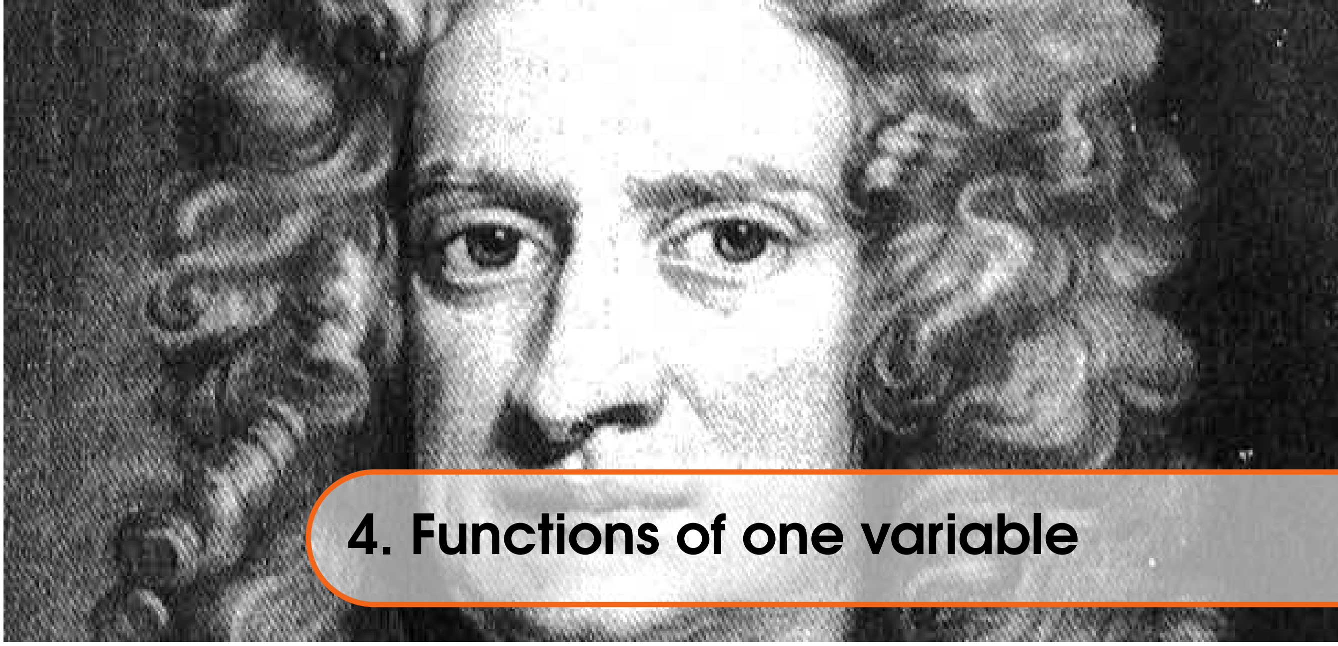


4. Functions of one variable

These lecture notes present my interpretation of Ruth Lawrence's lecture notes (in Hebrew)

[1]

In this chapter we are going to meet one of the most important concepts in mathematics: ***functions***.

4.1 Functions: definition and notations

What is a function? It is "something" that relates two sets, a first set called the ***domain*** (תחום) and a second set called the ***range*** (טווח). Crudely speaking, we can think of a function as a "machine" that when fed with an element of the domain returns an element of the range. Such a machine acts consistently: if the same element is inserted again the same element will come out—the input determines the output.

■ **Example 4.1** Consider the following two sets,

$$A = \{\text{Zambia, Uruguay, Kosovo}\}$$

and

$$B = \{\text{Pristina, Lusaka, Ramla, Paris, Montevideo}\}.$$

The function "Capital City" can be "fed" with any of the element of its domain, A, and "returns" an element of B. Feeding it with Zambia we get for output Lusaka. Feeding it with Uruguay we get for output Montevideo. Note that there are elements in B that are never returned (Paris and Ramla). ■

[1] Image of Sir Isaac Newton, 1642–1726

■ **Example 4.2** Take for domain the vector space $\mathbb{R}^2$. The function, which we denote by g, returns the size of the vector. The range is therefore the real numbers, $\mathbb{R}$. The action of the function is

$$g\begin{pmatrix} a \\ b \end{pmatrix} = \sqrt{a^2+b^2}.$$

■

In this chapter the domain and range will always be the set of real numbers, $\mathbb{R}$, or subsets of it. But keep in mind: functions can be defined for much more intricate domains and ranges.

Notations

To state that a function denoted by f has domain A and range B we write

$$f : A \to B$$

(in words: f acts from A to B, or f takes for input an element of A and returns for output an element of B). This notation tells us what are the elements that go into the function and what values may come out, but it gives no information as to what is the relation between the input and the output. The value returned by the function f if the input is a is denoted by

$$f(a).$$

Thus,

For all $a \in A$ the function f defines a corresponding $f(a) \in B$

To specify the values returned by f for *all* inputs, we have several alternatives. If the domain is a finite set we can prescribe all outputs using a ***table***. In certain cases, we can specify the action of a function using a ***formula***. Take for example a function f whose domain is $\mathbb{R}$ that returns the square of its input. We may denote the action of f either by

$$f(x) = x^2,$$

or by[2]

$$f : x \mapsto x^2$$

(in words: f maps x into x^2). Note that not all functions can be specified using formula!

■ **Example 4.3** Consider the function $h : \mathbb{N} \to \mathbb{N}$ (a function defined on the set of natural numbers),

$$h : n \mapsto \text{ the } n\text{-th decimal digit of } \pi.$$

[2]We can as well write $f : s \mapsto s^2$ or $f : \xi \mapsto \xi^2$.

Thus,

$$h(1) = 3 \qquad h(2) = 1 \qquad h(3) = 4 \qquad h(4) = 1 \qquad \text{etc.}$$

■

Comment 4.1 ***Important!*** A function $f : A \to B$ returns a value for *every* input in A, but not every element of B is required to be returned for some input in A.

Definition 4.1 Let $f : A \to B$. The set of values returned by f is called the ***image*** (תמונה) of f. It is denoted by

$$\text{Image}(f) = \{y \in B \mid \exists x \in A \ : \ f(x) = y\}.$$

(The set of all elements $y \in B$ for which there exists an element $x \in A$ for which $y = f(x)$.) When we know the image of a function we can limit its range to be its image.

■ **Example 4.4** You are acquainted with the following functions:

$$\begin{aligned} \sin &: \mathbb{R} \to [-1,1] \\ \cos &: \mathbb{R} \to [-1,1] \\ \exp &: \mathbb{R} \to (0,\infty) \\ \tan &: \mathbb{R} \setminus \{\pi/2 + k\pi \mid k \in \mathbb{Z}\} \to \mathbb{R}. \end{aligned}$$

In all these examples the range of the function coincides with its image. ■

■ **Example 4.5** A function you'll encounter later in the year (as a source of many counter examples) is known as the ***Dirichlet function***, $D : \mathbb{R} \to \{0,1\}$,

$$D : x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Note that we could have defined the range of this function to be $\mathbb{R}$, but once we know its image is the set $\{0,1\}$ we can limit to range to coincide with the image. ■

Definition 4.2 A function $f : A \to B$ is called ***one-to-one*** (חד חד ערכית) if $x \neq y$ implies that $f(x) \neq f(y)$, or equivalently if

$$f(x) = f(y)$$

implies that $x = y$. It is called ***onto*** (על) if its image and range coincide. That is,

$$\forall b \in B \quad \exists a \in A \ : \ f(a) = b.$$

4.2 The graph of a function

In mathematics you don't define new concepts as "machines". Thus, we need a "serious" definition of what a function is. A primitive notion to build upon is that of a set (קבוצה). It turns out that a function from A to B can be defined as a set. How? You actually know how. In high school you used to draw functions as ***graphs***. While you may think of a graph as a "drawing" (i.e., a piece of art), a graph is a well-defined entity. The graph of a function from A to B is a subset of the plane, and more precisely, a subset of the ***Cartesian product*** (מכפלה קרטזית)

$$A \times B = \{(a,b) \mid a \in A, b \in B\}.$$

■ **Example 4.6** Consider the function

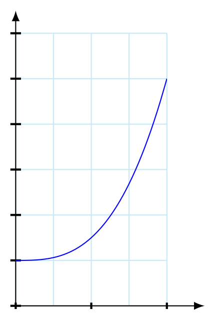
$$f : [0,2] \to \mathbb{R} \qquad f : t \mapsto 0.5t^3 + 1.$$

The Cartesian product of the domain and range of a is

$$[0,2] \times \mathbb{R} = \{(x,y) \mid 0 \le x \le 2\}.$$

The graph of f is a subset of the Cartesian product of its domain and range,

$$\text{Graph}(f) = \{(t, 0.5t^3 + 1) \mid t \in [0,2]\} \subset [0,2] \times \mathbb{R}.$$



■

In other words:

The graph of a function f is the set of all ordered pairs $(x, f(x))$.

A function $f : A \to B$ can be identified with a set $\text{Graph}(f) \subset A \times B$ provided that

$$\forall a \in A \quad \exists! b \in B : (a,b) \in \text{Graph}(f).$$

(The exclamation mark following the "exists" symbols means that there exists a *unique* b with the desired property.) The function f is one-to-one if

$$(a_1, b) \in \text{Graph}(f) \quad \text{and} \quad (a_2, b) \in \text{Graph}(f) \quad \text{implies} \quad a_1 = a_2.$$

The function f is onto if

$$\forall b \in B \quad \exists a \in A : (a,b) \in \text{Graph}(f).$$

4.3 Inverse functions

Suppose that $f : A \to B$ is both one-to-one and onto. This means that for every $b \in B$ there exists a unique $a \in A$ such that $f(a) = b$. Then we can define a function

$$g : B \to A$$

satisfying

$$g(b) = a \qquad \text{if and only if} \qquad f(a) = b.$$

The function g is called the ***inverse function*** (פונקציה הופכית) of f and it is denoted by f^{-1}.[3] By definition,

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y.$$

The relation between the graph of a function and the graph of its inverse is quite simple:

$$(a,b) \in \text{Graph}(f) \qquad \text{if and only if} \qquad (b,a) \in \text{Graph}(f^{-1}).$$

Geometrically, the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

■ **Example 4.7** The ***exponential function***

$$\exp : \mathbb{R} \to (0, \infty)$$

is one-to-one and onto. Therefore it has an inverse

$$\exp^{-1} : (0, \infty) \to \mathbb{R},$$

which is known as the ***natural logarithm***, and denoted ln. ■

■ **Example 4.8** The function $\sin : \mathbb{R} \to [-1, 1]$ is onto but not one-to-one (for example $\sin(0) = \sin(\pi)$). On the other hand, if we restrict the domain,

$$\sin : [-\pi/2, \pi/2] \to [-1, 1]$$

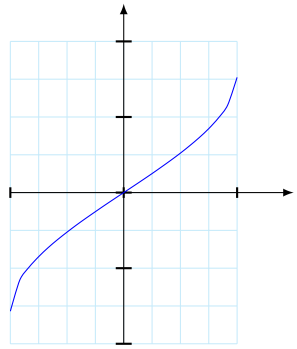
we obtain a function that is both one-to-one and onto. Therefore it has an inverse function, the ***arcsine***,

$$\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2],$$

whose graph we plot below.

---

[3] Do not confuse f^{-1} with $1/f$.



■

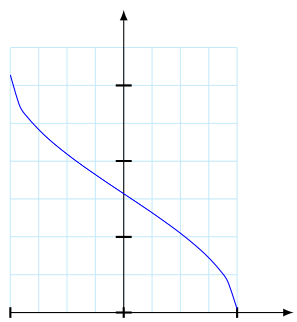
■ **Example 4.9** Similarly,

$$\cos : [0,\pi] \to [-1,1]$$

is one-to-one and onto. Therefore it has an inverse function, the ***arccosine***,

$$\cos^{-1} : [-1,1] \to [0,\pi],$$

whose graph we plot below:



■

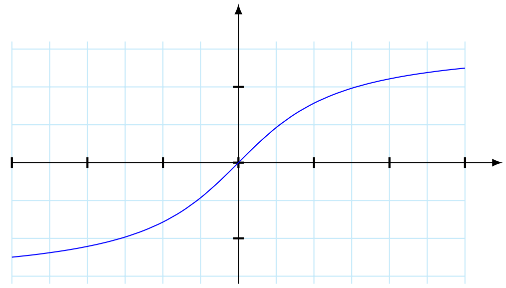
■ **Example 4.10** The function

$$\tan : (-\pi/2, \pi/2) \to \mathbb{R}$$

is one-to-one and onto. Therefore it has an inverse function, the ***arctangent***,

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2),$$

whose graph we plot below.



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4.4 The hyperbolic functions

In analogy with the trigonometric functions, we introduce the ***hyperbolic functions***,

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

And then,

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Recall the relation between the trigonometric function and complex exponentials,

$$\sin x = \frac{e^{\imath x} - e^{-\imath x}}{2} \qquad \text{and} \qquad \cos x = \frac{e^{\imath x} + e^{-\imath x}}{2},$$

hence we can relate the hyperbolic function to the trigonometric functions,

$$\sinh(\imath x) = \imath \sin x \qquad \text{and} \qquad \cosh(\imath x) = \cos x.$$

4.4.1 The origin of the name

What is hyperbolic about the hyperbolic functions? Consider first the unit circle,

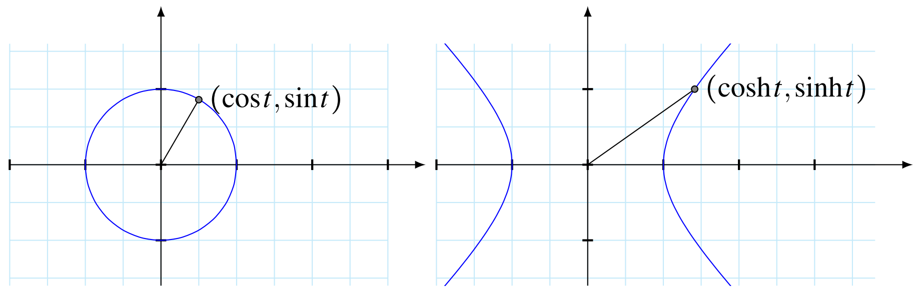
$$x^2 + y^2 = 1,$$

and a radius vector that rotates. If the radius sweeps an area of $t/2$ then the corresponding point on the circle has coordinates $(\cos t, \sin t)$.

Similarly, consider the unit hyperbola,

$$x^2 - y^2 = 1,$$

and a segment that points from the origin to a moving point on the hyperbola. If the segment sweeps an area of $t/2$ then the corresponding point on the hyperbola has coordinates $(\cosh t, \sinh t)$. By the way, there also exist ***elliptic functions***, but they are way more complicated.



4.4.2 Properties

The relations satisfied by the hyperbolic functions are strongly reminiscent of the relations satisfied by the trigonometric functions. All the properties of the hyperbolic functions follow directly from their definition. It is easy to see that,

$$\cosh^2 x - \sinh^2 x = 1.$$

Then,

$$\begin{aligned} 1 - \tanh^2 x &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \\ \coth^2 x - 1 &= \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x}. \end{aligned}$$

Further,

$$\begin{aligned} \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y, \end{aligned}$$

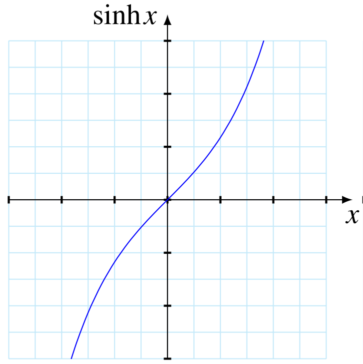
which can all be verified by direct substitution.

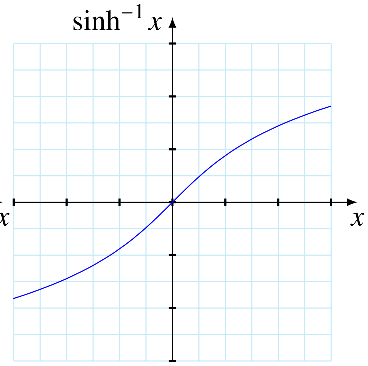
4.4.3 Inverse hyperbolic functions

The function sinh is anti-symmetric and monotonically increasing. It is a one-to-one and onto function $\mathbb{R} \to \mathbb{R}$. Thus, we can define its inverse,

$$\sinh^{-1} : \mathbb{R} \to \mathbb{R},$$

whose graph is displayed below:



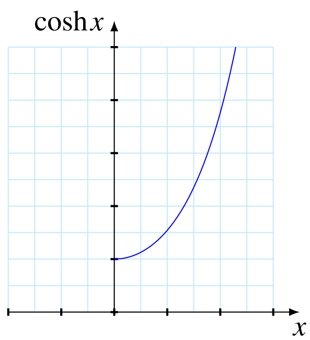


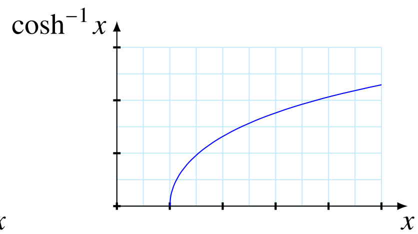
The function cosh on the other hand is symmetric and its image is $[1,\infty)$. A symmetric function cannot be one-to-one (because $f(x) = f(-x)$). The function cosh becomes one-to-one and onto if we restrict its domain as follows,

$$\cosh : [0,\infty) \to [1,\infty).$$

Then, we can define its inverse,

$$\cosh^{-1} : [1,\infty) \to [0,\infty).$$





The inverse hyperbolic functions can be expressed in terms of simpler functions. Since they are defined using the exponential, it shouldn't surprise you that their inverse involves the logarithm.

Start with,

$$y = \sinh x,$$

i.e.,

$$2y = e^x - e^{-x}.$$

Multiply both sides by e^x,

$$e^{2x} - 2ye^x - 1 = 0.$$

This is a quadratic equation for e^x, whose solution is

$$e^x = y \pm \sqrt{y^2 + 1}.$$

Since e^x is positive, only the plus sign remains, and we get that

$$x = \ln\left(y + \sqrt{y^2+1}\right),$$

which is the inverse relation sought.

Acting similarly with the hyperbolic cosine we obtain

$$\boxed{\begin{aligned} \sinh^{-1} u &= \ln\left(u + \sqrt{u^2+1}\right) \\ \cosh^{-1} u &= \ln\left(u + \sqrt{u^2-1}\right). \end{aligned}}$$

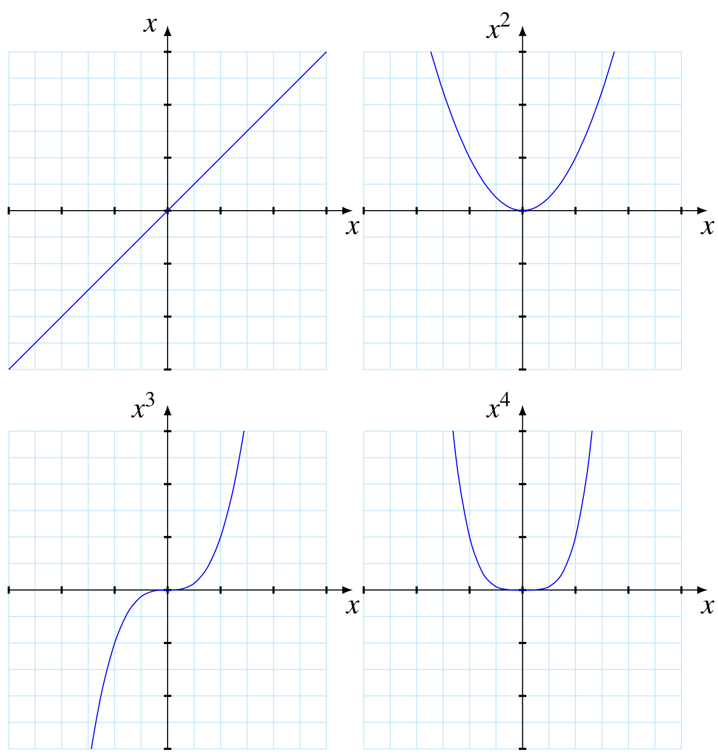
(Verify that the right-hand sides are indeed defined for any value in the domains of sinh and cosh.)

4.5 The power functions

For any $n \in \mathbb{N}$ define

$$f_n : \mathbb{R} \to \mathbb{R} \qquad f_n : s \mapsto s^n.$$

The graphs of the first few such functions are displayed below:



For n odd the function $f_n : t \mapsto t^n$ is anti-symmetric and monotonically increasing. We can define its inverse as a function $\mathbb{R} \to \mathbb{R}$,

$$\sqrt[n]{} : \mathbb{R} \to \mathbb{R}.$$

For n even the function $f_n : t \mapsto t^n$ is symmetric. To make it one-to-one we have to restrict its domain to the non-negative numbers. Then its inverse is a function,

$$\sqrt[n]{} : [0,\infty) \to [0,\infty).$$

The power functions satisfy the following algebraic properties,

$$x^m \cdot x^n = x^{m+n} \qquad (x^m)^n = x^{mn}.$$

These definitions call for the extension of the power function for n that is not a natural number. First, we extend it for any integer power,

$$x^{-n} = \frac{1}{x^n} \qquad \text{and} \qquad x^0 = 1.$$

Then, we identify unit fraction powers as roots,

$$x^{1/n} = \sqrt[n]{x}.$$

Next, for every rational number of the form p/q,

$$x^{p/q} = \left(\sqrt[q]{x}\right)^p.$$

(Note that we have to show that this definition does not depend on the representation of the rational number.) It only remains to define irrational powers. This is more delicate and requires a notion of limit, which is outside the scope of the present course.

4.6 The algebra of functions

Given two functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, we can form new functions by applying on f and g algebraic operations. We define

$$\begin{aligned} f+g : A \to \mathbb{R} \qquad & f+g : x \mapsto f(x)+g(x) \\ f\cdot g : A \to \mathbb{R} \qquad & f\cdot g : x \mapsto f(x)\cdot g(x) \\ af : A \to \mathbb{R} \qquad & af : x \mapsto a f(x). \end{aligned}$$

The set of functions $A \to \mathbb{R}$ form a vector space, and in addition have a product operation (both addition and multiplication make this set of functions an algebra).

There exists an additional way of creating new functions from existing ones. Suppose we have two functions,

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

We can create a new function, which given x will first generates $f(x)$, which in turn is fed to the function g, yielding $g(f(x))$. This operation is called ***composition*** (הרכבה), and is denoted,

$$g\circ f : A \to C \qquad g\circ f : s \mapsto g(f(s)).$$

Example 4.11 Let

$$f:\mathbb{R}\to\mathbb{R} \qquad f:x\to x^2$$
$$g:\mathbb{R}\to\mathbb{R} \qquad g:z\to \sin z,$$

then

$$(g\circ f)(x) = \sin(x^2) \qquad \text{and} \qquad (f\circ g)(x) = (\sin x)^2.$$

This example shows that ***composition in non-commutative***. ■

Example 4.12 Let

$$f:(0,\infty)\to\mathbb{R} \qquad f:x\to \ln x$$
$$g:\mathbb{R}\to\mathbb{R} \qquad g:z\to z+1,$$

then

$$g\circ f:(0,\infty)\to\mathbb{R} \qquad (g\circ f)(x) = 1+\ln x$$
$$f\circ g:(-1,\infty)\to\mathbb{R} \qquad (f\circ g)(x) = \ln(x+1).$$

■

4.7 Continuity

4.7.1 Definition

The notion of continuity is a central concept in the theory of functions. You will encounter it in depth in the next semester. Here we will make do with an intuitive interpretation of this concept.

> Let $f:A\to\mathbb{R}$ and $a\in A$ an ***internal point***. f is said to be ***continuous*** at a if we can make $f(x)$ deviates from $f(a)$ by an arbitrarily small amount by limiting x to a small enough neighborhood of a. We say that f is continuous everywhere if it is continuous at all points in its domain.

Example 4.13 Take the function $f:x\mapsto x^2$. Is it continuous at $a=3$? If we believe so, we have to show, for example, that there exists a small enough neighborhood of 3, such that for all x in that neighborhood,

$$|f(x)-9| < 0.000001.$$

Such a neighborhood does exist, for example, the segment $(2.9999999, 3.0000001)$. But that's not enough: we also have to show that there exists a small enough neighborhood of 3, such that for all x in that neighborhood,

$$|f(x)-9| < 10^{-31}.$$

Here again such a neighborhood exists. Since we can find such a neighborhood for *all* deviations of $f(x)$ from 9, no matter how small the deviation is, f is continuous at 3. ■

■ **Example 4.14** All the elementary functions, sin, cos, tan, exp, log, sinh, etc. are continuous everywhere they are defined. So is also the function

$$x \mapsto |x|.$$

■

■ **Example 4.15** An important function which is continuous everywhere except at a single point is the ***heaviside function***,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1. \end{cases}$$

■

■ **Example 4.16** A pathological function which is nowhere continuous is the ***Dirichlet function***,

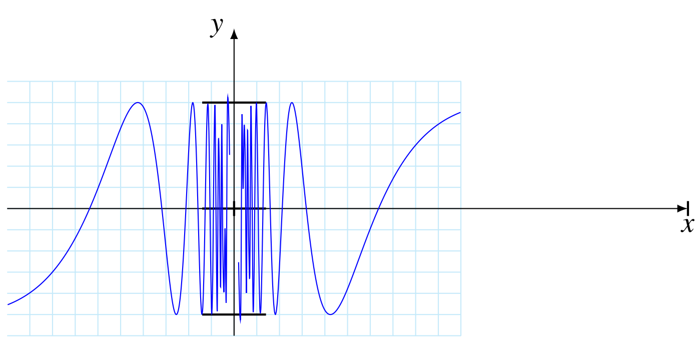
$$D(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

■

■ **Example 4.17** The function

$$f(x) = \sin\left(\frac{1}{x}\right)$$

is continuous everywhere except at the origin, and we can't “fix it", no matter how we define its value at zero.



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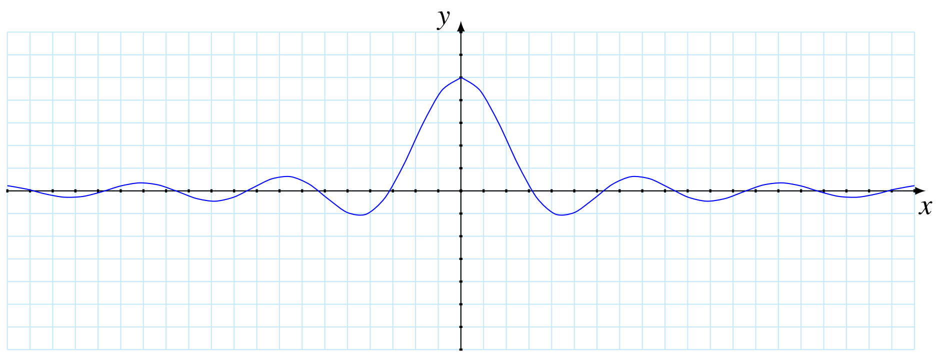
■ **Example 4.18** The function

$$f(x) = \frac{\sin x}{x}$$

is continuous everywhere except at the origin, but we can “fix it" into an everywhere continuous function by defining

$$f(x)=\begin{cases}\frac{\sin x}{x} & x\neq 0\\ 1 & x=0.\end{cases}$$



■

4.7.2 Properties of continuous functions

Continuous functions satisfy properties which you will prove in 80177. For example:

1. If f and g are continuous at a so is $f+g$.
2. If f and g are continuous at a so is $f\cdot g$.
3. If f is continuous at a and $f(a)\neq 0$ then $1/f$ is continuous at a.
4. If f is continuous at a and g is continuous at $f(a)$ then $g\circ f$ is continuous at a.

Theorem 4.1 — Mean value theorem. If $f:[a,b]\to\mathbb{R}$ is continuous and α is some number between $f(a)$ and $f(b)$ then there exists an $x\in[a,b]$ such that

$$f(x)=\alpha.$$

■ **Example 4.19** Consider the function

$$f(x)=\sin x-\frac{x}{2}.$$

Using a calculator you find that

$$f(1.8)=0.0738\ldots \qquad \text{and} \qquad f(1.9)=-0.0036\ldots$$

Since f is continuous in $[1.8, 1.9]$ there is a point x in this interval for which

$$\sin x = \frac{x}{2}.$$

■

Definition 4.3 Consider a function $f : A \to \mathbb{R}$. A point $a \in A$ is said to be a ***maximum point*** of f if

$$\forall x \in A \quad f(x) \le f(a).$$

A point b is said to be a ***minimum point*** of f if

$$\forall x \in A \quad f(x) \ge f(b).$$

We denote the minimum and maximum values assumed by f in the set A by

$$\min_{x \in A} f(x) \qquad \text{and} \qquad \max_{x \in A} f(x).$$

It is important to mention that a function does not necessarily have minima and maxima, however:

Theorem 4.2 — Weierstrass. If $f : [a, b] \to \mathbb{R}$ is continuous then there exist points x_1 and x_2 such that

$$\forall x \in [a, b] \qquad f(x_1) \le f(x) \le f(x_2).$$

Note that the last statement is not trivial. Consider the functions:

$$f : (0,1) \to \mathbb{R} \qquad f : x \mapsto x^2$$

$$g : \mathbb{R} \to \mathbb{R} \qquad g : x \mapsto \begin{cases} 1/x & x \ne 0 \\ 2 & x = 0 \end{cases}$$

Both don't have points of minima and maxima.

Comment 4.2 At this stage, the notion of minimum and maximum makes no mention of a vanishing derivative.