ is nonsingular

$$(\Psi^*)' = -\Psi^*A,$$

and it remains to take the Hermitian adjoint of both sides.
Conversely, if Ψ is a fundamental solution of the adjoint equation then

$$(\Psi^*\Phi)' = 0,$$

which concludes the proof. ■

Corollary 2.1 *Let A be anti-hermitian, i.e.,*

$$A(t) = -A^*(t),$$

then the linear homogeneous system and its adjoint coincide. In particular, if Φ *is a fundamental matrix for* (2.1) *then it is also a fundamental matrix for its adjoint, and*

$$\Phi^*(t)\Phi(t) = C.$$

The matrix $\Phi^*(t)\Phi(t)$ determines the magnitudes of the columns of $\Phi(t)$ and the angles between pairs of vectors. In particular, since any solution is of the form

$$y(t) = \sum_{j=1}^{n} \alpha_j \varphi_j(t),$$

then

$$\|y(t)\|^2 = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} (\varphi_i, \varphi_j) = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} C_{ij},$$

i.e., the norm of any solution is independent of t.

Example: Consider the linear system

$$y'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y(t).$$

Two independent solutions are

$$\varphi_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \qquad \text{and} \qquad \varphi_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

A fundamental matrix is therefore

$$\Phi(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}.$$

It follows that,

$$\Phi^*(t)\Phi(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any solution of this linear system is of the form

$$y(t) = \alpha \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \beta \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Now,

$$\|y(t)\|^2 = \alpha^2 \begin{pmatrix} \sin t & \cos t \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + 2\alpha\beta \begin{pmatrix} \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$
$$+ \beta^2 \begin{pmatrix} \cos t & -\sin t \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \alpha^2 + \beta^2.$$

▲▲▲

2.2.3 The solution operator

Consider the initial value problem,

$$y'(t) = A(t)y(t), \qquad y(t_0) = \text{given}.$$

If $\Phi(t)$ is a fundamental matrix, then the solution is of the form

$$y(t) = \Phi(t)c.$$

The vector c is found by substituting the initial data, $y(t_0) = \Phi(t_0)c$, i.e., $c = \Phi^{-1}(t_0)y(t_0)$, and the solution is therefore,

$$y(t) = \underbrace{\Phi(t)\Phi^{-1}(t_0)}_{\equiv R(t,t_0)} y(t_0).$$

Comments:

① It follows that the solution at time t depends linearly on the solution at time t_0, and the linear transformation between the solution at two different times is the matrix $R(t, t_0)$. We call the matrix $R(t, t_0)$ the ***solution operator*** (אופרטור הפתרון) between times t_0 and t.

② The solution operator seems to depend on the choice of the fundamental matrix, but it doesn't. It doesn't because the solution is unique. We can also see it as any other fundamental matrix $\Psi(t)$ can be represented as $\Psi(t) = \Phi(t)C$, and then

$$\Psi(t)\Psi^{-1}(t_0) = \Phi(t)CC^{-1}\Phi^{-1}(t_0) = \Phi(t)\Phi^{-1}(t_0).$$

③ For every t_0, t_1, t_2,

$$R(t_2, t_1)R(t_1, t_0) = R(t_2, t_0).$$

Why? Because for every $y(t_0)$,

$$y(t_2) = R(t_2, t_0)y(t_0),$$

but also,

$$y(t_2) = R(t_2, t_1)y(t_1) = R(t_2, t_1)R(t_1, t_0)y(t_0).$$

④ $R(t, t) = I$ for all t.

⑤ $R(t, t_0)$ is non-singular and

$$[R(t, t_0)]^{-1} = R(t_0, t).$$

Let us recapitulate: given a homogeneous linear system, we need to find n independent solutions. Then we can construct a fundamental matrix and the associated solution operator. This gives an explicit solution for any initial value problem. The catch is that in most cases we will not be able to find n independent solutions.

2.2.4 Order reduction

Consider an n-dimensional linear homogeneous system, and suppose that we have m linearly independent solutions

$$\varphi_1, \dots, \varphi_m,$$

with $m < n$.

Example: Consider the differential system in some interval that does not include the origin:

$$y'(t) = \begin{pmatrix} 0 & 1 \\ 3/2t^2 & -1/2t \end{pmatrix} y(t).$$

A solution to this system is

$$\varphi_1(t) = \begin{pmatrix} 1/t \\ -1/t^2 \end{pmatrix},$$

as

$$\varphi_1'(t) = \begin{pmatrix} -1/t^2 \\ 2/t^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3/2t^2 & -1/2t \end{pmatrix} \begin{pmatrix} 1/t \\ -1/t^2 \end{pmatrix}.$$

All we need in order to be able to solve an initial value problem is to find another independent solution. The knowledge of one solution enables us to get for the other solution an equation of lower order (in this case, a scalar equation). ▲▲▲

2.3 Non-homogeneous linear systems

Consider now a system of the form

$$y'(t) = A(t)y(t) + b(t),$$

where $A : I \to \mathbb{R}^{n\times n}$ and $b : I \to \mathbb{R}^n$ are continuous. Such a system is called a linear non-homogeneous system.

It turns out that if a fundamental matrix for the corresponding homogeneous system is known, then we can solve the non-homogeneous system in explicitly form.

Theorem 2.2 Let $\Phi(t)$ be a fundamental matrix for the homogenous system, then the solution for the non-homogeneous system is

$$y(t) = R(t,t_0)y(t_0) + \int_{t_0}^{t} R(t,s)b(s)\,ds,$$

where as above $R(t,t_0) = \Phi(t)\Phi^{-1}(t_0)$.

Proof: Note that the solution operator also satisfies the homogeneous system,

$$R'(t,t_0) = A(t)R(t,t_0)$$

Differentiating the above solution,

$$\begin{aligned} y'(t) &= R'(t,t_0)y(t_0) + \int_{t_0}^{t} R'(t,s)b(s)\,ds + R(t,t)b(t) \\ &= A(t)R(t,t_0)y(t_0) + A(t)\int_{t_0}^{t} R(t,s)b(s)\,ds + b(t) \\ &= A(t)y(t) + b(t). \end{aligned}$$

By uniqueness, this is *the* solution. ■

(10 hrs, (תשעב))

Look at the structure of the solution. It is a sum of the solution of the solution to the homogeneous problem and a term that "accumulates" the non-homogeneous term.

The space of solutions is an affine space Another point of practical interest is the following. Suppose that $y_{in}(t)$ is a solution to the non-homogeneous equation and let $y(t)$ by any other solution. Then,

$$(y - y_{in})' = A(t)(y - y_{in}),$$

which implies that any solution to the non-homogeneous system can be expressed as $y_{in}(t)$ plus a solution to the homogeneous system.

The method of variation of constants Another way to derive Theorem 2.2 is the method of ***variation of constants***. The solution to the homogeneous system is of the form

$$y(t) = \Phi(t)C.$$

We then look for a solution where C is a function of time,

$$y(t) = \Phi(t)C(t).$$

Differentiating we get

$$y'(t) = \underbrace{\Phi'(t)}_{A(t)\Phi(t)} C(t) + \Phi(t)C'(t) = A(t)y(t) + b(t) = A(t)\Phi(t)C(t) + b(t),$$

hence

$$\Phi(t)C'(t) = b(t) \qquad \text{or} \qquad C'(t) = \Phi^{-1}(t)b(t).$$

Integrating we find a solution to the non-homogeneous equation,

$$C(t) = \int_{t_0}^{t} \Phi^{-1}(s)b(s)\, ds \qquad \text{i.e.} \qquad y(t) = \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)\, ds.$$

This is not the most general equation: we can always add a solution to the homogeneous system:

$$y(t) = \Phi(t)C_1 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)\, ds,$$

and substituting the initial data we get that $C_1 = \Phi^{-1}(t_0)y(t_0)$.

Comment: The method shown here is a special case of something known as ***Duhammel's principle***. It is a general approach for expressing the solution of a linear non-homogeneous system in terms of the solution operator of the homogeneous system.

Example: Consider a forced linear oscillator,

$$y'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y(t) + \begin{pmatrix} 0 \\ \sin \omega t \end{pmatrix},$$

where ω is a frequency. This system describes a unit mass attached to a spring with unit force constant and forced externally with frequency ω. The entries of $y(t)$ are the displacement and the momentum.

A fundamental matrix for the homogeneous system is

$$\Phi(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}.$$

Since

$$\Phi^{-1}(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix},$$

then the solution operator is

$$R(t, t_0) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} \sin t_0 & \cos t_0 \\ \cos t_0 & -\sin t_0 \end{pmatrix} = \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix}.$$

Substituting into Theorem 2.2 we get

$$\begin{aligned} y(t) &= \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix} y(t_0) + \int_{t_0}^{t} \begin{pmatrix} \cos(t - s) & \sin(t - s) \\ -\sin(t - s) & \cos(t - s) \end{pmatrix} \begin{pmatrix} 0 \\ \sin \omega s \end{pmatrix} ds \\ &= \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix} y(t_0) + \int_{t_0}^{t} \begin{pmatrix} \sin \omega s \sin(t - s) \\ \sin \omega s \cos(t - s) \end{pmatrix} ds. \end{aligned}$$

Assume that $t_0 = 0$. Then

$$y(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} y(0) + \frac{1}{\omega^2 - 1} \begin{pmatrix} \omega \sin t - \sin \omega t \\ \omega(\cos t - \cos \omega t) \end{pmatrix}.$$

The solution splits into a part that depends on the initial condition plus a part that depends on the forcing. The expression for the non-homogeneous term holds only of $\omega \neq 1$. If $\omega = 1$, then the "natural frequency" coincides with the "forcing frequency, a phenomenon known as ***resonance***. If $\omega = 1$ then we get

$$y(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} y(0) + \frac{1}{2} \begin{pmatrix} \sin t - t\cos t \\ t \sin t \end{pmatrix},$$

i.e., the solution diverges linearly in time. ▲▲▲

2.4 Systems with constant coefficients

2.4.1 The Jordan canonical form

Let A be a (constant) n-by-n matrix. Recall that any matrix is similar to a matrix in ***Jordan form***: that is, there exists a non-singular complex matrix P such that

$$A = PJP^{-1}.$$

and J is of the form

$$J = \begin{pmatrix} J_0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix},$$

where

$$J_0 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix} \qquad \text{and} \qquad J_i = \begin{pmatrix} \lambda_{q+i} & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{q+i} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{q+i} \end{pmatrix},$$

for $i = 1, \dots, s$. If A is ***diagonalizable*** then it only has the diagonal part J_0. (Recall that a matrix is diagonalizable if it is ***normal***, i.e., it commutes with its adjoint.)

2.4.2 The exponential of a matrix

Let A be a (constant) n-by-n matrix. We define

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Is it well-defined? Yes, because,

$$\sum_{k>n} \frac{\|A^k\|}{k!} \le \sum_{k\ge n} \frac{\|A\|^k}{k!} = \sum_{k=0}^{\infty} \frac{\|A\|^{k+n}}{(k+n)!} \le \sum_{k=0}^{\infty} \frac{\|A\|^{k+n}}{n!\,k!} \le \frac{\|A\|^n}{n!} e^{\|A\|} \xrightarrow{n\to\infty} 0,$$

and by Cauchy's criterion the series converges absolutely.

What can we say about the exponential of a matrix? In general,

$$e^{A+B} \ne e^A e^B,$$

unless A and B commute.

If $A = PJP^{-1}$, where J is the Jordan canonical form, then

$$e^A = \sum_{n=0}^{\infty} \frac{(PJP^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{PJ^nP^{-1}}{n!} = Pe^JP^{-1},$$

which means that it is sufficient to learn to exponentiate matrices in Jordan canonical form.

Note that for a block-diagonal matrix J,

$$e^J = \begin{pmatrix} e^{J_0} & & & \\ & e^{J_1} & & \\ & & \ddots & \\ & & & e^{J_s} \end{pmatrix},$$

so that it is sufficient to learn to exponentiate diagonal matrices and matrices of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

The exponential of a diagonal matrix

$$J_0 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix} \qquad \text{is} \qquad e^{J_0} = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_q} \end{pmatrix}.$$

The non-diagonal blocks have the following structure:

$$J_i = \lambda_{q+i} I_{r_i} + E_{r_i},$$

where r_i is the size of the i-th block and

$$E_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Because I_{r_i} and E_i commute then

$$e^{J_i} = e^{\lambda_{q+i} I} e^{E_i} = e^{\lambda_{q+i}} e^{E_i}.$$

The matrix E_i is nilpotent. Suppose for example that $r_i = 4$, then

$$E_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_i^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_i^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_i^4 = 0.$$

Then,

$$e^{E_i} = \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & 1 & \frac{1}{2!} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For later use, we note that if we multiply J_j by a scalar t, then

$$e^{tJ_i} = e^{t\lambda_{q+i}} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Let

$$A = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}.$$

One way to calculate e^{tA} is to calculate powers of A:

$$A^2 = -t^2 I \qquad A^3 = -t^3 A \qquad A^4 = t^4 I,$$

hence

$$e^{tA} = I + tA - \frac{t^2}{2!} I - \frac{t^3}{3!} A + \frac{t^4}{4!} I + \frac{t^5}{5!} A - \frac{t^6}{6!} I - \cdots = \cos t\, I + \sin t\, A,$$

i.e.,

$$e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Alternatively, we diagonalize the matrix,

$$A = \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right] \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right],$$

and then

$$\begin{aligned} e^{tA} &= \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right] \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right] \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{it} & -ie^{it} \\ e^{-it} & ie^{-it} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{it} + e^{-it} & -ie^{it} + ie^{-it} \\ ie^{it} - ie^{-it} & e^{it} + e^{-it} \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

▲ ▲ ▲

2.4.3 Linear system with constant coefficients

We now consider homogeneous linear systems in which the coefficients are constant,

$$y'(t) = Ay(t),$$

where A is a (constant) n-by-n matrix. As we know, solving this system amounts to finding a fundamental matrix.

Proposition 2.7 *The matrix e^{tA} is a fundamental matrix for the system with constant coefficients A.*

Proof: From the definition of the derivative,

$$\lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h} = \left(\lim_{h\to 0} \frac{e^{hA} - I}{h} \right) e^{tA} = A\, e^{tA},$$

hence $y(t) = e^{tA}$ is a fundamental solution. ∎

Comments:

① Since A and e^{tA} commute it is also the case that

$$\frac{d}{dt} e^{tA} = e^{tA} A.$$

② It follows that the solution operator is

$$R(t, t_0) = e^{tA} e^{-t_0 A} = e^{(t-t_0)A},$$

where we used the fact that e^{tA} and e^{sA} commute for every t, s.

③ It follows that the solution of the inhomogenous equation $y'(t) = Ay(t) + b(t)$ is

$$y(t) = e^{(t-t_0)A} y(t_0) + \int_{t_0}^{t} e^{(t-s)A} b(s)\, ds.$$

④ Inspired by the case of $n = 1$, one may wonder whether the exponential is only a fundamental solution for constant coefficients. Isn't the solution to $y'(t) = A(t)y(t)$,

$$\Phi(t) = e^{\int_0^t A(s)\, ds} \; ?$$

Let's try to differentiate:

$$\frac{\Phi(t+h) - \Phi(t)}{h} = \frac{e^{\int_0^{t+h} A(s)\, ds} - e^{\int_0^t A(s)\, ds}}{h}.$$

However, in general

$$e^{\int_0^{t+h} A(s)\, ds} \neq e^{\int_0^t A(s)\, ds} e^{\int_t^{t+h} A(s)\, ds},$$

which is what prevents us from proceeding. These matrices commute either if A is constant, or if it is diagonal (in which case we have n independent equations).

⑤ If $A = PJP^{-1}$ then
$$e^{tA} = Pe^{tJ}P^{-1},$$
and by the properties of fundamental matrices,
$$e^{tA}P = Pe^{tJ}$$
is also a fundamental solution. This means that the t dependence of all the components of the fundamental matrix is a linear combinations of terms of the form $t^m e^{\lambda t}$, where λ is an eigenvalue of A and m is between zero is one minus the multiplicity of λ.

2.5 Linear differential equations of order n

2.5.1 The Wronskian

Let $a_0, a_1, \dots, a_n : I \to \mathbb{C}$ be continuous complex-valued functions defined on some interval. We define the linear ***differential operator***, $L_n : C^1(I) \mapsto C^0(I)$.

$$L_n(f) = a_0 f^{(n)} + a_1 f^{(n-1)} + \cdots + a_{n-1} f' + a_n f,$$

or in another notation,

$$L_n = a_0 \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \frac{d}{dt} + a_n.$$

(It will be assume that $a_0(t) \neq 0$ for all $t \in I$.)

A (scalar) differential equation of the form

$$L_n y = 0$$

is called a homogeneous linear equation of order n. It can also be written in the form

$$y^{(n)}(t) = -\frac{a_1(t)}{a_0(t)} y^{(n-1)}(t) - \frac{a_2(t)}{a_0(t)} y^{(n-2)}(t) - \cdots - \frac{a_{n-1}(t)}{a_0(t)} y'(t) - \frac{a_n(t)}{a_0(t)} y(t).$$

As usual, we can convert an n-th order equation into a first-order system of n equations,
$$Y'(t) = A(t)Y(t),$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n/a_0 & -a_{n-1}/a_0 & -a_{n-2}/a_0 & -a_{n-3}/a_0 & \cdots & -a_1/a_0 \end{pmatrix}.$$

This is a homogeneous linear system, hence we know a lot about the structure of its solutions. It is however a very special system, as the matrix A has most of its elements zero. In fact, every solution of the linear system has the following structure:

$$\begin{pmatrix} \varphi \\ \varphi' \\ \varphi'' \\ \vdots \\ \varphi^{(n-1)} \end{pmatrix}.$$

In particular, a fundamental matrix is determined by n independent scalar functions φ_j:

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdot & \cdot & \cdots & \cdot \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{pmatrix}.$$

The determinant of this matrix is called the ***Wronskian*** of the system $L_n y = 0$ with respect to the solutions $\varphi_1, \dots, \varphi_n$. We denote it as follows:

$$W(\varphi_1, \dots, \varphi_n)(t) = \det \Phi(t).$$

By Proposition 2.5

$$\begin{aligned} W(\varphi_1, \dots, \varphi_n)(t) &= W(\varphi_1, \dots, \varphi_n)(t_0) \exp\left(\int_{t_0}^{t} \operatorname{Tr} A(s)\, ds \right) \\ &= W(\varphi_1, \dots, \varphi_n)(t_0) \exp\left(- \int_{t_0}^{t} \frac{a_1(s)}{a_0(s)}\, ds \right). \end{aligned}$$

As we know, n solutions $\varphi_j(t)$ are linearly independent if the Wronskian (at any point t) is not zero.

The fact that for any n linearly independent solutions φ_j,

$$W(\varphi_1, \dots, \varphi_n)(t) = c \exp\left(-\int^t \frac{a_1(s)}{a_0(s)}\, ds \right)$$

is known as ***Abel's theorem***.

TA material 2.1 Use Abel's theorem to show how given one solution one can reduce the order of the equation for the remaining solutions. Show it explicitly for $n = 2$ and solve an example. Look for example at

`http://www.ux1.eiu.edu/˜wrgreen2/research/Abel.pdf`

Thus, given an n-th order linear operator L_n on I and n solutions $\varphi_j : I \to \mathbb{R}$ satisfying $L_n\varphi_j = 0$ for which $W(\varphi_1, \dots, \varphi_n)(t) \neq 0$, then every function $y : I \to \mathbb{R}$ satisfying $L_n y = 0$ is a linear combination of the φ_j's.

It turns out that the opposite is also true. To every n linearly independent functions corresponds a linear differential operator of order n for which they form a basis:

Theorem 2.3 *Let $\varphi_1, \dots, \varphi_n$ be n-times continuously differentiable functions on an interval I such that*

$$\forall t \in I \qquad W(\varphi_1, \dots, \varphi_n)(t) \neq 0.$$

Then there exists a unique linear differential operator of order n (assuming that the coefficient of the n-th derivative is one) for which these functions form a basis.

Proof: We start by showing the existence of such an equation. Consider the equation

$$(-1)^n \frac{W(y, \varphi_1, \dots, \varphi_n)(t)}{W(\varphi_1, \dots, \varphi_n)(t)} = 0.$$

This is an n-th order linear differential equation for y. Also the coefficient of $y^{(n)}$ is 1.

Clearly,

$$W(\varphi_j, \varphi_1, \dots, \varphi_n)(t) = 0$$

for all j's, which means that the φ_j's form a basis for this equation.

It remains to prove the uniqueness of this equation. The solutions φ_j provide us with a fundamental matrix Φ for the associated first-order system:

$$Y'(t) = A(t)Y(t).$$

I.e., the matrix $A(t)$ is uniquely determined, and this in turn uniquely determines the ratios $a_j(t)/a_0(t)$.

■

2.5.2 The adjoint equation

In the context of first-order linear systems we encountered the equation adjoint to a given equation:

$$Y'(t) = -A^*(t)Y(t).$$

The adjoint equation will become important later when we deal with boundary value problems.

Consider now the n-th order operator L_n, and assume that $a_0(t) = 1$. The adjoint system is

$$-A^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & \overline{a_n} \\ -1 & 0 & 0 & 0 & \cdots & 0 & \overline{a_{n-1}} \\ 0 & -1 & 0 & 0 & \cdots & 0 & \overline{a_{n-2}} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & \overline{a_2} \\ 0 & 0 & 0 & 0 & \cdots & -1 & \overline{a_1} \end{pmatrix}$$

Let us write down this equation in components:

$$\begin{aligned} y_1' &= \overline{a_n}y_n \\ y_2' &= -y_1 + \overline{a_{n-1}}y_n \\ y_3' &= -y_2 + \overline{a_{n-2}}y_n \\ &\vdots \\ y_{n-1}' &= -y_{n-2} + \overline{a_2}y_n \\ y_n' &= -y_{n-1} + \overline{a_1}y_n. \end{aligned}$$

Differentiate the last equation $(n-1)$ times,

$$y_n^{(n)} = -y_{n-1}^{(n-1)} + (\overline{a_1}y_n)^{(n-1)}.$$

Differentiating $(n-2)$ times the equation for y'_{n-1} and substituting:

$$y_n^{(n)} = y_{n-2}^{(n-2)} - (\overline{a_2} y_n)^{(n-2)} + (\overline{a_1} y_n)^{(n-1)}.$$

We proceed, and eventually get an n-th order scalar equation for y_n,

$$L_n^* y = 0,$$

where the adjoint operator L_n^* is given by

$$L_n^* = (-1)^n \frac{d^n}{dt^n} + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}} \overline{a_1} + (-1)^{n-2} \frac{d^{n-2}}{dt^{n-2}} \overline{a_2} - \cdots - \frac{d}{dt} \overline{a_{n-1}} + \overline{a_n}.$$

Theorem 2.4 (Lagrange identity) Suppose that the coefficients a_j in the operator L_n are differentiable sufficiently many times. Then for every pair of functions u, v that are n-times differentiable,

$$\overline{v}\, L_n u - u\, \overline{L_n^* v} = [uv]',$$

where $[uv]$ is defined as follows:

$$[uv] = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)} (a_{n-m} \overline{v})^{(j)}.$$

Proof: We will prove the case $n = 2$. The general proof is left as an exercise. For $n = 2$ (which is what we'll end up using later anyway):

$$L_2 = \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_2$$

and

$$L_2^* = \frac{d^2}{dt^2} - \frac{d}{dt} \overline{a_1} + \overline{a_2}.$$

Thus,

$$\begin{aligned}
\overline{v}\, L_n u - u\, \overline{L_n^* v} &= \overline{v}(u'' + a_1 u' + a_2 u) - u\overline{(v'' - (\overline{a_1} v)' + \overline{a_2} v)} \\
&= (u'' \overline{v} + a_1 u' \overline{v} + a_2 u \overline{v}) - (u (\overline{v})'' - u (a_1 \overline{v})' + u a_2 \overline{v}) \\
&= (u' \overline{v} - u \overline{v}' + u\, a_1 \overline{v})'.
\end{aligned}$$

■

Corollary 2.2 (Green's formula) For every t_1, t_2:

$$\int_{t_1}^{t_2} \left[\overline{v}\, L_n u - u\, \overline{L_n^* v} \right] dt = [uv](t_2) - [uv](t_1).$$

Proof: Obvious ∎

2.5.3 Non-homogeneous equation

We know how to express the solution of the non-homogeneous linear equation in terms of the solution operator of the homogeneous system for a general first-order system. In this section we derive an expression for the solution of an non-homogeneous n-th order linear equation,

$$(L_n y)(t) = b(t).$$

We may assume without loss of generality that $a_0(t) \equiv 1$. The corresponding first-order system is

$$Y'(t) = A(t)Y(t) + B(t),$$

where $A(t)$ is as above and

$$B(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}.$$

The differences between this problem and the general inhomogenous system is:

① The vector $B(t)$ has all entries but one zero.

② We are only looking for the 1st component of the solution.

Theorem 2.5 *Let $\varphi_1, \ldots, \varphi_n$ be n independent solution of the homogeneous equation $L_n y = 0$, then the solution of the initial-value problem*

$$(L_n y)(t) = b(t) \qquad Y(t_0) = Y_0,$$

is

$$y(t) = y_h(t) + \sum_{k=1}^{n} \varphi_k(t) \int_{t_0}^{t} \frac{W_k(\varphi_1, \cdots, \varphi_n)(s)}{W(\varphi_1, \cdots, \varphi_n)(s)} b(s)\, ds,$$

where $y_h(t)$ is the solution to the homogeneous initial-value problem, and

$$W_k(vp_1, \cdots, \varphi_n) = \begin{vmatrix} \varphi_1 & \cdots & \varphi_{k-1} & 0 & \varphi_{k+1} & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_{k-1}' & 0 & \varphi_{k+1}' & \cdots & \varphi_n' \\ \cdot & \cdots & \cdot & \vdots & \cdot & \cdots & \cdot \\ \varphi_1^{(n-1)} & \cdots & \varphi_{k-1}^{(n-1)} & 1 & \varphi_{k+1}^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

Proof: We know that for we can express the non-homogeneous part of the solution (in vector form) as

$$\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)\, ds.$$

Since we are only interested in the first component and by the structure of $B(t)$,

$$y(t) = y_h(t) = \sum_{j=1}^{n} \int_{t_0}^{t} \underbrace{\Phi_{1j}}_{\varphi_j(t)}(t)\Phi_{jn}^{-1}(s)b(s)\, ds$$

Thus, it only remains to show that

$$\Phi_{jn}^{-1} = \frac{W_k(\varphi_1, \cdots, \varphi_n)}{W(\varphi_1, \cdots, \varphi_n)},$$

but this follows from basic linear algebra (formula of the inverse by means of cofactors). ∎

TA material 2.2 Apply this formula for $n = 2$.

2.5.4 Constant coefficients

We now consider the case of an n-th order equation with constant coefficients:

$$L_n y = y^{(n)} + a_1 y^{(n+1)} + \cdots + a_n y,$$

where the a_j's are now constant. In matrix form,

$$Y'(t) = AY(t),$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-4} & \cdots & -a_1 \end{pmatrix}.$$

Lemma 2.1 The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

Proof: We prove it inductively on n. For $n = 2$

$$\lambda I - A = \begin{pmatrix} \lambda & -1 \\ a_2 & \lambda + a_1 \end{pmatrix},$$

and so

$$p_2(A) = \lambda(\lambda + a_1) + a_2 = \lambda^2 + a_1\lambda + a_2.$$

For $n = 3$,

$$\lambda I - A = \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda + a_1 \end{pmatrix}.$$

Hence,

$$p_3(\lambda) = \lambda p_2(\lambda) + a_3.$$

It is easy to see that $p_n(\lambda) = \lambda\, p_{n-1}(\lambda) + a_n$. ■

Theorem 2.6 *Let $\lambda_1, \dots, \lambda_s$ be distinct roots of the characteristic polynomial of A such that λ_j has multiplicity m_j,*

$$\sum_{j=1}^{s} m_j = n.$$

Then the space of solutions is spanned by the functions

$$e^{\lambda_j t}, t\, e^{\lambda_j t}, t^2\, e^{\lambda_j t}, \cdots, t^{m-1}\, e^{\lambda_j t}, \qquad j = 1, \dots, s.$$

Proof: We need to show that these n functions are indeed solutions and that they are linearly independent. First, note that for every λ (not necessarily a root of the characteristic polynomial):

$$L_n\left(e^{\lambda t}\right) = p_n(\lambda)\, e^{\lambda t}.$$

Let λ be a root of the characteristic polynomial of multiplicity m, and let $0 \le k \le m - 1$. Then,

$$L_n\left(t^k e^{\lambda t}\right) = L_n\left(\frac{d^k}{d\lambda^k} e^{\lambda t}\right) = \frac{d^k}{d\lambda^k} L_n\left(e^{\lambda t}\right) = \frac{d^k}{d\lambda^k}\left(p_n(\lambda) e^{\lambda t}\right).$$

If λ has multiplicity m and $k < m$, then

$$p_n(\lambda) = p_n'(\lambda) = \cdots = p_n^{(k)}(\lambda) = 0,$$

which proves that

$$L_n\left(t^k e^{\lambda t}\right) = 0.$$

It remains to prove that these solutions are linearly independent. Suppose that they were dependent. Then there would have existed constants c_{jk}, such that

$$\sum_{j=1}^{s} \underbrace{\sum_{k=0}^{m_j - 1} c_{jk} t^k}_{P_j(t)} e^{\lambda_j t} = 0.$$

Dividing by $e^{\lambda_1 t}$,

$$P_1(t) + e^{(\lambda_2 - \lambda_1)t} P_2(t) + \cdots + e^{(\lambda_s - \lambda_1)t} P_s(t) = 0.$$

The P_j are polynomials. Differentiate sufficiently many times until P_1 disappears. We will get an equation of the form:

$$e^{(\lambda_2-\lambda_1)t}Q_2(t) + \cdots + e^{(\lambda_s-\lambda_1)t}Q_s(t) = 0.$$

We then divide by $e^{(\lambda_2-\lambda_1)t}$, get

$$Q_2(t) + e^{(\lambda_3-\lambda_2)t}Q_3(t) + \cdots + e^{(\lambda_s-\lambda_2)t}Q_s(t) = 0.$$

We proceed this way until we get that $Q_s(t) = 0$. It follows then backwards that all the Q_j's (or the P_j's) vanish, i.e., all the c_j's were zero. ■

Example: Consider the equation

$$y'''' - y = 0.$$

The characteristic polynomial is

$$\lambda^4 - 1 = 0,$$

and its roots are $\lambda = \pm 1, \pm i$. This means that the linear space of solutions is spanned by

$$e^t, \quad e^{-t}, \quad e^{it} \quad \text{and} \quad e^{-it}.$$

Since

$$\cos t = \frac{e^{it} + e^{-it}}{2} \qquad \text{and} \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

we may also change basis and use instead the real-valued functions

$$e^t, \quad e^{-t}, \quad \cos t \quad \text{and} \quad \sin t.$$

▲▲▲

Example: Consider the equation

$$y'''' + 2y'' + y = 0.$$

The characteristic polynomial is

$$(\lambda^2 + 1)^2 = 0$$

and it has two roots of multiplicity 2: $\lambda = \pm i$. Thus, the space of solution is spanned by the basis

$$e^{it}, \quad te^{it}, \quad e^{-it}, \quad \text{and} \quad te^{-it},$$

to after a change in bases,

$$\cos t, \quad t\cos t, \quad \sin t, \quad \text{and} \quad t\sin t.$$

▲▲▲

2.6 Linear systems with periodic coefficients: Floquet theory

2.6.1 Motivation

Consider a linear homogeneous system of the form

$$y'(t) = A(t)y(t), \tag{2.3}$$

where for every t,

$$A(t+T) = A(t).$$

We will assume that T is the smallest number for which this holds. Note that A is automatically kT periodic for every integer k. Such equations are of interest in systems in which the dynamics have some inherent periodicity (e.g., dynamics of some phenomenon on earth that is affected by the position of the moon).

Another important application of linear systems with periodic coefficients is the ***stability of periodic solutions***. Consider a (generally nonlinear) differential system:

$$y'(t) = f(y(t)).$$

Note that f does not depend explicitly on t; such a system is called ***autonomous***.

Suppose that this system has a particular solution, $z(t)$, that is T-periodic, and that this solution passes through the point z_0. Since the system is autonomous, we can arbitrarily assume that $z(0) = z_0$. An important question is whether this periodic solution is ***stable***. The periodic solution is said to be stable if solutions with "slightly perturbed" initial data,

$$y(0) = z_0 + \varepsilon\eta_0,$$

where $\eta \ll 1$ do not diverge "too fast" from $z(t)$.

A standard way to analyze the stability of a particular solution is to represent the solution as a sum of this particular solution plus a perturbation. In this case we set

$$y(t) = z(t) + \varepsilon\eta(t).$$

Substituting into the equation we get

$$z'(t) + \varepsilon\eta'(t) = f(z(t) + \varepsilon\eta(t)).$$

We then use Taylor's theorem (assuming that f is twice differentiable):

$$z'(t) + \varepsilon \eta'(t) = f(z(t)) + \varepsilon \nabla f(z(t))\eta(t) + O(\varepsilon^2).$$

Since $z'(t) = f(z(t))$ we remain with

$$\eta'(t) = \nabla f(z(t))\eta(t) + O(\varepsilon).$$

This is a linear equation with T-periodic coefficients. The rationale of this approach is that if $\eta(t)$ remains bounded in time then for small enough ε, the perturbed solution will remain in an ε neighborhood of $z(t)$. Otherwise, the perturbed solution diverges away from the periodic solution.

The above is a very rough analysis intended to motivate the study of linear systems with periodic coefficients.

Might be interesting to read in this context: the ***van der Pol oscillator***. See:

`http://www.scholarpedia.org/article/Van_der_Pol_oscillator`

2.6.2 General theory

A natural question is whether the periodicity of the equation implies the periodicity of the solution.

Example: Consider the scalar equation:

$$y'(t) = (1 + \sin t)\, y(t),$$

which is 2π-periodic. One can check by direct substitution that the general solution is

$$y(t) = c \exp\left(t - \cos t\right),$$

which is not periodic (in fact, it diverges as $t \to \infty$). ▲▲▲

Theorem 2.7 Let $\Phi(t)$ be a fundamental matrix for the periodic equation, then $\Phi(t+T)$ is also a fundamental solution, and moreover, there exists a constant matrix B, such that

$$\Phi(t+T) = \Phi(t)B,$$

and

$$\det B = \exp\left(\int_0^T \operatorname{Tr} A(s)\, ds\right).$$

Proof: Let $\Phi(t)$ be a fundamental matrix and set $\Psi(t) = \Phi(t+T)$. Then,

$$\Psi'(t) = \Phi'(t+T) = A(t+T)\Phi(t+T) = A(t)\Psi(t),$$

which proves the first part.

As we know, if both $\Phi(t)$ and $\Psi(t)$ are fundamental matrices, then there exists a constant matrix B, such that

$$\Psi(t) = \Phi(t+T) = \Phi(t)B.$$

Finally, we use Proposition 2.5 to get that

$$\det\Phi(t+T) = \det\Phi(t)\left(\int_t^{t+T} \operatorname{Tr} A(s)\,ds\right),$$

and since $A(t)$ is periodic, we may as well integrate from 0 to T,

$$\det\Phi(t)\,\det B = \det\Phi(t)\left(\int_0^{T} \operatorname{Tr} A(s)\,ds\right),$$

which concludes the proof. ■

Note that in the last step we used the following property:

Lemma 2.2 Let f be periodic with period T, then for every t,

$$\int_t^{t+T} f(s)\,ds = \int_0^T f(s)\,ds.$$

The fact that $\Phi(t+T) = \Phi(t)B$ implies that the fundamental matrix is in general not periodic. Since this relation holds for every t, it follow that

$$B = \Phi^{-1}(0)\Phi(T).$$

Furthermore, we may always choose the fundamental matrix $\Phi(t)$ such that $\Phi(0) = I$, in which case $B = \Phi(T)$, i.e., for every t,

$$\Phi(t+T) = \Phi(t)\Phi(T).$$

This means that the knowledge of the fundamental matrix in the interval $[0,T]$ determines the fundamental matrix everywhere.

Moreover, it is easy to see by induction that for $k \in \mathbb{N}$:

$$\Phi(t + kT) = \Phi(t)B^k.$$

Thus, for arbitrary t,

$$\Phi(t) = \underbrace{\Phi(t \mod T)}_{T\text{-periodic}} B^{\lfloor t/T \rfloor},$$

where

$$t \mod T = t - \lfloor t/T \rfloor T.$$

It follows that the behaviour of solution for large t is dominated by powers of the matrix B. The forthcoming analysis will make this notion more precise.

Definition 2.1 The eigenvalues of $B = \Phi(T)$, ρ_i are called the ***characteristic multipliers*** *(כופלים אופיניים) of the periodic system. The constants μ_i defined by*

$$\rho_i = e^{\mu_i T}$$

are called the ***characteristic exponents*** *(אקספוננטים אופיניים) of the periodic linear system. (Note that the characteristic exponents are only defined modulo $2\pi ik/T$.)*

Theorem 2.8 Let ρ and μ be a pair of characteristic multiplier/exponent of the periodic linear system. There there exists a solution $y(t)$ such that for every t,

$$y(t + T) = \rho y(t).$$

Also, there exists a solution $y(t)$ such that for every t,

$$y(t + T) = e^{\mu t} p(t),$$

where $p(t)$ is T-periodic.

Proof: Let $Bu = \rho u$, and define $y(t) = \Phi(t)u$. By the definition of a fundamental matrix, $y(t)$ is a solution of (2.3) and

$$y(t + T) = \Phi(t + T)u = \Phi(t)Bu = \rho\Phi(t)u = \rho y(t),$$

which proves the first part.

Next define $p(t) = e^{-\mu t} y(t)$, then

$$p(t+T) = e^{-\mu(t+T)} y(t+T) = \underbrace{e^{-\mu T}\rho}_{1}\, e^{-\mu t} y(t) = p(t),$$

which concludes the proof. ■

Suppose now that B is diagonalizable (i.e., has a complete set of eigenvectors). Then there are n pairs (ρ_i, μ_i), and the general solution to the periodic linear system is

$$y(t) = \sum_{k=1}^{n} \alpha_k e^{\mu_k t} p_k(t),$$

where the functions $p_k(t)$ are T-periodic. In other words, every solution is of the form

$$y(t) = \underbrace{\begin{pmatrix} \vdots & \cdots & \vdots \\ p_1(t) & \cdots & p_n(t) \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} e^{\mu_1 t} & & \\ & \ddots & \\ & & e^{\mu_n t} \end{pmatrix}}_{\text{fundamental matrix}} C,$$

for some constant vector C.

An immediate corollary is that the system will have periodic solutions if one of the characteristic exponents of B is zero.

Comment: At this stage the theory we developed is not useful in finding the fundamental matrix. Its importance is so far theoretical in that it reveals the structure of the solution as a combination of exponential functions times periodic functions.

2.6.3 Hill's equation

I ended up not teaching this section; makes sense only with a good problem to study, like the stability of a limit cycle.

We will now exploit the results of the previous section to study a periodic (scalar) second order equation of the form,

$$y''(t) + a(t)y(t) = 0,$$

where $a(t+T) = a(t)$. Such equations arise in mechanics when then force on the particle is proportional to its location and the coefficient of proportionality is periodic in time (i.e., a pendulum in which the point of support oscillates vertically).

Note that there is no general explicit solution to a linear second-order equation with non-constant coefficients.

As usual, we start by rewriting this equation as a first order system,

$$y'(t) = A(t)y(t),$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

Note that $\operatorname{Tr} A(t) = 0$, which implies that $\det B = 1$, i.e., $\rho_1\rho_2 = 1$.

We are looking for a fundament matrix

$$\Phi(t) = \begin{pmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{pmatrix},$$

with $\Phi(0) = I$. Since,

$$\varphi'_{1j}(t) = \varphi_{2j}(t),$$

it follows that

$$\Phi(t) = \begin{pmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi'_{11}(t) & \varphi'_{12}(t) \end{pmatrix}.$$

Thus,

$$B = \begin{pmatrix} \varphi_{11}(T) & \varphi_{12}(T) \\ \varphi'_{11}(T) & \varphi'_{12}(T) \end{pmatrix}.$$

Moreover, since the Wronskian is constant, $W(t) = \det B = 1$, it follows that

$$\varphi_{11}(t)\varphi'_{12}(t) - \varphi_{12}(t)\varphi'_{11}(t) = 1.$$

The characteristic multiplier are the eigenvalues of B. For any 2-by-2 matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the eigenvalues are satisfy the characteristic equation

$$(a-\rho)(d-\rho) - bc = \rho^2 - (\operatorname{Tr} B)\rho + \det B = 0,$$

and since $\det B = 1$ we get that

$$\rho_{1,2} = \gamma \pm \sqrt{\gamma^2 - 1},$$

where $\gamma = \frac{1}{2}\operatorname{Tr} B = \frac{1}{2}[\varphi_{11}(T) + \varphi'_{12}(T)]$. Note that

$$\rho_1 + \rho_2 = 2\gamma \qquad \text{and} \qquad \rho_1\rho_2 = 1.$$

The characteristic exponents are $\rho_{1,2} = e^{\mu_{1,2}T}$, which implies that

$$e^{(\mu_1+\mu_2)T} = 1,$$

hence $\mu_1 + \mu_2 = 0$, and

$$2\gamma = e^{\mu_1 T} + e^{\mu_2 T} = 2\cosh \mu_1 T.$$

Thus, we are able to express the characteristic coefficients in terms of the single parameter γ:

$$\cosh \mu_1 T \qquad \text{and} \qquad \mu_1 + \mu_2 = 0.$$

Even though we do not have a general solution for this type of problems, it turns out that the range of possible solutions can be determined by the single parameter γ.

① ***Case 1:*** $\gamma > 1$ In this case $\rho_{1,2}$ are both real and positive, and furthermore,

$$0 < \rho_2 < 1 < \rho_1.$$

Thus, μ_1 and $\mu_2 = -\mu_1$ are both real-valued. The general solution is of the form

$$y(t) = c_1 e^{\mu_1 t} p_1(t) + c_2 e^{-\mu_1 t} p_2(t),$$

where $p_i(t)$ are periodic. It follows that there are no periodic solutions and that in general the solution diverges as $t \to \infty$.

② ***Case 2:*** $\gamma < -1$ In this case both $\rho_{1,2}$ are real and negative,

$$\rho_2 < -1 < \rho_1 < 0,$$

in which case

$$e^{\mu_1 T} = \rho_1$$

implies that

$$e^{(\mu_1 - \imath\pi/T)T} = |\rho_1|,$$

i.e.,

$$\mu_1 = \frac{i\pi}{T} + \log|\rho_1|.$$

Thus, the general solution is of the form

$$y(t) = c_1 e^{\log|\rho_1| t} \underbrace{e^{i\pi t/T} p_1(t)}_{\text{period } 2T} + c_2 e^{-\log|\rho_1| t} \underbrace{e^{-i\pi t/T} p_2(t)}_{\text{period } 2T}.$$

Note the doubling of the period. Once again there are no periodic solutions and that in general the solution diverges as $t \to \infty$.

③ ***Case 3:*** $-1 < \gamma < 1$ Now $\rho_{1,2}$ are both complex and located on the unit circle,

$$\rho_{1,2} = e^{\pm i\sigma T}$$

(this relation defines σ) and thus $\mu_{1,2} = \pm i\sigma$. In this case the general solution is of the form

$$y(t) = c_1 e^{i\sigma t} p_1(t) + c_2 e^{-i\sigma t} p_2(t),$$

with $p_{1,2}(t)$ periodic.

In this case the solutions remain bounded, however they will not be periodic, unless it happens that $\sigma T = 2\pi/m$ for some integer m.

④ ***Case 4:*** $\gamma = 1$

⑤ ***Case 5:*** $\gamma = -1$

Chapter 3

Boundary value problems

3.1 Motivation

Many problems in science involve differential equations with conditions that are specified at more than one point. Boundary value problems arise also in the study of partial differential equations.

Example: The equation that describes the motion of an elastic string is known as the ***wave equation*** (משוואת הגלים),

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where the unknown is a function $y(x,t)$ of space x and time t. The constant c is the wave speed that depends on the mass density and tension of the string.

Consider a finite string, $x \in [0, L]$ tied at its two ends, such that

$$(\forall t) \qquad y(0,t) = y(L,t) = 0.$$

In addition, one has to specify initial conditions,

$$y(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial y}{\partial t}(x,0) = g(x).$$

(It is a initial-boundary problem.)

This problem is analytically solvable. Note that this equation is linear, so that if $y_1(x,t)$ and $y_2(x,t)$ are solutions to the wave equation, then so is any linear combination. One method of solution is to look for solutions that have a ***separation of variables***:

$$y(x,t) = u(x)v(t).$$

Substituting into the wave equation:

$$u''(x)v(t) = c^2 u(x)v''(t),$$

and further,

$$\frac{u''(x)}{u(x)} = c^2 \frac{v''(t)}{v(t)}.$$

Since this equation holds for all x and t, it must be the case that both sides are equal to the same constant, namely, there exists a constant k, such that

$$u''(x) = ku(x) \qquad \text{and} \qquad v''(t) = \frac{k}{c^2} v(t).$$

Note that we may have many such solutions for various values of k, and any linear combination of these solutions is also a solution to the wave equation.

Consider the equation for $u(x)$. Because of the boundary conditions, we have a boundary value problem:

$$u''(x) = ku(x) \qquad u(0) = u(L) = 0.$$

Does it have a solution? Yes, $u \equiv 0$. Are there other solutions? yes, $u(x) = \sin(\sqrt{k}x)$, but only if $\sqrt{k} = \pi m/L$ for some integer m. Are there more solutions? Such questions are going to be addressed in the course of this chapter. ▲▲▲

Because the independent variable in boundary value problem is often space rather than time, we will denote it by x rather than t. Of course, a notation is nothing but a notation.

Let's get back to the same boundary value problem, this time on the interval $[0,1]$:

$$\begin{cases} Ly = \lambda y \\ y(0) = y(1) = 0, \end{cases}$$

where

$$L = -\frac{d^2}{dx^2}.$$

The parameter λ can be in general complex (the reason for the minus sign in L will be made clear later). We know that the general solution to the equation $Ly = \lambda y$ is

$$y(x) = A\sin\lambda^{1/2}x + B\cos\lambda^{1/2}x.$$

Since we require $y(0) = 0$ then $B = 0$. Since we further require $y(1) = 0$, it follows that

$$\lambda = \pi^2 m^2,$$

for $m = 1, 2, \dots$ ($m = 0$ is the trivial solution and we ignore it as of now).

What we see is that the equation $Ly = \lambda y$ with boundary values has a solution only for selected values of λ. Does it remind us something? L is a linear operator, λ is a scalar... yes, this looks like an eigenvalue problem! Indeed, the values of λ for which the boundary value problem has a solution are called ***eigenvalues***. The corresponding "eigenvectors" are rather called ***eigenfunctions*** (פונקציות עצמיות).

Like for vectors, we endow the space of functions on $[0,1]$ with an ***inner product***:

$$(f,g) = \int_0^1 f(x)\overline{g}(x)\,dx.$$

Then, the normalized eigenfunctions are

$$\psi_m(x) = \sqrt{2}\,\sin(m\pi x).$$

Note that this is an ***orthonormal set***,

$$(\psi_m,\psi_k) = 2\int_0^1 \sin(m\pi x)\,\sin(k\pi x)\,dx = \delta_{mk}.$$

The class of functions that are (well, Lebesgue...) square integrable, i.e,

$$\int_0^1 |f(x)|^2\,dx \quad \text{exists}$$

is known as $L^2[0,1]$. It is a ***Hilbert space***, which is an object that you learn about in Advanced Calculus 2. In case you didn't take this course, no worry. We will provide all the needed information.

For a function $f \in L^2[0,1]$, its inner-product with the elements of the orthogonal set ψ_k are called its ***Fourier coefficients***,

$$\hat{f}(k) = (f,\psi_k) = \sqrt{2}\int_0^1 f(x)\sin(k\pi x)\,dx.$$

The series

$$\sum_{k=1}^{\infty} \hat{f}(k)\psi_k(x)$$

converges in the $L^2[0,1]$ norm to f, namely,

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \hat{f}(k)\psi_k \right\| = 0.$$

Finally, the norm of f and its Fourier coefficients satisfy a relation known as the ***Parseval identity***:

$$\|f\|^2 = \sum_{k=1}^{\infty} |\hat{f}(k)|^2.$$

We have stated all these facts without any proof or any other form of justification. As we will see, these results hold in a much wider scope.

TA material 3.1 Solve the wave equation initial-value problem. Talk about harmonics. Talk about what happens when you touch the center of a guitar string.

Comment: An interesting connection between these infinite-dimensional boundary value problems and "standard" eigenvalue problems can be made by considering ***discretizations***. Suppose that we want to solve the boundary value problem

$$-y''(x) = \lambda y(x), y(0) = y(1) = 0.$$

We can approximate the solution by defining a mesh,

$$x_i = \frac{i}{N}, \qquad i = 0, \dots, N,$$

and look for $y_i = y(x_i)$, by approximating the differential equation by a ***finite-difference approximation***,

$$-\frac{y_{i-1} + y_{i+1} - 2y_i}{2\Delta x^2}, \qquad i = 1, \dots, N-1,$$

where $\Delta x = 1/N$ is the ***mesh size***. In addition we set $y_0 = y_N = 0$. We get a linear equation for the discrete vector (y_i),

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix}.$$

This is a "standard" eigenvalue problem, for which we expect N eigenvalues. As we ***refine*** the mesh and take $N \to \infty$, the number of eigenvalues is expected to tend to infinity.

✎ *Exercise 3.1* Find the eigenvalues and eigenvectors of this discretized system, and verify their dependence on N. Compare your results with the solution of the continuous equation.

3.2 Self-adjoint eigenvalue problems on a finite interval

3.2.1 Definitions

Consider a linear n-th order differential operator,

$$L = p_0(x)\frac{d^n}{dx^n} + p_2(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{d}{dx} + p_n(x),$$

where $p_j \in C^{n-j}(a,b)$ is in general complex-valued; we assume that $p_0(x) \neq 0$ for all $x \in (a,b)$. We also define a linear operator $U : C^n(a,b) \to \mathbb{R}^n$ of the form

$$U_k y = \sum_{j=1}^{n} \left(M_{kj} y^{(j-1)}(a) + N_{kj} y^{(j-1)}(b) \right), \qquad k = 1, \dots, n,$$

where the M_{jk} and N_{jk} are constant. That is,

$$\begin{aligned} U_k y = {} & M_{k1} y(a) + M_{k2} y'(a) + \cdots + M_{kn} y^{(n-1)}(a) \\ & + N_{k1} y(b) + N_{k2} y'(b) + \cdots + N_{kn} y^{(n-1)}(b). \end{aligned}$$

The equation

$$Ly = \lambda y, \qquad Uy = 0$$

is called an ***eigenvalue problem***.

Example: The example we studied in the previous section is such an instance, with $n = 2$,

$$p_0(x) = -1, \;\; p_1(x) = p_2(x) = 0,$$

and

$$M_{11} = N_{21} = 1, \text{ and other } M_{jk}, N_{jk} \text{ are zero,}$$

i.e.,

$$\begin{aligned} U_1 y &= M_{11} y(a) + N_{11} y(b) + M_{12} y'(a) + N_{12} y'(b) = y(a) \\ U_2 y &= M_{21} y(a) + N_{21} y(b) + M_{22} y'(a) + N_{22} y'(b) = y(b). \end{aligned}$$

▲▲▲

Example: Another class of examples for $n = 2$:

$$M_{11} = -N_{11} = 1, \quad M_{22} = -N_{22} = 1, \text{ and other } M_{jk}, N_{jk} \text{ are zero,}$$

i.e.,

$$\begin{aligned} U_1 y &= M_{11} y(a) + N_{11} y(b) + M_{12} y'(a) + N_{12} y'(b) = y(a) - y(b) \\ U_2 y &= M_{21} y(a) + N_{21} y(b) + M_{22} y'(a) + N_{22} y'(b) = y'(a) - y'(b). \end{aligned}$$

This corresponds to ***periodic boundary conditions***. ▲▲▲

Definition 3.1 The eigenvalue problem is said to be **self-adjoint** *(צמוד לעצמו) if*

$$(Lu, v) = (u, Lv)$$

for every $u, v \in C^n(a, b)$ that satisfy the boundary conditions $Uu = Uv = 0$.

Example: The two above examples are self-adjoint. In the first case $Uu = 0$ implies that $u(a) = u(b) = 0$, and then

$$(Lu, v) = -\int_a^b u''(x)\overline{v}(x)\,dx = \int_a^b u'(x)\overline{v}'(x)\,dx = -\int_a^b u(x)\overline{v}''(x)\,dx = (u, Lv).$$

▲▲▲

Recall the adjoint operator L^* derived in the previous chapter:

$$L^* = (-1)^n \frac{d^n}{dx^n}\overline{p_0}(x) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\overline{p_1}(x) + \cdots - \frac{d}{dx}\overline{p_{n-1}}(x) + \overline{p_n}(x).$$

The Green identity proved in the previous chapter showed that

$$(Lu, v) - (u, L^* v) = [uv](b) - [uv](a).$$

Thus, if $L^* = L$ and $Uu = Uv = 0$ implies that $[uv](b) = [uv](a)$, then the problem is self-adjoint.

A self-adjoint problem always has a trivial solution for any value of λ. Values of λ for which a non-trivial solution exists are called ***eigenvalues***; the corresponding functions are called ***eigenfunctions***.

3.2.2 Properties of the eigenvalues

Proposition 3.1 Consider a self-adjoint boundary value problem. Then,

① *The eigenvalues are real.*
② *Eigenfunctions that correspond to distinct eigenvalues are orthogonal.*
③ *The eigenvalues form at most a countable set that has no (finite-valued) accumulation point.*

Comment: At this stage there is no guarantee that eigenvalues exist, so as of now this theorem can hold in a vacuous sense.

Comment: The last part of the proof relies on complex analysis. Fasten your seat belts.

Proof: Let λ be an eigenvalue with eigenfunction ψ. Then,

$$\lambda(\psi,\psi) = (L\psi,\psi) = (\psi, L\psi) = \overline{\lambda}(\psi,\psi),$$

i.e.,

$$(\lambda - \overline{\lambda})(\psi,\psi) = 0,$$

which proves that ψ is real.

Let $\lambda \neq \mu$ be distinct eigenvalues,

$$L\psi = \lambda\psi, \qquad L\varphi = \mu\varphi.$$

Then,

$$\lambda(\psi,\mu) = (L\psi,\varphi) = (\psi, L\varphi) = \mu(\psi,\varphi),$$

i.e.,

$$(\lambda - \mu)(\psi,\varphi) = 0,$$

which implies that ψ and φ are orthogonal.

Let now $\varphi_j(x,\lambda),\ j = 1,\dots,n$, be solutions of the equation $L\varphi_j = \lambda\varphi_j$ with initial conditions,

$$\varphi_j^{(k-1)}(c,\lambda) = \delta_{jk},$$

where c is some point in (a,b). That is,

$$
\begin{aligned}
&\varphi_1(c) = 1 \quad \varphi_1'(c) = \varphi_1''(c) = \varphi_1'''(c) = \cdots = \varphi_1^{(n-1)}(c) = 0\\
&\varphi_2'(c) = 1 \quad \varphi_2(c) = \varphi_2''(c) = \varphi_2'''(c) = \cdots = \varphi_2^{(n-1)}(c) = 0\\
&\varphi_3''(c) = 1 \quad \varphi_3(c) = \varphi_3'(c) = \varphi_3'''(c) = \cdots = \varphi_3^{(n-1)}(c) = 0\\
&\qquad\qquad\qquad \text{etc.}
\end{aligned}
$$

Note, we are back in initial values problems; there no issue of existence nor non-uniqueness, i.e., the $\varphi_j(x,\lambda)$ are defined for every (complex) λ. Also, the functions φ_j are continuous both in x and λ. In fact, for fixed x the function $\lambda \mapsto \varphi(x,\lambda)$ is ***analytic***. [1]

The functions φ_j are linearly independent, hence every solution of the equation $Ly = \lambda y$ is of the form

$$y(x) = \sum_{j=1}^n c_j \varphi_j(x,\lambda).$$

For λ to be an eigenvalue we need

$$U_\ell y = \sum_{j=1}^n c_j U_\ell \varphi_j(\cdot,\lambda) = 0, \qquad \ell = 1,\dots,n,$$

i.e.,

$$\sum_{j=1}^n \underbrace{\sum_{k=1}^n \left(M_{\ell k}\varphi_j^{(k-1)}(a,\lambda) + N_{\ell k}\varphi_j^{(k-1)}(b,\lambda)\right)}_{\mathcal{A}_{\ell j}(\lambda)} c_j = 0, \qquad \ell = 1,\dots,n.$$

This is a homogeneous linear system for the coefficients c_j. For it to have a non-trivial solution the determinant of the matrix $\mathcal{A}_{\ell j}(\lambda)$ must vanish. That is, ***the eigenvalues*** λ ***are identified with the roots of*** $\det \mathcal{A}_{\ell j}(\lambda)$**.**

$\det \mathcal{A}_{\ell j}(\lambda)$ is analytic in the entire complex plane. It is not identically zero because it can only vanish on the real line. It is well-known (especially if you learn it...) that the zeros of an entire analytic function that it not identically zero do not have accumulation points (except at infinity). ■

[1] A concept learned in complex analysis; I will say a few words about it.

3.2.3 Non-homogeneous boundary value problem

We turn now to study the non-homogeneous equation

$$Ly = \lambda y + f, \qquad Uy = 0,$$

where $f \in C(a,b)$. What to expect? Think of the linear-algebraic analog,

$$Ax = \lambda x + b.$$

A (unique) solution exists if the matrix $A - \lambda I$ is not singular, i.e., if λ is not an eigenvalue of A. Similarly, we will see that the non-homogeneous boundary value problem has a solution for values of λ that are not eigenvalues of the homogeneous equation.

Example: Such boundary value problems may arise, for example, in electrostatics. The ***Poisson equation*** for the electric potential in a domain $\Omega \subset \mathbb{R}^n$ with a metallic boundary is

$$\Delta\varphi = 4\pi\rho, \qquad \varphi|_{\partial\Omega} = 0,$$

where Δ is the ***Laplacian*** and $\rho(x)$ is the ***charge density***. For $n = 1$ we get

$$\frac{d^2\varphi}{dx^2} = 4\pi\rho \qquad \varphi(0) = \varphi(L) = 0,$$

which is of the same type as above with $\lambda = 0$. ▲▲▲

Let $\varphi_j(x,\lambda)$ be as in the previous section solutions to $L\varphi_j(\cdot,\lambda) = \lambda\varphi_j(\cdot,\lambda)$, subject to the boundary conditions

$$\varphi_j^{(k-1)}(c,\lambda) = \delta_{jk}.$$

Note that

$$W(\varphi_1,\dots,\varphi_n)(c) = \det I = 1,$$

hence

$$W(\varphi_1,\dots,\varphi_n)(x) = \exp\left(-\int_c^x \frac{p_1(s)}{p_0(s)}\,ds\right).$$

Note that p_0 and p_1 are the same for L and $L - \lambda\,\mathrm{id}$.

We now define the following function:

$$K(x,\xi,\lambda) = \begin{cases} \dfrac{1}{p_0(\xi)W(\varphi_1,\dots,\varphi_n)(\xi)} \begin{vmatrix} \varphi_1(\xi,\lambda) & \cdots & \varphi_n(\xi,\lambda) \\ \varphi_1'(\xi,\lambda) & \cdots & \varphi_n'(\xi,\lambda) \\ \cdot & \cdots & \cdot \\ \varphi_1^{(n-2)}(\xi,\lambda) & \cdots & \varphi_n^{(n-2)}(\xi,\lambda) \\ \varphi_1(x,\lambda) & \cdots & \varphi_n(x,\lambda) \end{vmatrix} & \xi \le x \\ 0 & x < \xi. \end{cases}$$

This function depends on x and ξ, both taking values in $[a,b]$. It is apparently discontinuous, however note that

$$K(\xi^+,\xi,\lambda) = \frac{d}{dx}K(\xi^+,\xi,\lambda) = \cdots = \frac{d^{n-2}}{dx^{n-2}}K(\xi^+,\xi,\lambda) = 0,$$

hence K and its first $n-2$ derivatives with respect to x are continuous over all (x,ξ); in particular K is continuous for $n \ge 2$. The $(n-1)$-st and n-th derivatives are continuous in every subdomain. Also,

$$\frac{d^{n-1}}{dx^{n-1}}K(\xi^+,\xi,\lambda) = \frac{1}{p_0(\xi)}.$$

Finally, for $x > \xi$,

$$LK(x,\xi,\lambda) = \frac{1}{p_0(\xi)W(\varphi_1,\dots,\varphi_n)(\xi)} \begin{vmatrix} \varphi_1(\xi,\lambda) & \cdots & \varphi_n(\xi,\lambda) \\ \varphi_1'(\xi,\lambda) & \cdots & \varphi_n'(\xi,\lambda) \\ \cdot & \cdots & \cdot \\ \varphi_1^{(n-2)}(\xi,\lambda) & \cdots & \varphi_n^{(n-2)}(\xi,\lambda) \\ L\varphi_1(x,\lambda) & \cdots & L\varphi_n(x,\lambda) \end{vmatrix} = \lambda K(x,\xi,\lambda).$$

This relation holds trivially for $x < \xi$.

Define now the function

$$u(x,\lambda) = \int_a^b K(x,\xi,\lambda)f(\xi)\,d\xi = \int_a^x K(x,\xi,\lambda)f(\xi)\,d\xi.$$

We calculate its first n derivatives with respect to n,

$$u'(x,\lambda) = \int_a^x \frac{d}{dx}K(x,\xi,\lambda)f(\xi)\,d\xi + \underbrace{K(x,x^-,\lambda)f(x)}_{0},$$

$$u''(x,\lambda) = \int_a^x \frac{d^2}{dx^2} K(x,\xi,\lambda) f(\xi)\, d\xi + \underbrace{\frac{d}{dx} K(x,x^-,\lambda) f(x)}_{0},$$

and so until

$$u^{(n-1)}(x,\lambda) = \int_a^x \frac{d^{n-1}}{dx^{n-1}} K(x,\xi,\lambda) f(\xi)\, d\xi + \underbrace{\frac{d^{n-2}}{dx^{n-2}} K(x,x^-,\lambda) f(x)}_{0}.$$

Finally,

$$u^{(n)}(x,\lambda) = \int_a^x \frac{d^n}{dx^n} K(x,\xi,\lambda) f(\xi)\, d\xi + \underbrace{\frac{d^{n-1}}{dx^{n-1}} K(x,x^-,\lambda) f(x)}_{f(x)/p_0(x)}.$$

It follows that

$$Lu(x,\lambda) = \int_a^x LK(x,\xi,\lambda) f(\xi)\, d\xi + f(x) = \lambda u(x,\lambda) + f(x),$$

i.e., $u(x,\lambda)$ is a solution to the non-homogeneous equation.

We are looking for a solution of the non-homogeneous equation that satisfies the boundary conditions, $U_k y = 0$. Define

$$G(x,\xi,\lambda) = K(x,\xi,\lambda) + \sum_{j=1}^n c_j(\xi) \varphi_j(x,\lambda),$$

and set $c_j(\xi)$ such that for every $k = 1,\dots,n$ and every $\xi \in [a,b]$:

$$U_k G(\cdot,\xi,\lambda) = U_k K(\cdot,\xi,\lambda) + \sum_{j=1}^n c_j(\xi) U_k \varphi_j(\cdot,\lambda) = 0.$$

This linear system has a unique solution if $\det U_k \varphi_j(\cdot,\lambda) \neq 0$, i.e., if λ is not an eigenvalue of the homogeneous boundary value problem.

Set then

$$\begin{aligned} u(x,\lambda) &= \int_a^b G(x,\xi,\lambda) f(\xi)\, d\xi \\ &= \int_a^b K(x,\xi,\lambda) f(\xi)\, d\xi + \sum_{j=1}^n \varphi_j(x) \int_a^b c_j(\xi) f(\xi)\, d\xi. \end{aligned}$$

Then,

$$Lu(x,\lambda) = \lambda u(x,\lambda),$$

and

$$U_k u(\cdot,\lambda) = \int_a^b \left(U_k K(\cdot,\xi,\lambda) + \sum_{j=1}^n c_j(\xi) U_k \varphi(\cdot,\lambda) \right) f(\xi)\, d\xi = 0.$$

Thus, we found a solution to the non-homogeneous boundary value problem for every value of λ that is not an eigenvalue of the homogeneous boundary value problem.

✎ *Exercise 3.2* Solve the boundary value problem,

$$-y''(x) = \lambda y(x) + f(x), \qquad y(0) = y(1) = 0.$$

For what values of λ solutions exist? Repeat the above analysis (including the calculation of $K(x,\xi,\lambda)$ and $G(x,\xi,\lambda)$ for this particular example.

Solution 3.2: We start by constructing the two independent solution,

$$\begin{aligned} -\varphi_1''(x) &= \lambda \varphi_1(x), \qquad \varphi_1(0) = 1 \quad \varphi_1'(0) = 0 \\ -\varphi_2''(x) &= \lambda \varphi_2(x), \qquad \varphi_2(0) = 0 \quad \varphi_2'(0) = 1, \end{aligned}$$

namely,

$$\varphi_1(x) = \cos \lambda^{1/2} x \qquad \text{and} \qquad \varphi_2(x) = \lambda^{-1/2} \sin \lambda^{1/2} x.$$

Thus,

$$W(\varphi_1,\varphi_2)(x) = \begin{vmatrix} \cos \lambda^{1/2} x & \lambda^{1-0/2} \sin \lambda^{1/2} x \\ -\lambda^{1/2} \sin \lambda^{1/2} x & \cos \lambda^{1/2} x \end{vmatrix} = 1.$$

Next, since $p_0(\xi) = -1$,

$$K(x,\xi,\lambda) = \begin{cases} -\begin{vmatrix} \cos \lambda^{1/2}\xi & \sin \lambda^{1/2}\xi \\ \cos \lambda^{1/2} x & \sin \lambda^{1/2} x \end{vmatrix} = \sin \lambda^{1/2}(\xi - x) & \xi \le x \\ 0 & x < \xi \end{cases}.$$

Now,

$$G(x,\xi,\lambda) = K(x,\xi,\lambda) - c_1(\xi) \cos \lambda^{1/2} x - c_2(\xi) \lambda^{-1/2} \sin \lambda^{1/2} x.$$

The coefficients $c_{1,2}$ have to satisfy,

$$G(0,\xi,\lambda) = 0 - c_1(\xi) = 0,$$

and

$$G(1,\xi,\lambda) = \sin \lambda^{1/2}(\xi - 1) - c_1(\xi) \cos \lambda^{1/2} - c_2(\xi) \lambda^{-1/2} \sin \lambda^{1/2} = 0,$$

from which we get that

$$G(x,\xi,\lambda) = K(x,\xi,\lambda) - \frac{\sin\lambda^{1/2}(\xi-1)}{\sin\lambda^{1/2}}\sin\lambda^{1/2}x$$

Thus, the solution to the boundary value problem is

$$u(x,\lambda) = \int_0^x \sin\lambda^{1/2}(\xi-x)f(\xi)\,d\xi - \frac{\sin\lambda^{1/2}x}{\sin\lambda^{1/2}}\int_0^1 \sin\lambda^{1/2}(\xi-1)f(\xi)\,d\xi.$$

It is easy to see that the boundary conditions are indeed satisfied. You may now check by direct differentiation that the differential equation is indeed satisfied.

Suppose now that zero is *not* an eigenvalue of the homogeneous boundary value problem, and consider the boundary value problem

$$Ly = f \qquad Uy = 0.$$

The fact the we require zero not to be an eigenvalue of f will turn out not to be a real restriction since there exists a λ_0 that is not en eigenvalue and then $L - \lambda_0 I$ will be used as an operator for which zero is not en eigenvalue.

We will denote $G(x,\xi,0)$ simply by $G(x,\xi)$. Note that by definition $LG(x,\xi) = 0$ at all points where $x \neq \xi$.

Define the linear integral operator:

$$\mathcal{G} : f \mapsto \int_a^b G(\cdot,\xi)f(\xi)\,d\xi,$$

which maps the non-homogeneous term into the solution of the boundary value problem.

Proposition 3.2 The integral operator $\mathcal{G}$ (and the **Green function** *G) satisfies the following properties:*

① *$L\mathcal{G}f = f$ and $U\mathcal{G}f = 0$ for all $f \in C(a,b)$.*
② *$(\mathcal{G}f, g) = (f, \mathcal{G}g)$.*
③ *$G(x,\xi) = \overline{G(\xi,x)}$.*
④ *$\mathcal{G}Lu = u$ for all $u \in C^n(a,b)$ satisfying $Uu = 0$.*

(Note that L and $\mathcal{G}$ are almost inverse to each other.)

Proof: The first statement follows from the fact that $\mathcal{G}$ maps f into the solution of the boundary value problem. By the self-adjointness of L and the fact that $U\mathcal{G}f = U\mathcal{G}g = 0$,

$$(L\mathcal{G}f, \mathcal{G}g) = (\mathcal{G}f, L\mathcal{G}g),$$

hence by the first statement

$$(f, \mathcal{G}g) = (\mathcal{G}f, g),$$

which proves the second statement.

It follows from the second statement that

$$\int_a^b \left(\int_a^b G(x,\xi) f(\xi)\, d\xi \right) \overline{g(x)}\, dx = \int_a^b f(x) \overline{\left(\int_a^b G(x,\xi) g(\xi)\, d\xi \right)} dx.$$

Changing the order of integration and interchanging x and ξ in the second integral:

$$\int_a^b \int_a^b \left(G(x,\xi) - \overline{G(\xi,x)} \right) f(\xi) \overline{g(x)}\, dx d\xi = 0.$$

Since this holds for every f and g the third statement follows.

Finally, for every $g \in C(a,b)$ and $u \in C^n(a,b)$ satisfying $Uu = 0$,

$$(\mathcal{G}Lu, g) = (u, L\mathcal{G}g) = (u, g),$$

which implies that $\mathcal{G}Lu = u$. ∎

3.3 The existence of eigenvalues

So far we haven't shown that the eigenvalue problem

$$Ly = \lambda y, \qquad Uy = 0, \tag{3.1}$$

has solutions. We only know properties of solutions if such exist. In this section we will show that self-adjoint eigenvalue problems always have a countable set of eigenvalues.

Assuming that zero is not an eigenvalue we defined the Green function $G(x,\xi)$ and the operator $\mathcal{G}$. The first step will be to show that eigenfunctions of L coincide with eigenfunctions of $\mathcal{G}$:

Proposition 3.3 *φ is an eigenfunction of* (3.1) *with eigenvalue λ if and only if it is also an eigenfunction of $\mathcal{G}$ with eigenfunction $1/\lambda$, namely*

$$\mathcal{G}\varphi = \frac{1}{\lambda}\varphi, \tag{3.2}$$

Proof: Let (λ, φ) be eigensolutions of (3.1). Then,

$$\mathcal{G}\varphi = \frac{1}{\lambda}\mathcal{G}\lambda\varphi = \frac{1}{\lambda}\mathcal{G}L\varphi,$$

and since $U\varphi = 0$, it follows from the fourth statement of Proposition 3.2 that

$$\mathcal{G}\varphi = \frac{1}{\lambda}\varphi.$$

Conversely, suppose that (3.2) holds. Then,

$$L\varphi = \lambda L\frac{1}{\lambda}\varphi = \lambda L\mathcal{G}\varphi = \lambda\varphi,$$

and

$$U\varphi = \lambda U\mathcal{G}\varphi = 0,$$

where we used the fact that the range of $\mathcal{G}$ is functions that satisfy the boundary conditions. ■

Thus, in order to show that (3.1) has eigenvalues we can rather show that $\mathcal{G}$ has eigenvalues.

Lemma 3.1 *The set of functions*

$$\mathscr{X} = \{\mathcal{G}u : u \in C(0,1),\ \ \|u\| \le 1\}$$

is bounded (in the sup-norm) and equicontinuous (but the norm $\|\cdot\|$ is the L^2-norm).

Proof: We will consider the case of $n \geq 2$; the case of $n = 1$ requires a slightly different treatment. For $n \geq 2$ the function $G(x,\xi)$ is continuous on the square $(x,\xi) \in [a,b]^2$, hence it is uniformly continuous, and in particular,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : (\forall \xi \in [a,b]) \ (\forall x_1, x_2 : |x_1 - x_2| < \delta) \qquad |G(x_1,\xi) - G(x_2,\xi)| < \varepsilon.$$

Thus,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : (\forall u \in \mathscr{X}) \ (\forall x_1, x_2 : |x_1 - x_2| < \delta)$$

it holds that

$$|\mathcal{G}u(x_1) - \mathcal{G}u(x_2)| \leq \int_a^b |G(x_1,\xi) - G(x_2,\xi)||u(\xi)|\, d\xi \leq \varepsilon \int_a^b |u(\xi)|\, d\xi.$$

Using the Cauchy-Schwarz inequality,

$$|\mathcal{G}u(x_1) - \mathcal{G}u(x_2)| \leq \left(\int_a^b |u(\xi)|^2\, d\xi \right)^{1/2} \left(\int_a^b d\xi \right)^{1/2} = \varepsilon (b-a)^{1/2} \|u\|,$$

which proves the equicontinuity of $\mathscr{X}$.

Since $G(x,\xi)$ is continuous, it is also bounded; let K be a bound. Then, for all $u \in \mathscr{X}$ and $x \in [a,b]$,

$$|\mathcal{G}u(x)| \leq \int_a^b |G(x,\xi)||u(\xi)|\, d\xi \leq K(b-a)^{1/2} \|u\|,$$

which proves the uniform boundedness of $\mathscr{X}$. ∎

It follows from the Arzela-Ascoli theorem:

Corollary 3.1 Every sequence in $\mathscr{X}$ has a subsequence that converges uniformly on $[a,b]$.

The map $\mathcal{G}$ is actually a map from $L^2(a,b) \to L^2(a,b)$. [2] We therefore define its ***operator norm***:

$$\|\mathcal{G}\| = \sup_{\|u\|=1} \|\mathcal{G}u\| = \sup_{(u,u)=1} (\mathcal{G}u, \mathcal{G}u)^{1/2}.$$

It follows from Lemma 3.1 that $\|\mathcal{G}\| < \infty$ (i.e., $\mathcal{G}$ is a bounded linear operator).

[2] We defined it as a map from $C(a,b)$ to $C^n(a,b)$ to avoid measure-theoretic subtleties.

Proposition 3.4

$$\|\mathcal{G}\| = \sup_{\|u\|=1} |(\mathcal{G}u, u)|.$$

Proof: For every normalized u, it follows from the Cauchy-Schwarz inequality and the definition of the norm that

$$|(\mathcal{G}u, u)| \le \|\mathcal{G}u\| \|u\| \le \|\mathcal{G}\| \|u\|^2 = \|\mathcal{G}\|,$$

i.e.,

$$\sup_{\|u\|=1} |(\mathcal{G}u, u)| \le \|\mathcal{G}\|. \tag{3.3}$$

By the bilinearity of the inner-product and the self-adjointness of $\mathcal{G}$, for every u and v,

$$\begin{aligned}
(\mathcal{G}(u+v), u+v) &= (\mathcal{G}u, u) + (\mathcal{G}v, v) + (\mathcal{G}u, v) + (\mathcal{G}v, u) \\
&= (\mathcal{G}u, u) + (\mathcal{G}v, v) + 2\Re(\mathcal{G}u, v) \\
&\le \sup_{\|w\|=1} |(\mathcal{G}w, w)| \|u+v\|^2.
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{G}(u-v), u-v) &= (\mathcal{G}u, u) + (\mathcal{G}v, v) - (\mathcal{G}u, v) - (\mathcal{G}v, u) \\
&= (\mathcal{G}u, u) + (\mathcal{G}v, v) - 2\Re(\mathcal{G}u, v) \\
&\ge - \sup_{\|w\|=1} |(\mathcal{G}w, w)| \|u-v\|^2.
\end{aligned}$$

Subtracting the second inequality from the first,

$$4\Re(\mathcal{G}u, v) \le 2 \sup_{\|w\|=1} |(\mathcal{G}w, w)| \left(\|u\|^2 + \|v\|^2 \right).$$

Substitute now $v = \mathcal{G}u/\|\mathcal{G}u\|$. Then,

$$4\Re\left(\mathcal{G}u, \frac{\mathcal{G}u}{\|\mathcal{G}u\|}\right) \le 2 \sup_{\|w\|=1} |(\mathcal{G}w, w)| \left(\|u\|^2 + \left\| \frac{\mathcal{G}u}{\|\mathcal{G}u\|} \right\|^2 \right),$$

or,

$$4\|\mathcal{G}u\| \le 2 \sup_{\|w\|=1} |(\mathcal{G}w, w)| \left(\|u\|^2 + 1 \right).$$

For $\|u\| = 1$ we get

$$\|\mathcal{G}u\| \leq \sup_{\|w\|=1} |(\mathcal{G}w, w)|,$$

and since this holds for every such u,

$$\|\mathcal{G}\| = \sup_{\|u\|=1} \|\mathcal{G}u\| \leq \sup_{\|w\|=1} |(\mathcal{G}w, w)|. \tag{3.4}$$

Eqs. (3.3) and (3.4) together give the desired result. ■

Theorem 3.1 Either $\|\mathcal{G}\|$ or $-\|\mathcal{G}\|$ is an eigenvalue of $\mathcal{G}$.

Comments:

① This proves the existence of an eigenvalue to the boundary value problem.

② Since there exists an eigenfunction φ such that

$$\mathcal{G}\varphi = \|\mathcal{G}\|\varphi \qquad \text{or} \qquad \mathcal{G}\varphi = -\|\mathcal{G}\|\varphi,$$

it follows that

$$\|\mathcal{G}\varphi\| = \|\mathcal{G}\|\|\varphi\|,$$

thus the sup in the definition of $\|\mathcal{G}\|$ is attained.

③ Everything we have been doing holds in the finite-dimensional case. If A is an hermitian matrix then either $\|A\|$ or $-\|A\|$ is an eigenvalue.

Proof: Denote $\|\mathcal{G}\| = \mu_0$. then either

$$\mu_0 = \sup_{\|u\|=1} (\mathcal{G}u, u)$$

or

$$\mu_0 = -\sup_{\|u\|=1} (\mathcal{G}u, u).$$

Consider the first case; the second case is treated similarly.

By the definition of the supremum, there exists a sequence of functions (u_n) such that

$$\|u_n\| = 1 \qquad \lim_{n\to\infty} (\mathcal{G}u_n, u_n) = \mu_0.$$

By Corollary 3.1 the sequence $(\mathcal{G}u_n)$ has a subsequence (not relabeled) that converges uniformly on $[a,b]$; denote its limit by φ_0. In particular, uniform convergence implies L^2-convergence,

$$\lim_{n\to\infty} \|\mathcal{G}u_n - \varphi_0\| = 0.$$

Since

$$0 \leftarrow (\mathcal{G}u_n - \varphi_0, \mathcal{G}u_n - \varphi_0) = \|\mathcal{G}u_n\|^2 + \|\varphi_0\|^2 - 2\Re(\mathcal{G}u_n, \varphi_0) \to \|\mathcal{G}u_n\|^2 - \|\varphi_0\|^2,$$

we get the convergence of the norm,

$$\lim_{n\to\infty} \|\mathcal{G}u_n\| = \|\varphi_0\|.$$

(It is always true that convergence in norm implies the convergence of the norm.)

Now,

$$\|\mathcal{G}u_n - \mu_0 u_n\|^2 = \|\mathcal{G}u_n\|^2 + \mu_0^2 - 2\mu_0(\mathcal{G}u_n, u_n) \to \|\varphi_0\|^2 - \mu_0^2,$$

from which follows that $\varphi_0 \neq 0$ (we need this to claim that φ_0 is nontrivial).

Moreover, it follows that

$$0 \le \|\mathcal{G}u_n - \mu_0 u_n\|^2 \le 2\mu_0^2 - 2\mu_0(\mathcal{G}u_n, u_n) \to 0,$$

from which follows that

$$\lim_{n\to\infty} \|\mathcal{G}u_n - \mu_0 u_n\| = 0.$$

Finally, by the triangle inequality

$$\begin{aligned} \|\mathcal{G}\varphi_0 - \mu_0\varphi_0\| &\le \|\mathcal{G}\varphi_0 - \mathcal{G}(\mathcal{G}u_n)\| + \|\mathcal{G}(\mathcal{G}u_n) - \mu_0\mathcal{G}u_n\| + \|\mu_0\mathcal{G}u_n - \mu_0\varphi_0\| \\ &\le \|\mathcal{G}\|\,\|\varphi_0 - \mathcal{G}u_n\| + \|\mathcal{G}\|\,\|\mathcal{G}u_n - \mu_0 u_n\| + \mu_0\|\mathcal{G}u_n - \varphi_0\|. \end{aligned}$$

Letting $n \to \infty$ we get that

$$\mathcal{G}\varphi_0 = \mu_0\varphi_0,$$

which concludes the proof. ■

Theorem 3.2 *$\mathcal{G}$ has an infinite number of eigenfunctions.*

Proof: Let φ_0 and μ_0 be as above and assume that φ_0 has been normalized, $\|\varphi_0\| = 1$. Define a new kernel

$$G_1(x,\xi) = G(x,\xi) - \mu_0\varphi_0(x)\overline{\varphi}_0(\xi),$$

and the corresponding operator,

$$\mathcal{G}_1 f(x) = \int_a^b G_1(x,\xi) f(\xi)\, d\xi.$$

This operator is also self-adjoint,

$$(\mathcal{G}_1 u, v) = (\mathcal{G}u, v) + \mu_0 \iint_a^b \varphi_0(x)\overline{\varphi}_0(\xi) u(\xi)\overline{v}(x)\, d\xi dx = (u, \mathcal{G}_1 v).$$

Hence, unless $\mathcal{G}_1 = 0$ (the zero operator), it has an eigenvalue μ_1, whose absolute value is equal to $\|\mathcal{G}_1\|$. Let,

$$\mathcal{G}_1\varphi_1 = \mu_1\varphi_1 \qquad \|\varphi_1\| = 1.$$

Note that $\mathcal{G}_1\varphi_0 = 0$, hence

$$(\varphi_1, \varphi_0) = \frac{1}{\mu_1}(\mathcal{G}_1\varphi_1, \varphi_0) = \frac{1}{\mu_1}(\varphi_1, \mathcal{G}_1\varphi_0) = 0,$$

i.e., φ_0 and φ_1 are orthogonal. Then,

$$\mathcal{G}\varphi_1 = \mathcal{G}_1\varphi_1 + \mu_0\varphi_1(\varphi_1, \varphi_0) = \mathcal{G}_1\varphi_1 = \mu_1\varphi_1,$$

which means that φ_1 is also an eigenfunction of $\mathcal{G}$. By the maximality of μ_0,

$$|\mu_1| \le |\mu_0|.$$

We continue this way, defining next

$$G_2(x,\xi) = G_1(x,\xi) - \mu_1\varphi_1(x)\overline{\varphi}_1(\xi)$$

and find a third orthonormal eigenfunction of $\mathcal{G}$.

The only way this process can stop is if there is a $\mathcal{G}_m = 0$ for some m. Suppose this were the case, then for every $f \in C(a,b)$,

$$\mathcal{G}_m f(x) = \mathcal{G}f(x) - \sum_{j=1}^{m-1} \mu_j\varphi_j(x)(f, \varphi_j) = 0.$$

Applying L on both sides, using the fact that $L\mathcal{G}f = f$ and $L\varphi_j = \mu_j^{-1}\varphi_j$,

$$f(x) = \sum_{j=1}^{m-1} \varphi_j(x)(f,\varphi_j).$$

This would imply that every continuous function is in the span of m differentiable functions, which is wrong. ■

3.4 Boundary value problems and complete orthonormal systems

We have thus proved the existence of a countable orthonormal system, φ_j, with corresponding eigenvalues

$$|\mu_0| \ge |\mu_1| \ge |\mu_2| \ge \cdots.$$

Lemma 3.2 (Bessel's inequality) Let $f \in L^2(a,b)$. Then the sequence

$$\sum_{j=0}^{\infty} |(f,\varphi_j)|^2$$

converges and

$$\sum_{j=0}^{\infty} |(f,\varphi_j)|^2 \le \|f\|^2.$$

Proof: For every $m \in \mathbb{N}$,

$$0 \le \left\| f - \sum_{j=0}^{m} (f,\varphi_j)\varphi_j \right\|^2 = \|f\|^2 - \sum_{j=0}^{m} |(f,\varphi_j)|^2,$$

which proves both statements. ■

Theorem 3.3 *Let $f \in C^n(a,b)$ and $Uf = 0$, then*

$$f = \sum_{j=0}^{\infty} (f, \varphi_j)\varphi_j,$$

where the convergence is uniform.

Corollary 3.2 *An immediate consequence is* **Parseval's identity**. *Multiplying this equation by $\bar{f}$ and integrating over $[a,b]$, interchanging the order of integration and summation, we get*

$$\|f\|^2 = \sum_{j=0}^{\infty} |(f, \varphi_j)|^2$$

Proof: For every $j \in \mathbb{M}$ and $x \in [a,b]$,

$$(\mathcal{G}\varphi_j)(x) = \int_a^b G(x,\xi)\varphi_j(\xi)\, d\xi = \mu_j \varphi_j(x).$$

Taking the complex conjugate and using the fact that $\overline{G(x,\xi)} = G(\xi, x)$,

$$(G(x,\cdot), \varphi_j) = \int_a^b G(\xi, x)\overline{\varphi_j}(\xi)\, d\xi = \mu_j \overline{\varphi_j}(x).$$

Squaring and summing over j,

$$\sum_{j=0}^{m} |(G(x,\cdot), \varphi_j)|^2 = \sum_{j=0}^{m} |\mu_j|^2 |\varphi_j(x)|^2 = \sum_{j=0}^{m} |\mu_j|^2.$$

By Bessel's inequality applied to $G(x,\cdot)$,

$$\sum_{j=0}^{m} |\mu_j|^2 \leq \|G(x,\cdot)\|^2 = \int_a^b |G(x,\xi)|^2\, d\xi.$$

Integrating over x,

$$\sum_{j=0}^{m} |\mu_j|^2 \leq \iint_a^b |G(x,\xi)|^2\, d\xi dx \leq K^2(b-a)^2,$$

where K is a bound on $|G(x,\xi)|$ It follows at once that $\sum_{j=0}^{\infty} |\mu_j|^2$ converges and in particular $\mu_j \to 0$.

For $m \in \mathbb{N}$ consider

$$G_m(x,\xi) = G(x,\xi) - \sum_{j=0}^{m-1} \mu_j \varphi_j(x)\overline{\varphi_j}(\xi).$$

By the way we constructed the sequence of eigenfunctions, we know that

$$\|\mathcal{G}_m\| = |\mu_m|.$$

Thus, for every $u \in C(a,b)$,

$$\|\mathcal{G}_m u\| = \left\| \mathcal{G}u - \sum_{j=0}^{m-1} \mu_j (u,\varphi_j)\varphi_j \right\| \le |\mu_m| \|u\|.$$

Letting $m \to \infty$,

$$\lim_{m\to\infty} \left\| \mathcal{G}u - \sum_{j=0}^{m-1} \mu_j (u,\varphi_j)\varphi_j \right\| = 0.$$

This means that $\sum_{j=0}^{\infty} \mu_j (u,\varphi_j)\varphi_j$ converges to $\mathcal{G}u$ in L^2. We will show that it converges in fact uniformly. For $q > p$,

$$\sum_{j=p}^{q} \mu_j (u,\varphi_j)\varphi_j = \mathcal{G}\left(\sum_{j=p}^{q} (u,\varphi_j)\varphi_j \right).$$

Hence,

$$\left| \sum_{j=p}^{q} \mu_j (u,\varphi_j)\varphi_j \right| \le K(b-a)^{1/2} \left\| \sum_{j=p}^{q} (u,\varphi_j)\varphi_j \right\| = K(b-a)^{1/2} \left(\sum_{j=p}^{q} |(u,\varphi_j)|^2 \right)^{1/2},$$

where we used the Cauchy-Schwarz inequality. Since the right hand side vanishes as $p,q \to \infty$, it follows by Cauchy's criterion that $\sum_{j=p}^{q} \mu_j (u,\varphi_j)\varphi_j$ converges uniformly. Since it converges to $\mathcal{G}u$ in L^2 and the latter is continuous, it follows that

$$\sum_{j=0}^{\infty} \mu_j (u,\varphi_j)\varphi_j = \mathcal{G}u,$$

where the convergence is uniform.

Let now $f \in C^n(a,b)$, $Uf = 0$. Then, Lf is continuous, and

$$\sum_{j=0}^{\infty} \mu_j (Lf,\varphi_j)\varphi_j = \mathcal{G}Lf.$$

Since $\mathcal{G}Lf = f$ and $L\varphi_j = \mu_j^{-1}\varphi_j$,

$$\sum_{j=0}^{\infty}(f,\varphi_j)\varphi_j = f.$$

■

Chapter 4

Stability of solutions

4.1 Definitions

Consider the following nonlinear system:

$$\begin{aligned} y_1' &= \tfrac{1}{3}(y_1 - y_2)(1 - y_1 - y_2) \\ y_2' &= y_1(2 - y_2). \end{aligned}$$

This system does not depend explicitly on t; such systems are called ***autonomous***. More generally, in a system

$$y'(t) = f(t, y(t)),$$

the family of vector fields $f(t, \cdot)$ is called autonomous if it does not depend on t, i.e., $f : \mathbb{R}^n \to \mathbb{R}^n$.

Note that if φ is a solution to the autonomous initial data problem,

$$y'(t) = f(y(t)), \qquad y(0) = y_0,$$

then $\psi(t) = \varphi(t - t_0)$ is a solution to the initial data problem

$$y'(t) = f(y(t)), \qquad y(t_0) = y_0.$$

Fixed points For the above system, four points in the y-plane that are of special interest:

$$y = (0,0) \quad y = (2,2) \quad y = (0,1) \quad \text{and} \quad y = (-1,2).$$

These are ***fixed points*** (נקודות שבת), or ***equilibrium points*** (נקודות שיווי משקל), or ***stationary points***. Each such point corresponds to a solution to the system that does not change in time. If any of these points is prescribed as an initial value then the solution will be constant.

The above system is nonlinear and cannot be solved by analytical means for arbitrary initial values. Yet, in many cases, the global behaviour of solutions can be inferred by knowing the behaviour of solutions in the vicinity of fixed points. Roughly speaking, a fixed point is ***stable*** if any solution starting "close" to it will be attracted to it asymptotically, and it is ***unstable*** if in any neighborhood there exist points such that a solution starting at this point is repelled from the fixed point.

The notion of stability can be made more precise and more general:

Definition 4.1 Let $u(t)$ be a solution to the autonomous system $y'(t) = f(y(t))$. It is called **stable in the sense of Lyapunov** *if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for every solution $y(t)$ satisfying $\|y(t_0) - u(t_0)\| < \delta$,*

$$\|y(t) - u(t)\| < \varepsilon \qquad \forall t > t_0.$$

It is called **asymptotically stable** *if there exists a δ such that $\|y(t_0) - u(t_0)\| < \delta$ implies that*

$$\lim_{t\to\infty} \|y(t) - u(t)\| = 0.$$

Comment: The stability of fixed points is a particular case of stability of solutions as defined above.

We will be concerned in this chapter with the ***global behaviour of solutions***. here is a list of possible behaviors:

① Fixed point.
② Asymptotic convergence to a fixed point.
③ Limit cycle (periodic solution).
④ Asymptotic convergence to a limit cycle.
⑤ Quasi-periodic.
⑥ Chaotic.
⑦ Tends asymptotically to infinity.

4.2 Linearization

Consider an autonomous system

$$y'(t) = f(y(t)), \tag{4.1}$$

and a particular solution $u(t)$. Since we are interested in understanding the behaviour of solutions in the vicinity of $u(t)$, we set

$$y(t) = u(t) + z(t).$$

Substituting in the equation we get

$$u'(t) + z'(t) = f(u(t) + z(t)).$$

Since $u'(t) = f(u(t))$, we get a differential equation for the deviation $z(t)$ of the solution $y(t)$ from the solution $u(t)$:

$$z'(t) = f(u(t) + z(t)) - f(u(t)).$$

If f is twice continuously differentiable, then as long as $z(t)$ is in some compact set $U \subset \mathbb{R}^n$.

$$z'(t) = Df(u(t))z(t) + O(|z(t)|^2).$$

It is sensible to think that if $z(t_0)$ is "very small", then at least as long as it remains so, the quadratic correction to the Talor expansion is negligible, hence the devision of the solution from $u(t)$ is governed by the ***linearized equation***,

$$y'(t) = Df(u(t))y(t), \tag{4.2}$$

hence the stability of $u(t)$ in the nonlinear equation (4.1) can be determined by the stability of of the solution $y = 0$ of the linear equation (4.2).

The task is therefore 2-fold:

① Show that the zero solution of the linearized equation (4.2) is stable/unstable; a solution $u(t)$ of (4.1) for which the zero solution of the linearized equations is stable is called ***linearly stable*** (יציבה ליניארית).

② Show that the stability/instability of the zero solution of (4.2) implies the stability/instability of the solution $u(t)$ of the nonlinear equation (4.1). (That is, that linear stability implies ***nonlinear stability***.)

The first task is generally difficult, since we do not have a general way of solving linear equations with non-constant coefficients. The second task is in fact not always possible; one can find situations in which a solution is linearly stable but nonlinearly unstable.

The first task is however easy if $u(t)$ is a constant solution, i.e., if u is a fixed point. In this case, we know the solution to the linearized equation,

$$y(t) = e^{Df(u)\,t} y(0).$$

Moreover, we know the form of the exponential $e^{Df(u)\,t}$, and can therefore determine the following:

Theorem 4.1 A fixed point u of (4.1) *is* **linearly** *stable if and only if (i) all the eigenvalues of $Df(u)$ have non-positive real parts, and (ii) if an eigenvalue has a zero real-part then it is of multiplicity one.*

While we already know this is true, we will also prove it using a new technique.

Definition 4.2 A fixed point u is called **hyperbolic** *if none of the eigenvalues of $Df(u)$ has real part equal to zero. It is called a* **sink** *(בולען) if all the eigenvalues have negative real parts; it is called a* **source** *(מקור) all the eigenvalues have positive real parts; it is called a* **center** *(מרכז) if all the eigenvalues are imaginary; it is called a* **saddle** *(אוכף) if certain eigenvalues have positive real parts while other have negative real parts.*

Example: Consider the ***unforced Duffing oscillator***,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 - y_1^3 - \delta y_2, \end{aligned}$$

where $\delta \geq 0$. This describes a particle that is repelled from the origin when close it, but attracted to it when far away from it.

This system has three fixed points,

$$(0,0) \qquad \text{and} \qquad (\pm 1, 0).$$

Let $u = (u_1, u_2)$ be the fixed point. The linearized equation is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 - 3u_1^2 y_1 - \delta y_2, \end{aligned}$$

i.e.,

$$Df(u) = \begin{pmatrix} 0 & 1 \\ 1 - 3u_1^2 & -\delta \end{pmatrix}.$$

The eigenvalues satisfy

$$\lambda(\lambda + \delta) - (1 - 3u_1^2) = 0,$$

which implies that

$$\lambda = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2 + 4(1 - 3u_1^2)}\right).$$

For the fixed point $(0,0)$ there is a positive and a negative eigenvalue, which implies that it is unstable. For the fixed point $(\pm 1, 0)$ both eigenvalues have negative real parts, which implies that these fixed points are stable. ▲▲▲

4.3 Lyapunov functions

Theorem 4.2 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be twice differentiable; let u be a fixed point of f. Suppose that all the eigenvalues of $Df(u)$ have negative real parts. Then u is asymptotically stable.

Comment: This is a situation in which linear stability implies nonlinear stability.

Comment: Without loss of generality we may assume that $u = 0$, for define $z(t) = y(t) - u$. Then,

$$z'(t) = f(z + u) \equiv F(z).$$

Zero is a fixed point of F, which is stable if and only if u is a stable fixed point of f; furthermore,

$$DF(0) = Df(u).$$

Proof: The idea of the proof is based on a so-called ***Lyapunov function***. Suppose we construct a function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions:

① $V(y) \geq 0$ for all y.

② $V(y) = 0$ iff $y = 0$.

③ The sets
$$\mathscr{K}_c = \{y \in \mathbb{R}^n : V(y) \leq c\}$$
are compact, connected and contain the origin as internal point.

④ Close enough to the fixed point, the vector field f points "inward" on the level sets of V. That is, there exists a neighborhood U of the origin, such that
$$\forall y \in U \qquad (DV(y), f(y)) \leq 0.$$
Then, solutions are "trapped" within those level sets.

Back to the proof. There exists a neighborhood U of the origin and a constant $C > 0$, such that
$$\forall y \in U \qquad \|f(y) - Df(0)y\| \leq C\|y\|^2.$$
Define the Lyapunov function
$$V(y) = \frac{1}{2}\|y\|^2,$$
and set
$$g(t) = V(y(t)).$$
By the chain rule,
$$g'(t) = (y(t), f(y(t))) = (y(t), Df(0)y(t)) + \underbrace{(y(t), f(y(t)) - Df(0)y(t))}_{\equiv R(t)}.$$

Since all the eigenvalues of $Df(0)$ have negative real part, there exists a constant $\alpha > 0$ such that
$$(y, Df(0)y) < -\alpha\|y\|^2,$$
i.e.,
$$g'(t) < -2\alpha g(t) + R(t).$$
Moreover, if $y(t) \in U$, then
$$\|R(t)\| \leq \|y(t)\| \cdot C\|y(t)\|^2 \leq C_1 g^{3/2}(t).$$

In particular, there exists a constant $k > 0$ such that $\mathscr{K}_c \subset U$, and as long as $y(t) \in U'$

$$g'(t) < -\alpha g(t).$$

If follows that if $g(0) < c$ then $g(t) < c$ for all $t > 0$.

In addition,

$$\frac{d}{dt}\left(e^{\alpha t} g(t)\right) < 0,$$

and further,

$$e^{\alpha t} g(t) < g(0),$$

which implies that

$$\lim_{t \to \infty} g(t) = 0,$$

and in turn this implies that $y(t) \to 0$, i.e., the solution is asymptotically stable. ■

✎ *Exercise 4.1* Show that if A is a real matrix whose eigenvalues have all negative real parts, then there exists a constant $\beta > 0$, such that

$$\forall x \in \mathbb{R}^n \qquad (Ay, y) < -\beta \|y\|^2.$$

Example: An important use of Lyapunov functions is in cases where linear stability cannot conclusively determine the (nonlinear) stability of a fixed point. Consider the following system,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 - \varepsilon y_1^2 y_2. \end{aligned}$$

The point $(0,0)$ is a fixed point. The linearized equation about the origin is

$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y,$$

and the eigenvalues are $\pm i$. This means that $(0,0)$ is a center and we have no current way to determine its (nonlinear) stability.

Consider then the Lyapunov function

$$V(y) = \frac{1}{2}(y_1^2 + y_2^2).$$

Then,

$$(DV(y), f(y)) = (y_1, y_2)\begin{pmatrix} y_2 \\ -y_1 - \varepsilon y_1^2 y_2 \end{pmatrix} = -\varepsilon y_1^2 y_2^2$$

The origin is therefore stable for $\varepsilon > 0$.

▲▲▲

Example: Consider the system

$$\begin{aligned} y_1' &= -3y_2 \\ y_2' &= y_1 + \alpha(2y_2^3 - y_2), \end{aligned}$$

where $\alpha > 0$. The origin is a fixed point, and

$$Df(0) = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are $\pm\sqrt{3}i$, i.e., the origin is a center. Consider the Lyapunov function

$$V(y) = \frac{1}{2}(y_1^2 + 3y_2^2).$$

Setting $g(t) = V(y(t))$,

$$g'(t) = y_1(-3y_2) + 3y_2\left[y_1 + \alpha(2y_2^3 - y_2)\right] = -\alpha(3y_2^2 - 6y_2^4).$$

It follows that $g'(t) \le 0$ as long as $y_2^2 < \frac{1}{2}$, i.e., if

$$g(0) < 3,$$

then $g'(t) \le 0$ for all t. ▲▲▲

Theorem 4.3 Let V be a Lyapunov function, i.e., $V(y) \ge 0$ with $V(y) = 0$ iff $y = 0$; also the level sets of V are closed. If

$$\frac{d}{dt}V(y(t)) < 0,$$

then the origin is asymptotically stable.

Proof: Suppose that $y(t)$ does not converge to zero. Since $V(y(t))$ is monotonically decreasing it must converge to a limit $\delta > 0$. That is,

$$\forall t > 0 \qquad \delta \le V(y(t)) \le V(y_0).$$

The set

$$C = \{y : \delta \le V(y) \le V(y_0)\}$$

is closed and bounded and hence compact. Thus,

$$(DV(y), f(y))$$

must attain its maximum on C,

$$\max_{y \in C}(DV(y), f(y)) \equiv -\alpha < 0.$$

Then,

$$\frac{d}{dt}V(y(t)) = (DV(y(t)), f(y(t)) \le -\alpha,$$

and therefore

$$V(y(t)) \le V(y_0) - \alpha t,$$

which leads to a contradiction for large t. ∎

✎ *Exercise 4.2* Consider the nonlinear system

$$\begin{aligned} x' &= y \\ y' &= -ky - g(x), \end{aligned}$$

where g is continuous and satisfies $xg(x) > 0$ for $x \neq 0$ and $k > 0$. (This might represent the equation of a mass and nonlinear spring subject to friction.) Show that the origin is stable by using the Lyapunov function

$$V(x, y) = \frac{1}{2} + \int_0^x g(s)ds.$$

✎ *Exercise 4.3* Consider the system

$$\begin{aligned} x' &= -x^3 - y^2 \\ y' &= xy - y^3. \end{aligned}$$

Prove that the origin is stable using the Lyapunov function

$$V(x, y) = -x - \log(1 - x) - y - \log(1 - y).$$

4.4 Invariant manifolds

Definition 4.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an autonomous vector field. This vector field induces a* **flow map** *(העתקת זרימה),*

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$$

via the trajectories of the solutions, namely

$$y_1 = \Phi_t(y_0),$$

if y_1 is the solution at time t of the differential system,

$$y' = f(y), \qquad y(0) = y_0.$$

Another way to write it is

$$\frac{d}{dt}\Phi_t(y) = f(\Phi_t(y)) \qquad \Phi_0(y) = y.$$

Definition 4.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an autonomous vector field. A set $S \subset \mathbb{R}^n$ is said to be* **invariant** *if*

$$\Phi_t(y) \in S \qquad \text{for all } y \in S \text{ and for all } t \in \mathbb{R}.$$

That is, a trajectory that starts in S has always been in S and remains in S. It is said to be **positively invariant** *(אינווריאנטית חיובית) if*

$$\Phi_t(y) \in S \qquad \text{for all } y \in S \text{ and for all } t > 0.$$

It is said to be **negatively invariant** *(אינווריאנטית שלילית) if*

$$\Phi_t(y) \in S \qquad \text{for all } y \in S \text{ and for all } t < 0.$$

Example: Any **trajectory**,

$$O(y) = \{\Phi_t(y) : t \in \mathbb{R}\}$$

is an invariant set. Any union of trajectories,

$$\bigcup_{y \in A} O(y)$$

is an invariant set, and conversely, any invariant set is a union of trajectories. Any ***forward trajectory***,

$$O^+(y) = \{\Phi_t(y) : t \geq 0\}$$

is a positively-invariant set, and any ***backward trajectory***,

$$O^+(y) = \{\Phi_t(y) : t \leq 0\}$$

is a negatively-invariant set. ▲▲▲

Example: A fixed point is an invariant set (no matter if it is stable or not). ▲▲▲

Definition 4.5 An invariant set S is said to be an ***invariant manifold*** *(יריעה אינווריאנטית) if "looks" locally like a smooth surface embedded in $\mathbb{R}^n$.* [1]

Consider again an autonomous system,

$$y' = f(y),$$

and let u be a fixed point. The behavior of the solution near the fixed point is studied by means of the linearized equation,

$$y' = Ay,$$

where $A = Df(u)$. The solution of the linearized system is

$$y(t) = e^{At}y_0,$$

i.e., the flow map of the linearized system is given by the linear map

$$\Phi_t = e^{At}.$$

As we know from linear algebra, we may decompose $\mathbb{R}^n$ as follows:

$$\mathbb{R}^n = E_s \oplus E_u \oplus E_c,$$

where

$$E_s = \mathrm{Span}\{e_1, \dots, e_s\}$$

[1] This is of course a non-rigorous definition; for a rigorous definition you need to take a course in differential geometry.

is the span of (generalized) eigenvectors corresponding to eigenvalues with negative real part,

$$E_u = \operatorname{Span}\{e_{s+1}, \dots, e_{s+u}\}$$

is the span of (generalized) eigenvectors corresponding to eigenvalues with positive real part, and

$$E_c = \operatorname{Span}\{e_{s+u+1}, \dots, e_{s+u+c}\}$$

is the span of (generalized) eigenvectors corresponding to eigenvalues with zero real part.

These three linear subspaces are examples of invariant manifolds (of the linearized system!). The characterization of the ***stable manifold*** E_s, is that

$$\lim_{t\to\infty} \Phi_t(y) = 0 \qquad \forall y \in E_s.$$

The characterization of the ***unstable manifold*** E_u, is that

$$\lim_{t\to-\infty} \Phi_t(y) = 0 \qquad \forall y \in E_u.$$

Example: Let $n = 3$ and suppose that A has three distinct real eigenvalues,

$$\lambda_1, \lambda_2 < 0 \qquad \text{and} \qquad \lambda_3 > 0.$$

Let e_1, e_2, e_3 be the corresponding normalized eigenvectors. Then,

$$A = \begin{pmatrix} \vdots & \vdots & \vdots \\ e_1 & e_2 & e_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \cdots & e_1 & \cdots \\ \cdots & e_2 & \cdots \\ \cdots & e_3 & \cdots \end{pmatrix}^{-1} \equiv T\Lambda T^{-1}.$$

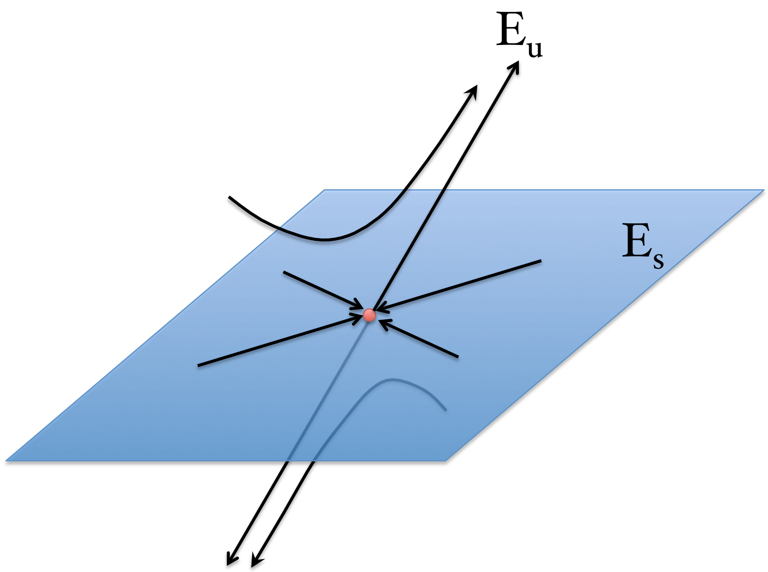
As we know,

$$y(t) = e^{At} y_0 = T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^{-1} y_0.$$

In this case

$$E_s = \operatorname{Span}\{e_1, e_2\} \qquad \text{and} \qquad E_u = \operatorname{Span}\{e_3\}$$

are invariant manifolds. The geometry of the trajectories is depicted in the following figure:



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Example: Suppose now that

$$\lambda_{1,2} = \rho \pm i\omega \qquad \text{and} \qquad \lambda_3 > 0,$$

where $\rho < 0$. Then, the corresponding eigenvectors are complex, however

$$A(e_1 + e_2) = (\rho + i\omega)e_1 + (\rho - i\omega)e_2 = \rho(e_1 + e_2) + i\omega(e_1 - e_2)$$
$$A(e_1 - e_2) = (\rho + i\omega)e_1 - (\rho - i\omega)e_2 = \rho(e_1 - e_2) + i\omega(e_1 + e_2),$$

or,

$$A\frac{e_1 + e_2}{\sqrt{2}} = \rho\frac{e_1 + e_2}{\sqrt{2}} - \omega\frac{e_1 - e_2}{\sqrt{2}i}$$
$$A\frac{e_1 - e_2}{\sqrt{2}i} = \rho\frac{e_1 - e_2}{\sqrt{2}i} + \omega\frac{e_1 + e_2}{\sqrt{2}}.$$

Thus, there exists a transformation of variables $A = T\Lambda T^{-1}$, in which

$$\Lambda = \begin{pmatrix} \rho & -\omega & 0 \\ \omega & \rho & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

The solution to this system is

$$y(t) = T\begin{pmatrix} e^{\rho t}\cos\omega t & -e^{\rho t}\sin\omega t & 0 \\ e^{\rho t}\sin\omega t & e^{\rho t}\cos\omega t & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^{-1} y_0.$$

Here again there is a two-dimensional stable subspace and a one-dimensional unstable subspace.

SHOW FIGURE. ▲▲▲

Example: Consider now the case where $\lambda_1 < 0$ is an eigenvalue of multiplicity two and $\lambda_3 > 0$. Then, $A = T\Lambda T^{-1}$ with

$$\Lambda = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and the solution is

$$y(t) = T \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} T^{-1} y_0.$$

In this transformed system $E_s = \text{Span}\{e_1, e_2\}$ and $E_u = \text{Span}\{e_3\}$. This is a deficient system, hence has only one direction "entering" the fixed point.

SHOW FIGURE.

▲▲▲

✎ *Exercise 4.4* Consider the following differential systems:

① $y' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} y \qquad \lambda < 0, \mu > 0.$

② $y' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} y \qquad \lambda < 0, \mu < 0.$

③ $y' = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} y \qquad \lambda < 0, \omega > 0.$

④ $y' = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} y \qquad \lambda < 0.$

⑤ $y' = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} y \qquad \lambda > 0.$

⑥ $y' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} y.$

In each case calculate ***all*** the trajectories and illustrate them on the plane. Describe the stable and unstable manifolds of the origin.

✎ *Exercise 4.5* Consider the nonlinear systems:

① $\begin{cases} y_1' = y_2 \\ y_2' = -\delta y_2 - \mu y_1 - y_1^2 \end{cases}$.

② $\begin{cases} y_1' = -y_1 + y_1^3 \\ y_2' = y_1 + y_2 \end{cases}$.

Find all the fixed points and determine their linear stability. Can you infer from this analysis the nonlinear stability?

The question is what happens in the nonlinear case. How does the structure of the linearized equation affect the properties of the solutions in the vicinity of the fixed point?

Theorem 4.4 Suppose that the vector field f is k-times differentiable and let the origin be a fixed point. Then there exists a neighborhood of the origin in which there are C^k-invariant manifold,

$$W^s, W^u \quad \text{and} \quad W^c,$$

that intersect at the origin are are tangent at the origin to the invariant manifolds of the linearized invariant manifolds. Moreover, the nonlinear stable and unstable manifolds retain the asymptotic properties of the linearized manifolds, namely,

$$\forall y \in W^s \qquad \lim_{t\to\infty} \Phi_t(y) = 0,$$

and

$$\forall y \in W^u \qquad \lim_{t\to-\infty} \Phi_t(y) = 0,$$

Proof: Not in this course. ■

Example: Consider again the Duffing oscillator,

$$\begin{aligned} x' &= y \\ y' &= x - x^3 - \delta y. \end{aligned}$$

The linearized vector field is

$$y' = \begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix} y,$$

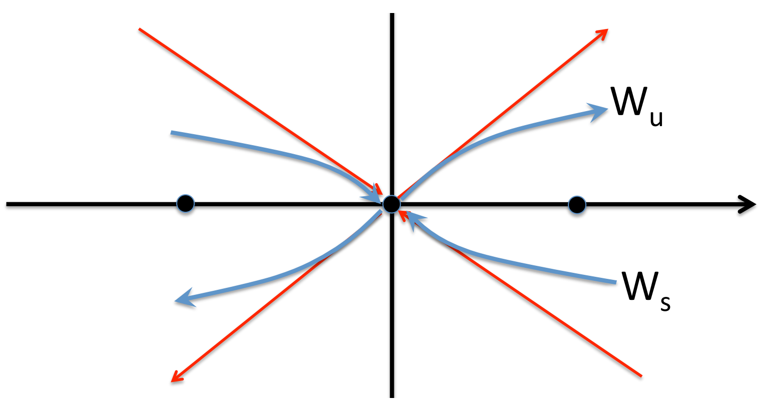
with eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\pm\sqrt{\delta^2+4} - \delta\right).$$

The corresponding (linear) stable and unstable manifolds are

$$y = \lambda_{1,2} x.$$

The origin is a saddle, i.e., it has a one-dimensional stable manifold and a one dimensional unstable manifold.



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Example: Consider the following artificial system,

$$\begin{aligned} x' &= x \\ y' &= -y + x^2. \end{aligned}$$

The origin is a fixed point and the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e., $E_s = \text{Span}\{e_2\}$ and $E_u = \text{Span}\{e_1\}$ are the stable and unstable manifolds of origin in the linearized system.

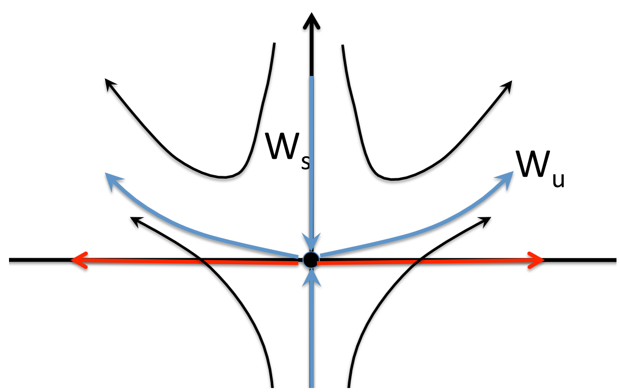
In this case we can actually compute the stable and unstable manifolds of the origin in the nonlinear system. We look for trajectories $y(x)$. These satisfy the differential equation,

$$y'(x) = \frac{y'(t)}{x'(t)} = -\frac{y}{x} + x.$$

This is a linear system whose solution is

$$y(x) = \frac{C}{x} + \frac{x^2}{3}.$$

For $C = 0$, we get a one-dimensional manifold which is invariant (since it is a trajectory) and tangent to the unstable manifold of the linearized system. The stable manifold of the nonlinear system is the y-axis, $x = 0$.



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4.5 Periodic solutions

Definition 4.6 A solution through a point y is said to be periodic with period $T > 0$ if

$$\Phi_t(y) = y.$$

We will now concentrate on periodic solutions of autonomous systems in the plane.

Theorem 4.5 (Bendixon's criterion) Consider the planar system,

$$y' = f(y)$$

where f is at least C^1. If in a simply connected domain D

$$\operatorname{div} f = \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2}$$

is not identically zero and does not change sign, then there are no closed orbits lying entirely in D.

Proof: Suppose there was a closed orbit Γ that encloses a domain $S \subset D$. By Green's theorem,

$$\iint_S \operatorname{div} f \, dy = \int_\Gamma (f, n) \, d\ell = 0,$$

where n is the unit normal to Γ, and $(f, n) = 0$ because f is tangent to Γ. If, as assumed $\operatorname{div} f$ is neither identically zero nor changes sign, then this is impossible. ■

Example: For the Duffing oscillator,

$$\operatorname{div} f = -\delta,$$

hence this system cannot have any closed orbit. ▲▲▲

4.6 Index theory

In this section we are concerned with autonomous differential systems in the plane:

$$x' = f(x, y)$$
$$y' = g(x, y).$$

Let Γ be a closed curve in the plane. If the curve does not intersect a fixed point, then at any point along the curve the vector field makes an angle ϕ with the x axis,

$$\phi(x, y) = \tan^{-1} \frac{g(x, y)}{f(x, y)}.$$

When we complete a cycle along Γ the angle ϕ must have changed by a multiple of 2π. We define the ***index of the curve*** by

$$i(\Gamma) = \frac{1}{2\pi} \int_\gamma d\phi.$$

It turns out that the index of a curve satisfies the following properties:

① The index of a curve does not change when it is deformed continuously, as long as it does not pass through a fixed point.

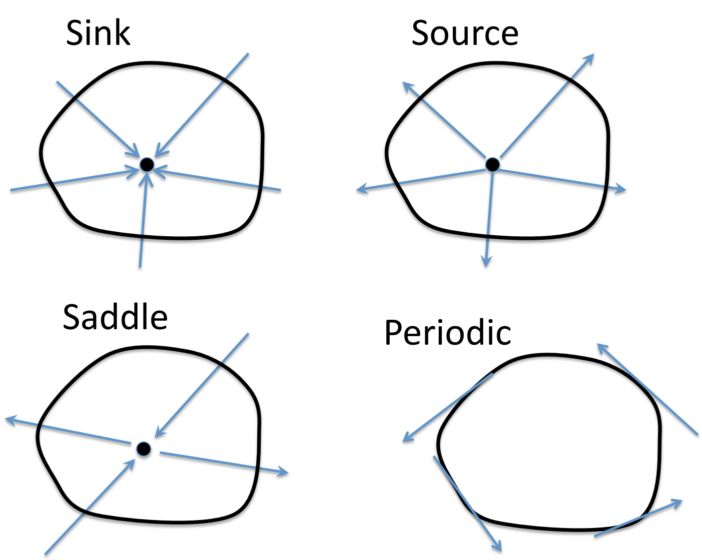
② The index of a curve that encloses a single sink, source, or center is 1.

③ The index of a periodic orbit is 1.

④ The index of a saddle is -1.

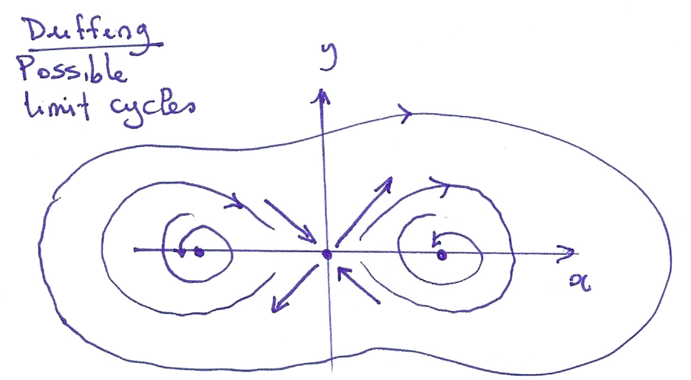
⑤ The index of a curve that does not enclose any fixed point is zero.

⑥ The index of a curve is equal to the sum of the indexes of the fixed points that it encloses.



Corollary 4.1 Every periodic trajectory encloses at least one fixed point. If it encloses a single fixed point then it is either a sink, a source, or a center. If all the enclosed fixed points are hyperbolic, then there must be $2n + 1$ of those with n saddles and $n + 1$ sources/sinks/centers.

Example: According to index theory, possible limit cycles in the Duffing oscillator are as follows:



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4.7 Asymptotic limits

We are back to autonomous systems in $\mathbb{R}^n$; we denote by $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ the flow map. The trajectory through a point y is the set:

$$O(y) = \{\Phi_t(y) : t \in \mathbb{R}\}$$

We also have the positive and negative trajectories:

$$O^+(y) = \{\Phi_t(y) : t \geq 0\}$$
$$O^-(y) = \{\Phi_t(y) : t \leq 0\}.$$

Definition 4.7 Let $y \in \mathbb{R}^n$. *A point* $\tilde{y}$ *belongs to the* ω***-limit*** *of* y,

$$\tilde{y} \in \omega(y),$$

if there exists a sequence $t_n \to \infty$ *such that*

$$\lim_{n\to\infty} \Phi_{t_n}(y) = \tilde{y}.$$

It belongs to the α***-limit*** *y,*

$$\tilde{y} \in \alpha(y),$$

if there exists a sequence $t_n \to -\infty$ *such that*

$$\lim_{n\to\infty} \Phi_{t_n}(y) = \tilde{y}.$$

Thus the ω***-set*** (קבוצת אומגה) $\omega(y)$ denotes the collection of all "future partial limits" of trajectories that start at y, and $\alpha(y)$ denotes the collection of all "past partial limits" of trajectories that pass through y.

Example: Let u be a fixed point and let y belong to the stable manifold of u. Then,

$$\omega(y) = \{u\}.$$

If z belongs to the unstable manifold of u, then

$$\alpha(z) = \{u\}.$$

▲ ▲ ▲

Example: Let $y : \mathbb{R} \to \mathbb{R}^n$ be a periodic solution. Then for every t_0,

$$\omega(y(t_0)) = \alpha(y(t_0)) = O(y(t_0)).$$

▲ ▲ ▲

4.8 The Poincaré-Bendixon theorem

We consider again two-dimensional systems,

$$y' = f(y).$$

As before we denote the flow function by $\Phi_t(x, y)$.

Proposition 4.1 Let $\mathcal{M}$ *be a compact positively invariant set (i.e., trajectories that start in* $\mathcal{M}$ *remain in* $\mathcal{M}$ *at all future times). Let* $p \in \mathcal{M}$*. Then,*

① $\omega(p) \neq \varnothing$.
② $\omega(p)$ *is closed.*
③ $\omega(p)$ *is invariant, i.e., it is a union of trajectories.*
④ $\omega(p)$ *is connected.*

Proof:

① Take any sequence $t_n \to \infty$. Since $\Phi_{t_n}(p) \in \mathcal{M}$ and $\mathcal{M}$ is compact this sequence has a converging subsequence,

$$\lim_{k\to\infty} \Phi_{t_{n_k}}(p) = q \in \mathcal{M}.$$

By definition $q \in \omega(p)$, which is therefore non-empty.

② Let $q \notin \omega(p)$. By the definition of the ω-set, there exists a neighborhood U of q and a time $T > 0$ such that

$$\Phi_t(p) \notin U \qquad \forall t > T.$$

Since U is open, every point $z \in U$ has a neighborhood $V \subset U$, and

$$\Phi_t(p) \notin V \qquad \forall t > T,$$

which implies that $z \notin \omega(p)$, i.e.,

$$(\omega(p))^c \quad \text{is open},$$

hence $\omega(p)$ is closed.

③ Let $q \in \omega(p)$. By definition, there exists a sequence $t_n \to \infty$, suchthat

$$\lim_{n\to\infty} \Phi_{t_n}(p) = q,$$

Let $\tau \in \mathbb{R}$ be arbitrary. Then,

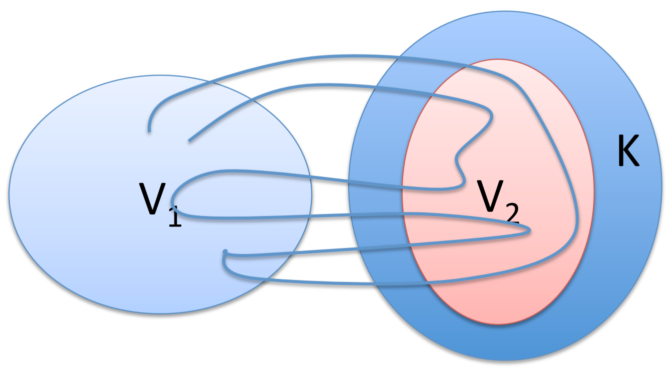
$$\lim_{n\to\infty} \Phi_{t_n+\tau}(p) = \lim_{n\to\infty} \Phi_\tau\left(\Phi_{t_n}(p)\right) = \Phi_\tau\left(\lim_{n\to\infty} \Phi_{t_n}(p)\right) = \Phi_\tau(q),$$

which proves that $\Phi_\tau(q) \in \omega(p)$ for all $\tau \in \mathbb{R}$, and hence $\omega(p)$ is invariant.

④ Suppose by contradiction that $\omega(p)$ was not connected. Then there exist disjoint open sets V_1, V_2 such that

$$\omega(p) \subset V_1 \cup V_2 \qquad \omega(p) \cap V_1 \neq \varnothing \quad \text{and} \quad \omega(p) \cap V_2 \neq \varnothing.$$

It follows that the trajectory $O(p)$ must pass from V_1 to V_2 and back infinitely often. Consider a compact set K that encloses V_1 and not intersect V_2. The trajectory must pass infinitely often in the compact set $K \smallsetminus V_1$, hence must have there an accumulation point, in contradiction to $\omega(p)$ being contained in $V_1 \cup V_2$.



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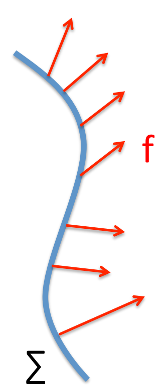
Definition 4.8 Let Σ be a continuous and connected arc in $\mathbb{R}^2$ (i.e., a map $I \to \mathbb{R}^2$). It is called **transversal** *to the flow field f neither vanishes on Σ nor is tangent to Σ, namely,*

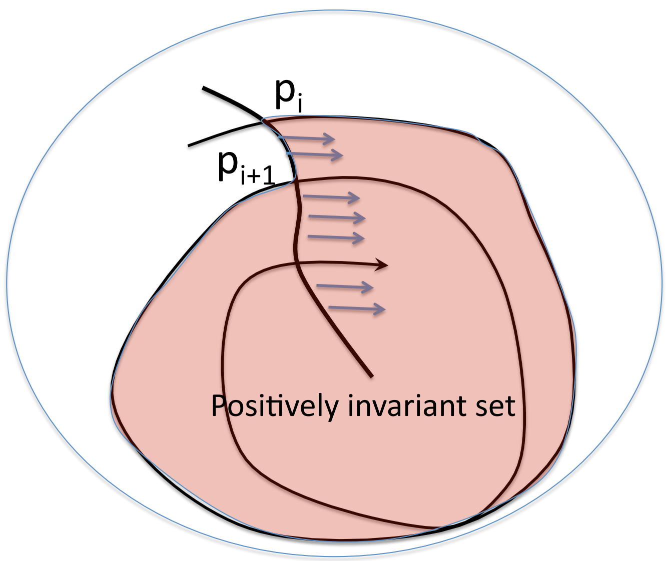
$$(\Sigma'(t), f(\Sigma(t)) \neq \|\Sigma'(t)\| \, \|f(\Sigma(t)\|.$$



Lemma 4.1 Let $\mathcal{M} \subset \mathbb{R}^2$ be a positively invariant set and let Σ be a transversal arc in $\mathcal{M}$. Then for every $p \in \mathcal{M}$, the positive trajectory $O^+(p)$ intersects Σ monotonically, that is, if $p_i = \Sigma(s_i)$ is the i-th intersection of $O^+(p)$ and Σ, then the sequence s_i is monotonic.

Comment: The lemma does not say that $O^+(p)$ must intersect Σ at all, or that it has to intersect it more than once.

Proof: Proof by sketch:



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Corollary 4.2 Let $\mathcal{M} \subset \mathbb{R}^2$ be a positively invariant set and let Σ be a transversal arc in $\mathcal{M}$. Then for every $p \in \mathcal{M}$, $\omega(p)$ intersects Σ at at most one point,

$$\#|\Sigma \cap \omega(p)| \leq 1.$$

Proof: The existence of two intersection points would violate the previous lemma, for suppose that

$$p_1, p_2 \in \Sigma \cap \omega(p).$$

Then, $O^+(p)$ would intersect Σ at infinitely many points in disjoint neighborhoods of p_1 and p_2.

Note that this point is somewhat subtle. If $p_1 \in \Sigma \cap \omega(p)$ we are guaranteed that the trajectory will visit infinitely often any neighborhood of p_1, but does it mean that it has to intersect Σ each time? Yes. Take any neighborhood U of p_1 and consider

$$U \cap \Sigma,$$

which is an open neighborhood of p_1 in Σ. By transversality,

$$\{\Phi_t(U \cap \Sigma) : -\varepsilon < t < e\}$$

is an open neighborhood of p_1 in $\mathbb{R}^2$. Any trajectory visiting this neighborhood intersects $U \cap \Sigma$. ■

Proposition 4.2 Let $\mathcal{M}$ be a positively invariant set and let $p \in \mathcal{M}$. If $\omega(p)$ does not contain any fixed point, then $\omega(p)$ is a periodic trajectory.

Proof: Let $q \in \omega(p)$ and let $x \in \omega(q)$ (we know that both are non-empty sets). Since $O(q) \subset \omega(p)$ (an ω-limit set is invariant) and $\omega(p)$ is closed, then

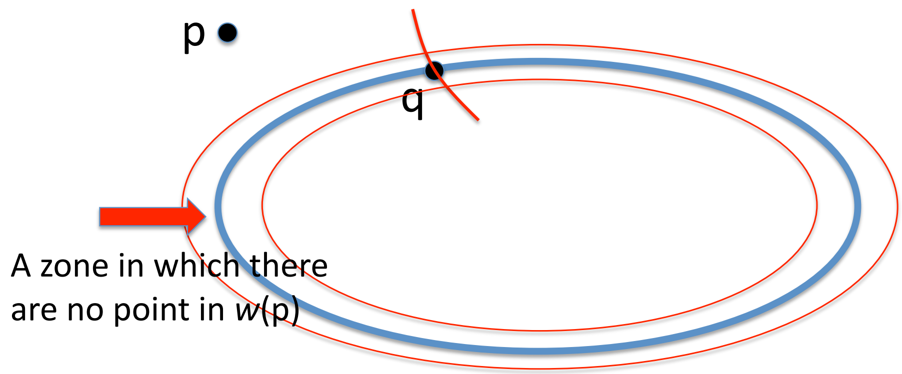
$$\omega(q) \subset \omega(p).$$

It follows that $\omega(q)$ does not contain fixed points, i.e., x is not a fixed point.

Since x is not a fixed point, we can trace through it a transversal arc Σ. The trajectory $O^+(q)$ will intersect Σ monotonically at points $q_n \to x$, but since $q_n \in \omega(p)$, then all the q_n must coincide with x. It follows that $O(q)$ intersects x more than once, hence it is periodic.

It remains to show that $O(q)$ coincides with $\omega(p)$. Let Σ now be a transversal arc at q. We claim that $O(q)$ has a neighborhood in which there are no other points that belong to $\omega(p)$, for if every neighborhood of $O(q)$ contained points in $r \in \omega(p)$, $O(r) \subset \omega(p)$, and there would be such an r for which $O(r)$ intersects Σ.

Since $\omega(p)$ is connected, it follows that $\omega(p) = O(q)$.



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Definition 4.9 A **heteroclinic orbit** *is a trajectory that connects two fixed points, i.e., a trajectory $O(y)$, such that*

$$\alpha(O(y)) = \{p_1\} \qquad \text{and} \qquad \omega(O(y)) = \{p_2\},$$

where p_1 and p_2 are fixed points. If $p_1 = p_2$ (a saddle), then it is called a ***homoclinic orbit****.*

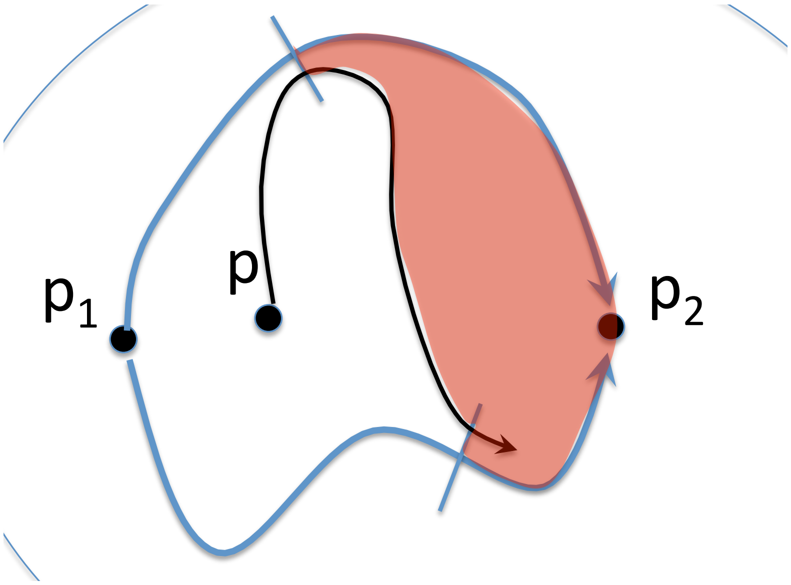
Proposition 4.3 *Let $\mathcal{M}$ be a positively invariant set. Let p_1, p_2 be fixed points and*

$$p_1, p_2 \in \omega(p)$$

for some $p \in \mathcal{M}$. Since $\omega(p)$ is connected and invariant (a union of trajectories). There exists a heteroclinic orbit that connects p_1 and p_2 (or vice versa). There exists at most one trajectory, $\gamma \subset \omega(p)$, such that

$$\{p_1\} = \alpha(\gamma) \qquad \text{and} \qquad \{p_2\} = \omega(\gamma).$$

Proof: Proof by sketch:



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Proposition 4.4 (Poincaré-Bendixon) Let $\mathcal{M}$ be a positively invariant set that contains a finite number of fixed points. Let $p \in \mathcal{M}$. Then, one and only one of the following occurs:

① *$\omega(p)$ is a single fixed point.*
② *$\omega(p)$ is a limit cycle.*
③ *$\omega(p)$ is a union of fixed points p_i and heteroclinic orbits γ_j that connect them.*

Proof: If $\omega(p)$ contains only fixed points then it only contains one due to connectedness.

If $\omega(p)$ does not contain any fixed point then it is a limit cycle (we proved it).

Remains the case where $\omega(p)$ contains fixed points and points that are not fixed points; let $q \in \omega(p)$ and consider the set $\omega(q)$. We will show that it contains a single fixed point, hence q is on a trajectory that converges asymptotically to a fixed point in $\omega(p)$. Similarly, $\alpha(q)$ contains a single fixed point, so that q lies, as claimed, on a trajectory that connects between two fixed points in $\omega(p)$.

So let $x \in \omega(q)$. Suppose that x was not a fixed point, then we would trace a transversal arc Σ through r. The trajectory $O(q)$ must intersect this arc infinitely often, but since $O(q) \in \omega(p)$, it must intersect Σ at a single point, which implies that $O(q)$ is a periodic orbit, but this is impossible because $\omega(p)$ is connected. It follows that $\omega(q)$ contains only fixed points, but then we know that it can only contain a single fixed point.

■