7. Indefinite Integrals

These lecture notes present my interpretation of Ruth Lawrence's lecture notes (in Hebrew)

7.1 Problem statement

By the fundamental theorem of calculus, to calculate an integral

$$\int_a^b f$$

we need to find a function F satisfying $F' = f$. Then,

$$\int_a^b f = F(b) - F(a).$$

Such a function F is called a ***primitive function*** (פונקציה קדומה), an ***anti-derivative*** or an ***indefinite integral*** (אינטגרל לא מסוים) of f. This chapter presents techniques for calculating indefinite integrals. Given a function f, we denote its indefinite integral by

$$\int f.$$

To denote that we evaluate a primitive function of f at x we write

$$\int^x f.$$

A more standard notation is

$$\int^x f(t)\,dt.$$

The standard notation is useful when rather than putting the name of a function in the integrand we write its functional form. For example, for $f(x) = x^2$ we may either write

$$\int f \qquad \text{or} \qquad \int t^2\,dt.$$

Recall that an indefinite integral is only determined up to an additive constant. Thus, when we write an identity like

$$\int f = \int g$$

we mean that any primitive of f is also a primitive of g, and when we write an identity like

$$\int f = h + \int g$$

we mean that any primitive of g plus h is a primitive of f.

Comment 7.1 It is not generally true that if F and G are primitive functions of f then they differ by a constant. Consider the function

$$f(x) = -\frac{1}{x^2} \qquad x \in \mathbb{R} \setminus \{0\}.$$

Then

$$F(x) = \frac{1}{x} \qquad \text{and} \qquad G(x) = \begin{cases} \frac{1}{x} + 1 & x < 0 \\ \frac{1}{x} + 5 & x > 0 \end{cases}$$

are both primitive functions of f. Generally, if a function is defined on a domain consisting of ***disconnected components***, its indefinite integral is unique up to a different additive constant in each connected component.

The most elementary method for finding indefinite integrals is to differentiate functions and create a table of functions and their derivative. Then, we can read this table backward. This is not a very practical mean, but still, we have at the outset a few functions whose indefinite integrals we know.

■ **Example 7.1** Since we know that $\sin' x = \cos x$, it follows that

$$\int^x \cos t\,dt = \sin x.$$

It is also true that

$$\int^x \cos t\,dt = 5 + \sin x,$$

but it is certainly not true that $\sin x = 5 + \sin x$. Recall that the equality really means that sin belongs to the set of primitive functions of cos. In fact, we should have written

$$\sin \in \int \cos,$$

but we won't be doing it. ■

We present below a table of "elementary integrals", which are obtained by differentiation:

$\int^x f$	$f(x)$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$
x^a	$a x^{a-1}$
$\ln\lvert x\rvert$	$1/x$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$
$\tan^{-1} x$	$1/(1+x^2)$
$\ln(\sin x)$	$\cot x$
$\ln(\cos x)$	$-\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1-\tanh^2 x$
$\sinh^{-1} x$	$1/\sqrt{x^2+1}$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$
$\tanh^{-1} x$	$1/(1-x^2)$

This table is an starting point for calculating indefinite integrals of many more functions.

7.2 Elementary functions

The functions for which we have names—powers, trigonometric functions, hyperbolic functions, the exponential and the logarithm, roots, inverse trigonometric functions and inverse hyperbolic functions—are called commonly ***elementary functions***. So is any function that can be obtained by adding, multiplying, dividing and composing elementary functions (note the recursive definitions).

■ **Example 7.2** The function

$$f(x) = \frac{\sqrt{\sin^2 e^x + \sin^{-1}(\tanh \sqrt[3]{x})}}{3x^5 + \ln(\tan^{-1}(x))}$$

is an elementary function. ■

By the laws of derivation we know how to differentiate every elementary function, getting again an elementary function. Note, however, that there are many more functions that are not elementary. We simply don't have names for them (sometimes we do). It turns out that many elementary functions don't have elementary indefinite integrals. A theorem by Liouville in 1835 provided the first proof that non-elementary anti-derivatives of elementary functions exist.

■ **Example 7.3** The following elementary functions don't have elementary anti-derivatives:

$$f(x) = \sqrt{1-x^4} \qquad f(x) = \ln(\ln x) \qquad f(x) = \frac{\sin x}{x} \quad \text{and} \quad f(x) = e^{-x^2}.$$

■

It should be clear, however, that these functions have anti-derivatives. We simply can't express them as elementary functions. In fact, the function

$$\int e^{-t^2}\,dt$$

is sufficiently important in statistics that it has a name – the ***error function***.

7.3 Integration by parts (אינטגרציה בחלקים)

Recall the Leibniz rule,

$$(uv)' = u'v + uv'.$$

Since uv is a primitive function of $(uv)'$, we have

$$uv = \int u'v + \int uv',$$

or

$$\boxed{\int u'v = uv - \int uv'} \tag{7.1}$$

which in more standard notation reads

$$\int^x u'(t)v(t)\,dt = u(x)\,v(x) - \int^x u(t)v'(t)\,dt.$$

This innocent-looking identity is a useful starting point for evaluating indefinite integrals.

■ **Example 7.4** Consider the indefinite integral

$$I(x) = \int^x \ln t\,dt.$$

We evaluate it as follows: set $u(x) = x$ and $v(x) = \ln x$. Then,

$$I(x) = \int \underbrace{1}_{u'(t)}\,\underbrace{\ln t}_{v(t)}\,dt = \underbrace{x}_{u(x)}\,\underbrace{\ln x}_{v(x)} - \int^x \underbrace{t}_{u(t)}\,\underbrace{\frac{1}{t}}_{v'(t)}\,dt = x\ln x - x.$$

Note that we could have as well taken $u(x) = x+5$, in which case we would have gotten

$$I(x) = \int^x \underbrace{1}_{u'(t)} \underbrace{\ln t}_{v(t)}\, dt = \underbrace{(x+5)}_{u(x)} \underbrace{\ln x}_{v(x)} - \int^x \underbrace{(t+5)}_{u(t)} \underbrace{\frac{1}{t}}_{v'(t)}\, dt = x\ln x - x,$$

so as expected, it doesn't matter which anti-derivative of u' we take. ■

■ **Example 7.5** For $n \in \mathbb{N}$ consider the indefinite integrals

$$I_n(x) = \int^x t^n \ln t\, dt.$$

Here we'll take $u'(x) = x^n$ and $v(x) = \ln x$, hence

$$I_n(x) = \int^x t^n \ln t\, dt = \frac{x^{n+1}}{n+1}\ln x - \int^x \frac{t^{n+1}}{n+1}\frac{1}{t}\, dt = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2}.$$

■

■ **Example 7.6** For $n \in \mathbb{N}$ consider the indefinite integrals

$$I_n(x) = \int^x t^n e^t\, dt.$$

Taking $e^x = u'(x)$ and $x^n = v(x)$ we get

$$I_n(x) = x^n e^x - n\int^x t^{n-1} e^t\, dt = x^n e^x - n I_{n-1}(x).$$

This is a ***recurrence relation*** (יחס נסיגה). For $n = 0$,

$$I_0(x) = e^x.$$

Then,

$$\begin{aligned} I_1 &= xe^x - e^x = (x-1)e^x \\ I_2 &= x^2e^x - 2(x-1)e^x = (x^2-2x+2)e^x \\ I_3 &= x^3e^x - 3(x^2-2x+2)e^x = (x^3-3x^2+6x-6)e^x \\ \vdots &= \vdots \end{aligned}$$

It is easy to see that

$$I_n(x) = n!\left(\frac{x^n}{n!} - \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} - \cdots + (-1)^n\right)e^x = n!\, e^x \sum_{k=0}^{n} \frac{(-1)^{n-k}x^k}{k!}.$$

■

■ **Example 7.7** Consider the sequence of indefinite integrals

$$I_n(x) = \int^x \sin^n t\, dt.$$

We treat it as follows:

$$\begin{aligned} I_n(x) &= \int^x \sin^{n-1} t \sin t \, dt \\ &= -\sin^{n-1} x \cos x + (n-1) \int^x \sin^{n-2} t \cos^2 t \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int^x \sin^{n-2} t (1 - \sin^2 t) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) I_{n-2}(x) - (n-1) I_n(x). \end{aligned}$$

Hence, we get a recurrence relation

$$I_n(x) = \frac{n-1}{n} I_{n-2}(x) - \frac{1}{n} \sin^{n-1} x \cos x.$$

This time we need both I_0 and I_1:

$$I_0(x) = x \qquad \text{and} \qquad I_1(x) = -\cos x.$$

Then,

$$\begin{aligned} I_2(x) &= \frac{1}{2} I_0(x) - \frac{1}{2} \sin^1 x \cos x \\ I_3(x) &= \frac{2}{3} I_1(x) - \frac{1}{3} \sin^2 x \cos x \\ I_4(x) &= \frac{3}{4} I_2(x) - \frac{1}{4} \sin^3 x \cos x, \end{aligned}$$

and so on. ■

7.4 Substitution method (שיטת ההצבה)

Recall the chain rule,

$$(F \circ g)' = (F' \circ g) \, g'$$

from which follows that

$$F \circ g = \int (F' \circ g) \, g'$$

For $F = \int^x f(t) \, dt$,

$$\boxed{\int^{g(x)} f = \int^x (f \circ g) \, g'} \tag{7.2}$$

and it more standard notation,

$$\int^x f(g(t)) \, g'(t) \, dt = \int^{g(x)} f(t) \, dt.$$

As we will see, this identity is useful for finding many indefinite integrals.

7.4.1 Substitution of linear functions

Suppose that we know a primitive function of f and we want to evaluate

$$I(x) = \int^x f(at+b)\,dt,$$

where a is a constant.

Set $g(x) = ax+b$, and use the substitution formula (7.2),

$$I(x) = \frac{1}{a}\int^x f(g(t))\,g'(t)\,dt = \frac{1}{a}\int^{g(x)} f(t)\,dt = \frac{1}{a}\int^{ax+b} f(t)\,dt.$$

Comment 7.2 The "physicists" way of doing the same procedure goes as follows: they define, say,

$$u = ax+b,$$

and then write

$$du = \frac{du}{dx}dx = a\,dx.$$

Then,

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du,$$

and it is understood that the right hand side is evaluated at $u = ax+b$.

■ **Example 7.8**

$$\int^x \cos(at+b)\,dt = \frac{1}{a}\sin(ax+b).$$

■

■ **Example 7.9** Find

$$I(x) = \int^x \frac{dt}{\sqrt{a^2-t^2}}.$$

First note that

$$I(x) = \frac{1}{a}\int^x \frac{dt}{\sqrt{1-(t/a)^2}}.$$

Setting

$$g(x) = \frac{x}{a},$$

we use the substitution formula (7.2) to get

$$I(x) = \frac{1}{a}\int^x \frac{dt}{\sqrt{1-g^2(t)}} = \int^x \frac{g'(t)dt}{\sqrt{1-g^2(t)}}$$
$$= \int^{g(x)} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(g(x)) = \sin^{-1}\left(\frac{x}{a}\right).$$

■

■ **Example 7.10** As a similar example, consider

$$I(x) = \int^x \frac{dt}{\sqrt{8+2t-t^2}}.$$

The trick here is to first complete the square under the root,

$$I(x) = \int^x \frac{dt}{\sqrt{9-(t-1)^2}} = \frac{1}{3}\int^x \frac{dt}{\sqrt{1-((t-1)/3)^2}}.$$

Set then

$$g(x) = \frac{x-1}{3}.$$

Then,

$$I(x) = \frac{1}{3}\int^x \frac{dt}{\sqrt{1-g^2(t)}} = \int^x \frac{g'(t)dt}{\sqrt{1-g^2(t)}} = \int^{g(x)} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}\left(\frac{x-1}{3}\right).$$

■

7.4.2 Other examples

■ **Example 7.11** Consider again the integrals

$$I_n(x) = \int^x \sin^n t\, dt,$$

this time only for odd values of $n = 2m+1$. We've already solved these integrals using integration by parts. This time we will use the method of substitution. Start with

$$I_n(x) = \int^x \sin t (\sin^2 t)^m\, dx = \int^x \sin t (1-\cos^2 t)^m\, dx.$$

Setting

$$f(x) = (1-x^2)^m \qquad \text{and} \qquad g(x) = \cos x,$$

we get

$$I_n(x) = -\int^x f(g(t))\, g'(t)\, dt = -\int^{\cos x} (1-t^2)^m\, dt.$$

Take for example $n = 3$. Then,

$$I_3(x) = -\left(\cos x - \frac{\cos^3 x}{3}\right)$$

■

■ **Example 7.12** Consider integrals of the form

$$I(x) = \int^x \frac{r(t)\, dt}{\sqrt{1-t^2}},$$

where $r(x)$ is a rational function. Setting

$$g(x) = \sin^{-1} x \qquad \text{or equivalently} \qquad x = \sin g(x),$$

we get

$$I(x) = \int^x r(\sin(g(t)))\, g'(t)\, dt = \int^{\sin^{-1} x} r(\sin(t))\, dt.$$

Take for example $r(x) = x^2$. Then,

$$\int^x r(\sin(t))\, dt = \int^x \sin^2 t\, dt = \frac{1}{2}\int^x (1-\cos 2t)\, dt = \frac{x}{2} - \frac{1}{4}\sin 2x = \frac{1}{2}(x - \sin x \cos x).$$

Thus,

$$\int^x \frac{t^2\, dt}{\sqrt{1-t^2}} = \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right).$$

■

7.5 Integrals of rational functions

Consider integrals of the form

$$\int^x \frac{p(t)}{q(t)}\, dt,$$

where p and q are polynomials (a function that is the ratio of two polynomials is called a ***rational function***). If $\deg p \ge \deg q$ then we can divide the polynomials to get

$$\frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{q(x)},$$

where r and s are polynomials, and $\deg s < \deg q$. Since we know how to integrate polynomials, it remains to learn how to integrate rational functions with $\deg p < \deg q$.

7.5.1 Linear denominators

When $q(x)$ is a linear function, we end with an integral of the form

$$\boxed{\int^x \frac{a\, dt}{t+b} = a \ln|x+b|}$$

■ **Example 7.13** Consider the following indefinite integral

$$I(x) = \int^x \frac{t^3\, dt}{t+5}.$$

Since

$$\begin{aligned}\frac{x^3}{x+5} &= \frac{(x+5)x^2 - 5x^2}{x+5} \\ &= \frac{(x+5)x^2 - 5x(x+5) + 25x}{x+5} \\ &= \frac{(x+5)x^2 - 5x(x+5) + 25(x+5) - 125}{x+5} \\ &= x^2 - 5x + 25 - \frac{125}{x+5},\end{aligned}$$

it follows that

$$I(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 25x - 125 \ln|x+5|.$$

■

7.5.2 Quadratic denominators

When $q(x)$ is a quadratic function there are three possibilities:

1. q has two distinct real-valued roots, in which case we obtain an integral of the form
$$\int^x \frac{at+b}{(t-c)(t-d)}\, dt,$$
2. The two roots of q coincide, in which case we obtain an integral of the form
$$\int^x \frac{at+b}{(t-c)^2}\, dt$$
3. q is irreducible, i.e., has two complex-valued root, then the integral can be brought to the form
$$\int^x \frac{at+b}{(t-c)^2+d^2}\, dt$$

Case 1: Two distinct real-valued roots

In this case we split the the integrand as follows:

$$\frac{at+b}{(t-c)(t-d)} = \frac{A}{t-c} + \frac{B}{t-d} = \frac{A(t-d)+B(t-c)}{(t-c)(t-d)}.$$

Matching coefficients we obtain

$$A + B = a \qquad \text{and} \qquad -Ad - Bc = b.$$

This is a linear system that always has a solution provided $c \neq d$:

$$A = \frac{ac+b}{c-d} \qquad \text{and} \qquad B = \frac{ad+b}{d-c}$$

Then,

$$\boxed{\int^x \frac{(at+b)\,dt}{(t-c)(t-d)} = \frac{ac+b}{c-d}\ln|x-c| + \frac{ad+b}{d-c}\ln|x-d|}$$

■ **Example 7.14** Consider the case $a=0$, $b=1$, $c=-1$ and $d=-2$,

$$\int^x \frac{dt}{(t+1)(t+2)} = \ln|x+1| - \ln|x+2|$$

■

Case 2: Two equal roots

We split the integrand as follows,

$$\frac{at+b}{(t-c)^2} = \frac{a(t-c)+(b+ac)}{(t-c)^2}.$$

Thus,

$$\int^x \frac{(at+b)\,dt}{(t-c)^2} = a\int^x \frac{dt}{t-c} + (ac+b)\int^x \frac{dt}{(t-c)^2},$$

namely,

$$\boxed{\int^x \frac{(at+b)\,dt}{(t-c)^2} = a\ln|x-c| - \frac{ac+b}{x-c}}$$

■ **Example 7.15**

$$\int^x \frac{2t+5}{t^2+4t+4}\,dt = \int^x \frac{2t+5}{(t+2)^2}\,dt = 2\ln|x+2| - \frac{1}{x+2}.$$

■

Case 3: Two complex-valued roots

Remains the case where q is irreducible (with real numbers). In this case we can bring q by a square completion to the form

$$q(t) = (t-c)^2 + d^2.$$

Now,

$$\frac{at+b}{(t-c)^2+d^2} = \frac{a(t-c)+(ac+b)}{(t-c)^2+d^2}.$$

Hence,

$$\int^x \frac{(at+b)\,dt}{(t-c)^2+d^2} = a\int^x \frac{(t-c)\,dt}{(t-c)^2+d^2} + (ac+b)\int^x \frac{dt}{(x-c)^2+d^2}.$$

The first integral equals

$$\frac{a}{2}\ln|(t-c)^2+d^2|,$$

whereas the second integral equals, using the method of substitution

$$\frac{ac+b}{d^2}\int^x \frac{dt}{[(t-c)/d]^2+1} = \frac{ac+b}{d}\tan^{-1}\left(\frac{x-c}{d}\right).$$

To conclude,

$$\boxed{\int^x \frac{(at+b)\,dt}{(t-c)^2+d^2} = \frac{a}{2}\ln|(x-c)^2+d^2| + \frac{ac+b}{d}\tan^{-1}\left(\frac{x-c}{d}\right)}$$

7.5.3 Denominators of higher degree

Let's consider an example. Other follow a similar route:

■ **Example 7.16**

$$\int^x \frac{(t+5)\,dt}{t^3-t^2-t+1}$$

To do something we must be able to decompose the denominator. In the present case

$$t^3-t^2-t+1 = (t+1)(t-1)^2.$$

Since we have a simple root, (-1), and a double root, 1, we look for a decomposition,

$$\frac{t+5}{t^3-t^2-t+1} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1}.$$

Matching the numerators

$$t+5 = A(t^2-1) + B(t+1) + C(t^2-2t+1),$$

from which we get that

$$A = -C \qquad B-2C = 1 \qquad \text{and} \qquad -A+B+C = 5.$$

Then,

$$A = -1 \qquad B = 3 \qquad \text{and} \qquad C = 1,$$

from which we conclude that

$$\int^x \frac{(t+5)\,dt}{t^3-t^2-t+1} = -\ln|x-1| - \frac{3}{x-1} + \ln|x+1|.$$

■

7.6 Definite integrals

We now turn to evaluate definite integrals. By the Newton-Leibniz theorem it suffices to find a primitive function F of f, in which case

$$\int_a^b f = F|_a^b.$$

In this section we will see how to apply the integration methods for the calculation of definite integrals.

7.6.1 Integration by parts

Since

$$uv|_a^b = \int_a^b (uv)' = \int_a^b u'v + \int_a^b uv',$$

it follows that

$$\boxed{\int_a^b u'(x)v(x)\,dx = u(x)v(x)|_a^b - \int_a^b u(x)v'(x)\,dx}$$

■ **Example 7.17** Calculate

$$\int_5^6 \ln.$$

■

7.6.2 Method of substitution

Let $F' = f$. Since

$$F \circ g|_a^b = \int_a^b (F \circ g)' = \int_a^b f(g(t))g'(t)\,dt,$$

If follows that

$$\int_a^b f(g(t))g'(t)\,dt = F|_{g(a)}^{g(b)},$$

i.e.,

$$\boxed{\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(t)\,dt}$$

■ **Example 7.18** Calculate

$$I = \int_0^2 \frac{t\,dt}{(1+t^2)(2+t^2)}.$$

Letting $g(t) = t^2$, we have

$$I = \frac{1}{2}\int_0^2 \frac{g'(t)\,dt}{(1+g(t))(2+g(t))} = \frac{1}{2}\int_{g(0)}^{g(2)} \frac{dt}{(1+t)(2+t)} = \frac{1}{2}\int_0^4 \frac{dt}{(1+t)(2+t)}.$$

We proceed by writing

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} = \frac{A(2+t)+B(1+t)}{(1+t)(2+t)},$$

obtaining $A = -B = 1$. Hence

$$I = \frac{1}{2}\left(\int_0^4 \frac{dt}{1+t} - \int_0^4 \frac{dt}{2+t}\right) = \frac{1}{2}(\ln 5 - \ln 1 - \ln 6 + \ln 2) = \frac{1}{2}\ln\frac{5}{3}.$$

■

7.7 Improper integrals

Improper integrals (אינטגרלים לא אמיתיים) are definite integrals in which "something infinite" takes place: either the domain of integration is infinite or the integrand diverges within the domain of integration.

The first instance comprises of integrals of the form

$$\int_a^\infty f = \lim_{x\to\infty} \int_a^x f$$
$$\int_{-\infty}^a f = \lim_{x\to-\infty} \int_x^a f$$
$$\int_{-\infty}^\infty f = \lim_{x\to\infty} \lim_{y\to-\infty} \int_y^x f.$$

■ **Example 7.19** Compare the two cases

$$\int_1^\infty \frac{dt}{t^2} \qquad \text{and} \qquad \int_1^\infty \frac{dt}{t}.$$

Note that the Riemann sums that correspond to $\Delta t = 1$ are the respectively convergent and divergent series,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

and

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

■

The second instance comprises the examples

$$\int_0^1 \frac{dt}{\sqrt{t}} \qquad \text{and} \qquad \int_0^1 \frac{dt}{t},$$

which we interpret as

$$\lim_{x\to 0^+} \int_x^1 \frac{dt}{\sqrt{t}} \qquad \text{and} \qquad \lim_{x\to 0^+} \int_x^1 \frac{dt}{t}.$$

■ **Example 7.20** Consider the integral

$$\int_{-\infty}^\infty \sin t \, dt.$$

Since sin is anti-symmetric, for every x

$$\int_{-x}^x \sin t \, dt = 0.$$

It may seem therefore that the integral of sin over the whole line is zero, but this is not the case as

$$\int_y^x \sin t \, dt = \cos y - \cos x$$

does not have a limit as $y \to -\infty$ and $x \to \infty$ ■