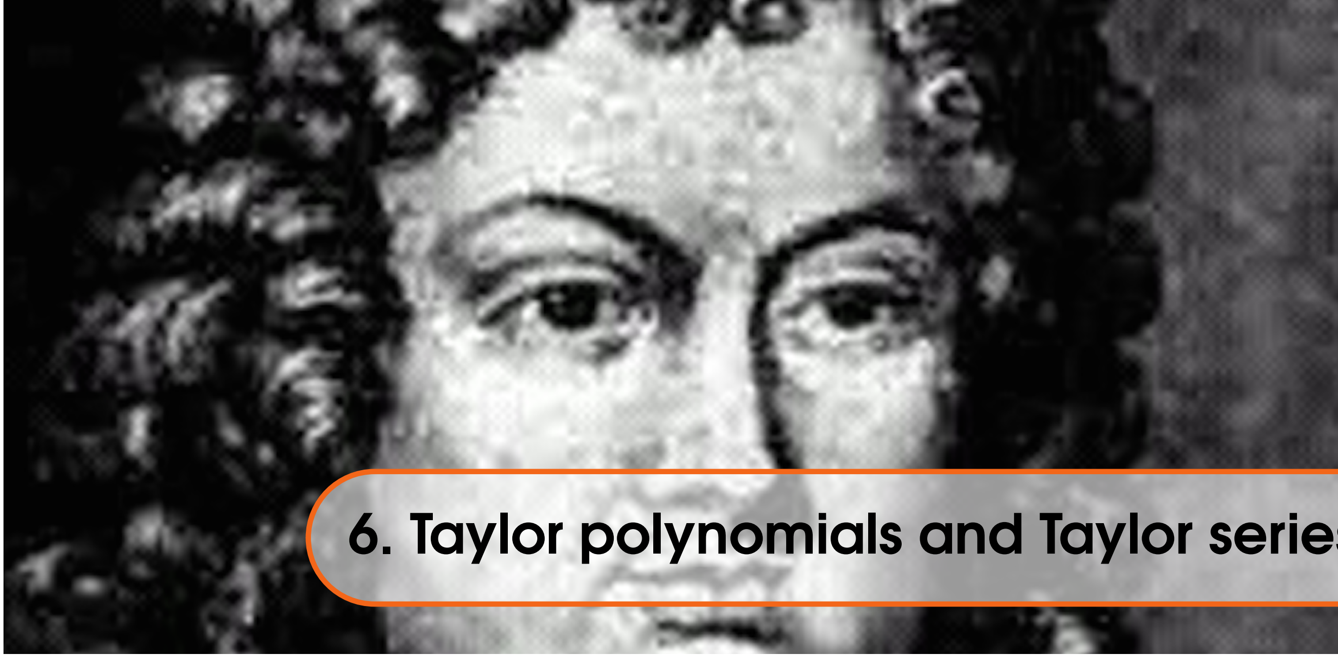


6. Taylor polynomials and Taylor series

These lecture notes present my interpretation of Ruth Lawrence's lecture notes (in Hebrew)

[1]

6.1 Preliminaries

6.1.1 Polynomials

A ***polynomial of degree*** n (פולינום) is a function of the form

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0,$$

where $b_n \neq 0$. It is customary to denote

$$\Pi_n = \{\text{all polynomials of degree up to } n\}.$$

This is a set of functions closed under addition and scalar multiplication (in fact, it is a vector space), but it is not closed under multiplication, as the product of two polynomials of degree n is a polynomial of degree $2n$ (but the set of all polynomials of any degree is closed under multiplication).

Let's calculate all the derivatives of p. First,

$$p'(x) = nb_n x^{n-1} + (n-1)b_{n-1} x^{n-2} + \cdots + 2b_2 x + b_1,$$

so that the derivative of a polynomial of degree n is a polynomial of degree $n-1$. Then,

$$\begin{aligned} p''(x) &= n(n-1)b_n x^{n-2} + (n-1)(n-2)b_{n-1} x^{n-3} + \cdots + 3\cdot 2b_3 x + 2b_2 \\ p'''(x) &= n(n-1)(n-2)b_n x^{n-3} + (n-1)(n-2)(n-3)b_{n-1} x^{n-4} + \cdots + 3\cdot 2b_3. \end{aligned}$$

[1] Image of Brook Taylor, 1685–1731

Finally,

$$p^{(n)}(x) = n(n-1)(n-2)\cdot 3\cdot 2\cdot b_n,$$

which is a constant function, and then $p^{(n+1)}(x) = 0$, as well as all higher derivatives.

We then evaluate p and all its derivatives at the origin:

$$p(0) = b_0 \quad p'(0) = b_1 \quad p''(0) = 2b_2 \quad p'''(0) = 3\cdot 2b_3 \quad \cdots \quad p^{(n)}(0) = n!\,b_n,$$

i.e.,

$$p^{(k)}(0) = \begin{cases} k!\,b_k & k \le n \\ 0 & k > n. \end{cases}$$

Let a be an arbitrary number and consider the function

$$q(x) = b_n(x-a)^n + b_{n-1}(x-a)^{n-1} + \cdots + b_1(x-a) + b_0.$$

We first argue that q is a polynomial of degree n (just open the brackets). In fact, it is identical to p up to a translation of x by a. Hence, the derivatives of q at a must be identical to the derivatives of p at zero,

$$q^{(k)}(a) = \begin{cases} k!\,b_k & k \le n \\ 0 & k > n, \end{cases}$$

which you can also check by a direct calculation. We can invert this relation,

$$b_k = \frac{q^{(k)}(a)}{k!}, \qquad k \le n.$$

6.1.2 Linear approximation revisited

Let f be a function differentiable at a. Recall that the ***linear approximation*** of f about a is a polynomial of degree one,

$$p(x) = f(a) + f'(a)(x-a).$$

The values of p and its first derivative at a are

$$p(a) = f(a) \qquad \text{and} \qquad p'(a) = f'(a).$$

That is, the linear approximation of f at a is a polynomial of degree one that has the same value as well as the same first derivative as f at a (but only there). It is easy to see that it is the *only* polynomial of degree up to one that satisfies this property.

In what sense is p an approximation to f in the vicinity a? Clearly, if we look at the difference between f and p (the ***remainder*** (שארית)),

$$R(x) = f(x) - p(x) = f(x) - f(a) - f'(a)(x-a),$$

then

$$\lim_{x\to a} R(x) = 0.$$

But the extent to which p approximates f near a is even stronger. Consider the ratio

$$\frac{R(x)}{x-a} = \frac{f(x)-f(a)-f'(a)(x-a)}{x-a} = \Delta_{f,a}(x) - f'(a).$$

Even though the denominator tends to zero as $x \to a$, we nevertheless get that

$$\lim_{x\to a} \frac{R(x)}{x-a} = \lim_{x\to a} \Delta_{f,a}(x) - f'(a) = 0.$$

Definition 6.1 Let g and h be two functions. We say that

$$g(x) = o(h(x)) \qquad \text{as } x \to 0$$

if

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = 0.$$

How is this definition relevant to the notion of a linear approximation? We have just shown that

$$R(x) = f(x) - f(a) - f'(a)(x-a) = o(|x-a|),$$

or,

$$\boxed{f(x) = f(a) + f'(a)(x-a) + o(|x-a|) \qquad \text{as } x \to a}$$

■ **Example 6.1** Consider the function $f(x) = e^x$. Its linear approximation about 1 is

$$p(x) = e + e(x-1).$$

Then,

$$e^x = e + e(x-1) + o(|x-1|) \qquad \text{as } x \to 1,$$

which means that

$$\lim_{x\to 1} \frac{e^x - (e + e(x-1))}{x-1} = 0.$$

■

6.2 Taylor polynomials

The idea of approximating a function by a polynomial of degree one calls for a generalization. Suppose that f were twice differentiable at a. Can we find a polynomial that has the same value as f at a, as well as the same first two derivatives?

We can find plenty of such polynomials, but there is only one of *minimal* degree. By Section 6.1.1, the polynomial

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

satisfies,

$$p(a) = f(a) \qquad p'(a) = f'(a) \quad \text{and} \quad p''(a) = f''(a).$$

If f were differentiable three times at a we could construct a polynomial of degree three that has the same value as f at a, as well as the same first three derivatives. More generally:

Definition 6.2 Suppose that f is n-times differentiable at a. Its ***Taylor polynomial*** (**פולינום טיילור**) of degree n about a is given by

$$P_{f,n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The short-hand notation is

$$\boxed{P_{f,n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

where, by convention, $f^{(0)} = f$. This polynomial is the only one of degree (up to) n that agrees with the value of f and with the value of its first n derivatives at the point a. But please note: unless f was a polynomial to start with, $f(x) \neq P_{f,n,a}(x)$.

■ **Example 6.2** Consider the Taylor polynomial of degree n of the function exp about the point $a = 0$. Since for all k, $\exp^{(k)} = \exp$ it follows that $\exp^{(k)}(0) = 1$. Then,

$$P_{\exp,n,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

■

■ **Example 6.3** Consider the Taylor polynomial of degree n of the sine function about the point $a = 0$. Since $\sin' = \cos$, $\sin'' = -\sin$, $\sin''' = -\cos$ and $\sin'''' = \sin$ we have a periodic pattern,

$$\sin^{(k)}(0) = \begin{cases} 0 & k = 2j \\ (-1)^{j-1} & k = 2j+1. \end{cases}$$

It follows that the Taylor polynomial of degree n includes only odd terms. We have

$$P_{\sin,2n+1,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n}(-1)^{k-1}\frac{x^{2k+1}}{(2k+1)!}.$$

■

■ **Example 6.4** Consider the Taylor polynomial of degree n of the cosine function about the point $a = 0$. Since $\cos' = -\sin$, $\cos'' = -\cos$, $\cos''' = \sin$ and $\cos'''' = \cos$ we have a periodic pattern,

$$\cos^{(k)}(0) = \begin{cases} 0 & k = 2j+1 \\ (-1)^j & k = 2j. \end{cases}$$

It follows that the Taylor polynomial of degree n includes only even terms,

$$P_{\cos,2n,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

■

■ **Example 6.5** Consider the Taylor polynomial of degree n of the natural logarithm about the point $a = 1$. We have

$$\ln'(x) = \frac{1}{x} \quad \ln''(x) = -\frac{1}{x^2} \quad \ln'''(x) = \frac{2}{x^3} \quad \ln''''(x) = -\frac{3!}{x^4},$$

so that $\ln(1) = 0$ and

$$\ln^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

Hence,

$$P_{\ln,n,1}(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!\,(x-1)^k}{k!} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1} \frac{(x-1)^n}{n}.$$

Equivalently, the Taylor polynomial of degree n of the function

$$f(x) = \ln(x+1)$$

about $a = 0$ is

$$P_{f,n,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}$$

■

■ **Example 6.6** Consider now the function

$$f(x) = \frac{1}{1+x}$$

and its Taylor polynomial about the point $a = 0$ (note that this is the derivative of the previous example!). We have

$$f'(x) = -\frac{1}{(1+x)^2} \qquad f''(x) = \frac{2}{(1+x)^3},$$

and generally,

$$f^{(k)}(0) = (-1)^k k!.$$

Then,

$$P_{f,n,0}(x) = \sum_{k=0}^{n} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^n.$$

■

Like for differentiation, we can make the calculation of Taylor polynomials easier by showing that they satisfy certain algebraic properties:

Proposition 6.1 Let f, g be n-times differentiable at a. Then,

$$P_{f+g,n,a} = P_{f,n,a} + P_{g,n,a}.$$

Proposition 6.2 Let f, g be n-times differentiable at a. Then,

$$P_{fg,n,a} = [P_{f,n,a} P_{g,n,a}]_n,$$

where the notation $[\,]_n$ stands for truncation at the n-th degree.

Example 6.7 Consider the Taylor polynomial of degree four of the function

$$f(x) = \sin x \ln(1+x).$$

about $a = 0$. Since

$$P_{\sin,4,0}(x) = x - \frac{x^3}{6} \qquad \text{and} \qquad P_{\ln(1+\cdot),4,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

Then,

$$\begin{aligned} P_{f,4,0}(x) &= \left[\left(x - \frac{x^3}{6} \right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \right) \right]_4 \\ &= x \left(x - \frac{x^2}{2} + \frac{x^3}{3} \right) - \frac{x^3}{6} x = x^2 - \frac{x^3}{2} + \frac{x^4}{6}. \end{aligned}$$

■

And finally the composition rule:

Proposition 6.3 Let f be n-times differentiable at a and let g be n times differentiable at $f(a)$. Then,

$$P_{g\circ f,n,a} = [P_{g,n,f(a)} \circ P_{f,n,a}]_n.$$

Example 6.8 Consider the function

$$f(x) = \ln(\cos(x)).$$

We want to calculate its Taylor polynomial of degree 4 about the point 0. Recall that

$$P_{\cos,4,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!},$$

and

$$P_{\ln,4,1}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

Then,

$$\begin{aligned}
P_{f,4,0}(x) &= [P_{\ln,4,1}(P_{\cos,4,0}(x))]_4 \\
&= \left[(P_{\cos,4,0}(x)-1) - \frac{(P_{\cos,4,0}(x)-1)^2}{2} + \frac{(P_{\cos,4,0}(x)-1)^3}{3} - \frac{(P_{\cos,4,0}(x)-1)^4}{4}\right]_4 \\
&= \left[(P_{\cos,4,0}(x)-1) - \frac{(P_{\cos,4,0}(x)-1)^2}{2}\right]_4 \\
&= -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{(-x^2/2)^2}{2} = -\frac{x^2}{2} - \frac{x^4}{12}.
\end{aligned}$$

■

We will need below the following straightforward fact:

Lemma 6.4 Let f be n times differentiable at a. The Taylor polynomial of degree n of f about a satisfies the following property,

$$P'_{f,n,a} = P_{f',n-1,a}. \tag{6.1}$$

Proof. This is immediate as

$$P_{f,n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

hence

$$\begin{aligned}
P'_{f,n,a}(x) &= \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} k(x-a)^{k-1} = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}(x-a)^{k-1} \\
&= \sum_{j=0}^{n-1} \frac{f'^{(j)}(a)}{j!}(x-a)^j = P_{f',n-1,a}(x).
\end{aligned}$$

■

6.3 Properties of Taylor polynomials

The first questions to be asked about the Taylor polynomial $P_{f,n,a}$ are:

1. How well does the polynomial approximate the function in the vicinity of a? In particular, how does the quality of this approximation depend on n?

2. How well does the polynomial approximate the function away from a? In particular, how does the quality of this approximation depend on n?
3. What happens as we let $n \to \infty$?

Before we answer those questions, let us learn a useful theorem. Suppose that f and g both vanish at a and have continuous derivatives in a neighborhood of a; moreover, assume that g' does not vanish in this neighborhood. Fix a point b in this neighborhood and consider the function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

This function satisfies $h(a) = h(b)$. By the mean-value theorem there exists a point x between a and b where $h'(x) = 0$, i.e.,

$$f'(x)g(b) - g'(x)f(b) = 0,$$

or

$$\frac{f(b)}{g(b)} = \frac{f'(x)}{g'(x)}.$$

What happens as $b \to a$? Then, also $x \to a$. One can then show that:

Theorem 6.5 — l'Hopital's rule (הכלל של לופיטל). Suppose that f and g both vanish at a and have continuous derivatives in a neighborhood of a; moreover, assume that neither g nor g' vanish in this neighborhood. If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \qquad \text{exists,}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

■ **Example 6.9** Consider the function

$$h(x) = \frac{1 - \cos x}{\sin x}.$$

Does this function have a limit as $x \to 0$ (note the zero divided by zero)? L'Hopital's rule tell us that

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{(1-\cos)'(x)}{\sin' x} = \lim_{x\to 0} \frac{\sin x}{\cos x},$$

provided that the right-hand side exists. In the present case,

$$\lim_{x\to 0} h(x) = 0.$$

■

L'Hopital's rule is a useful tool for calculating limits. In the present context, it is helpful in the following sense. Consider the difference between a function and its linear approximation divided by the distance of the point of evaluation from the point of expansion:

$$\frac{f(x) - P_{f,1,a}(x)}{x-a}.$$

L'Hopital's rule tells us that the limit as $x \to a$ exists is the limit of the ratios of the derivatives does. The latter is equal to

$$\frac{f'(x) - P_{f',0,a}(x)}{1},$$

which tends to zero as $x \to 0$. Thus, we recover once again the property of the linear approximation,

$$\lim_{x\to a} \frac{f(x) - P_{f,1,a}(x)}{x-a} = 0.$$

Consider now the Taylor polynomial of degree 2, and consider the ratio

$$\frac{f(x) - P_{f,2,a}(x)}{(x-a)^2}.$$

By l'Hopital's rule,

$$\lim_{x\to a} \frac{f(x) - P_{f,2,a}(x)}{(x-a)^2} = \lim_{x\to a} \frac{f'(x) - P_{f',1,a}(x)}{2(x-a)}$$

provided that the right-hand side exists, but by the very same rule,

$$\lim_{x\to a} \frac{f'(x) - P_{f',1,a}(x)}{2(x-a)} = \lim_{x\to a} \frac{f''(x) - P_{f'',0,a}(x)}{2}$$

provided that the right-hand side exists. It does; it is zero. Hence,

$$\lim_{x\to a} \frac{f(x) - P_{f,2,a}(x)}{(x-a)^2} = 0.$$

Equivalently, we an use the "little o" notation,

$$f(x) = P_{f,2,a}(x) + o((x-a)^2) \qquad \text{as } x \to a.$$

Proceeding inductively we obtain the following important characterization of the Taylor polynomial:

Theorem 6.6 — Taylor. Let f be n times differentiable at a. Then

$$\lim_{x\to a} \frac{f(x)-P_{f,n,a}(x)}{(x-a)^n} = 0,$$

or using the “little o" notation,

$$f(x) = P_{f,n,a}(x) + o((x-a)^n) \qquad \text{as } x\to a.$$

Taylor’s theorem states that the difference between $P_{f,n,a}(x)$ and $f(x)$ tends to zero as $x\to a$ “faster" than $(x-a)^n$. Important: in Taylor’s theorem n is fixed and we consider the limit $x\to a$. This is in contrast with the next section, where we keep x fixed and let $n\to\infty$.

Can we be more explicit about the deviation of $P_{f,n,a}(x)$ from $f(x)$? It turns out that by a generalization of the mean-value theorem, the following can be proved:

Theorem 6.7 — Remainder of Taylor’s polynomial. Suppose that f is differentiable $(n+1)$ times in a neighborhood of a. Then for every x in this neighborhood there exists a point c between a and x such that

$$f(x)-P_{f,n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

(This expression is known as ***Lagrange’s form of the remainder***).

■ **Example 6.10** Consider the sine function. Its Taylor polynomial of degree 2 about zero is

$$P_{\sin,2,0}(x) = x.$$

By the remainder theorem, there exists for every x a point c (that depends on x), such that

$$\sin x - x = \frac{-\cos c}{3!}x^3.$$

Thus,

$$|\sin x - x| \le \frac{|x|^3}{6}.$$

For example,

$$|\sin(0.1)-0.1| \le \frac{0.001}{6}.$$

The Taylor polynomial of degree four is

$$P_{\sin,4,0}(x) = x - \frac{x^3}{6}.$$

There exists for every x a point c (that depends on x), such that

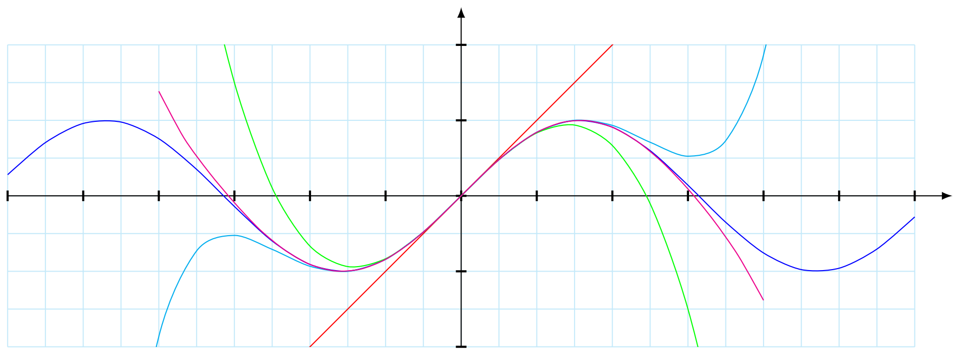
$$\sin x - \left(x - x^3/6\right) = \frac{\cos c}{5!} x^5.$$

For example,

$$|\sin(0.1) - (0.1 - 0.001/6)| \le \frac{0.00001}{120} < 10^{-7}.$$

The formula for the remainder gives us a practical tool for estimating the difference between a function and its approximation by a Taylor polynomial. ■

A comparison between the sine function and its first few Taylor polynomials about $x = 0$ is displayed below:



6.4 Taylor series

In the previous section we characterized the Taylor polynomial for n fixed and $x \to a$. Now we ask a totally different question: for fixed x, how does $P_{f,n,a}(x)$ behave as $n \to \infty$ (assuming, of course, that f is differentiable at a infinitely many times, without which the question is meaningless). There are different possible scenarios:

1. The limit of $P_{f,n,a}(x)$ as $n \to \infty$ does not exist.
2. The limit of $P_{f,n,a}(x)$ as $n \to \infty$ exists and equals $f(x)$.
3. The limit of $P_{f,n,a}(x)$ as $n \to \infty$ exists but does not equals $f(x)$.

If the second or third scenario occurs, we denote

$$\lim_{n\to\infty} P_{f,n,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This infinite sum is called a ***Taylor series*** (טור טיילור).

Example 6.11 Consider the sine function. We know that for every x and n,

$$|\sin x - P_{\sin,2n,0}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

For fixed x and $n \to \infty$ the right-hand side vanishes. Hence, the Taylor series of the sine function exists everywhere, and

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

■

Definition 6.3 Let f be infinitely many times differentiable at a. A number R, which is the largest number for which

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$$

for every $x \in (x-R, x+R)$ is called the ***radius of convergence*** of the Taylor series. A function that has an infinite radius of convergence is called ***real analytic*** (**אנליטית**).

Example 6.12 The classical example of a function that has a zero radius of convergence is the following:

$$f(x) = \begin{cases} e^{-1/x^2} & x \ne 0 \\ 0 & x = 0. \end{cases}$$

It can be shown that all the derivative of f vanish at zero. As a result, its Taylor polynomial of any degree about zero is identically zero,

$$P_{f,n,0}(x) = 0.$$

It follows that,

$$\lim_{n\to\infty} P_{f,n,0}(x) = 0 \ne f(x),$$

no matter has close x is to zero. Please note that this does not contradict Theorem 6.6 which states that for every n,

$$\lim_{x\to 0} \frac{f(x) - P_{f,n,0}(x)}{x^n} = 0.$$

■