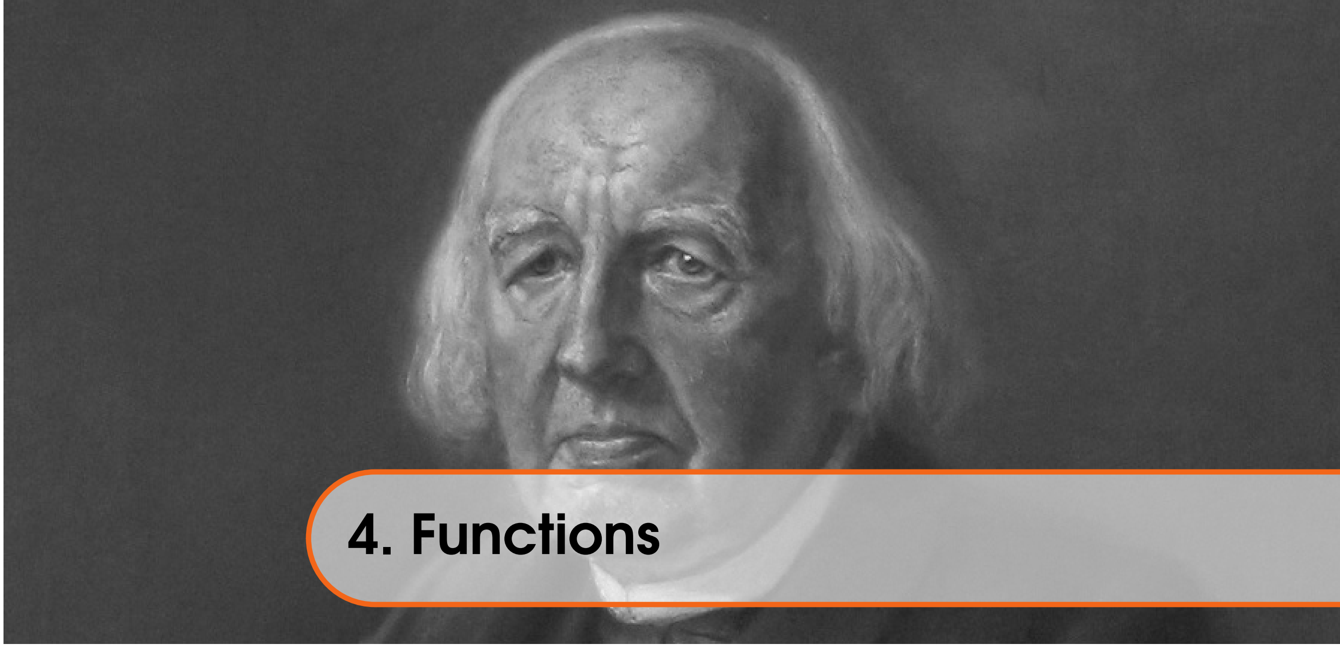


4. Functions

(This chapter should take 18 hours.)

4.1 Basic definitions

What is a function? There is a formal way of defining functions, but at this point we will deliberately be a little less formal, and introduce a function as a “machine", which when provided with a number, returns a number—only one, and always the same for the same input.

A function is characterized by three components:

1. A ***domain*** (תחום): a subset of $\mathbb{R}$. The numbers which may be “fed into the machine".
2. A ***range*** (טווח): another subset of $\mathbb{R}$. Numbers that may be “emitted by the machine". We do not exclude the possibility that some of these numbers may never be emitted. We only require that every number returned by the function belongs to its range.
3. A ***transformation rule*** (כלל העתקה). The crucial point is that to every number in its domain corresponds one and only one number in its range (חד ערכיות).

We denote functions by letters, like we do for real numbers (and for any other mathematical entity). The most common notation for a function is the letter f, but of course, there is nothing special about this letter, except for being the fist letter of the word “function". If $A \subseteq \mathbb{R}$ is the domain of f, and $B \subseteq \mathbb{R}$ is its range, we write $f : A \to B$ (f maps the set A into the set B). The transformation rule specifies what number in B is assigned by the function for each number $x \in A$. We denote the assignment by $f(x)$ (the function f evaluated at x). That the assignment rule is “assign $f(x)$ to x" is denoted by $f : x \mapsto f(x)$ (pronounced “f maps x to $f(x)$").

Comment 4.1 Programmers: think of $f : A \to B$ as defining the "type" or "syntax" of the function, and of $f : x \mapsto f(x)$ as defining the "action" of the function.

Definition 4.1 Let $f : A \to B$ be a function. Its ***image*** (תמונה) is the subset of B of numbers that are actually assigned by the function. That is,

$$\text{Image}\, f = \{y \in B : \exists x \in A,\ f(x) = y\}.$$

The function f is said to be ***onto*** B (על) if B is its image. f is said to be ***one-to-one*** (חד חד ערכית) if to each number in its image corresponds a unique number in its domain, i.e.,

$$(\forall y \in \text{Image}\, f)(\exists! x \in A)(f(x) = y).$$

■ **Examples 4.1**

1. A function that assigns to every real number its square. If we denote the function by f, then

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{and} \qquad f : x \mapsto x^2.$$

We may also write $f(x) = x^2$.
We should not say, however, that "the function f is x^2". In particular, we may use any letter other than x as an argument for f. Thus, the functions $f : \mathbb{R} \to \mathbb{R}$, defined by the transformation rules $f(x) = x^2$, $f(t) = t^2$, $f(\alpha) = \alpha^2$ and $f(\xi) = \xi^2$ are identical.
The function f returns only non-negative numbers, and

$$\text{Image}\, f = [0, \infty).$$

There is however nothing wrong with the definition of the range as the whole of $\mathbb{R}$. We could limit the range to be the set $[0, \infty)$, but not to the set $[1, 2]$.

2. A function that assigns to every $w \neq \pm 1$ the number $(w^3 + 3w + 5)/(w^2 - 1)$. If we denote this function by g, then

$$g : \mathbb{R} \setminus \{\pm 1\} \to \mathbb{R} \qquad \text{and} \qquad g : w \mapsto \frac{w^3 + 3w + 5}{w^2 - 1}.$$

3. A function that assigns to every $-17 \le x \le \pi/3$ its square. This function differs from the function in the first example because the two functions do not have the same domain (different "syntax" but same "routine").
4. A function that assigns to every real number the value zero if it is irrational and one if it is rational. This function is known as the ***Dirichlet function*** (named after Peter Gustav Lejeune Dirichlet, 1805–1859). Thus,

$$f : \mathbb{R} \to \{0, 1\} \qquad f : x \mapsto \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

(The Dirichlet function is going to be the course's favorite to display counter examples.)

5. A function defined on the domain

$$A = \{2, 17, \pi^2/17, 36/\pi\} \cup \{a + b\sqrt{2} \,:\, a, b \in \mathbb{Q}\},$$

such that

$$x \mapsto \begin{cases} 5 & x = 2 \\ 36/\pi & x = 17 \\ 28 & x = \pi^2/17 \text{ or } 36/\pi \\ 16 & \text{otherwise.} \end{cases}$$

The range may be taken to be $\mathbb{R}$, but the image is $\{6, 16, 28, 36/\pi\}$.

6. A function defined on $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers), assigning to x the number of 7's in its decimal expansion, if this number is finite. If this number is infinite, then it returns $-\pi$. This example differs from the previous ones in that we do not have an assignment rule in closed form (how can we compute $f(x)$?). Nevertheless it provides a legal assignment rule.
   Note that we limited the domain of the function to the irrational numbers because rational numbers may have a non-unique decimal representation, e.g.,

$$0.7 = 0.6999999\ldots.$$

7. For every $n \in \mathbb{N}$ we may define the n-th power function $f_n : \mathbb{R} \to \mathbb{R}$, by $f_n : x \mapsto x^n$. Here again, we will avoid referring to "the function x^n". The function $f_1 : x \mapsto x$ is known as the ***identity function*** (**פונקציית הזהות**), often denoted by Id, namely

$$\text{Id} : \mathbb{R} \to \mathbb{R}, \qquad \text{Id} : x \mapsto x.$$

8. There are many functions that you all know since high school, such as ***polynomials***, ***rational functions***, the ***sine***, the ***cosine***, the ***exponential***, the ***logarithm*** and many more. These function as well as functions generated by algebraic combinations thereof are called ***elementary functions***. . In this course we assume that the definition and properties of elementary functions (along with their domains and ranges) are known.

■

4.2 Sums, products and compositions of functions

Given several functions, they can be combined together to form new functions. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be given functions, and $a, b \in \mathbb{R}$. We may define a new function,

$$af + bg : A \cap B \to \mathbb{R}, \qquad af + bg : x \mapsto af(x) + bg(x).$$

(This is not trivial. We are adding “machines", not numbers.) Functions $\mathbb{R} \to \mathbb{R}$ form a vector space over the field of real numbers, whose zero element is the zero function, $x \mapsto 0$; given a function f, its inverse is $-f = (-1)f$.

Moreover, functions form an ***algebra***. We may define the product of two functions,

$$f \cdot g : A \cap B \to \mathbb{R}, \qquad f \cdot g : x \mapsto f(x)g(x),$$

as well as their quotient,

$$f/g : A \cap \{z \in B \ : \ g(z) \neq 0\} \to \mathbb{R}, \qquad f/g : x \mapsto f(x)/g(x).$$

A third operation that combines two functions is ***composition*** (הרכבה). Let $f : A \to B$ and $g : B \to C$. We define

$$g \circ f : A \to C, \qquad g \circ f : x \mapsto g(f(x)).$$

For example, if f is the sine function and g is the square function, then

$$g \circ f : \xi \mapsto \sin^2 \xi \quad \text{and} \quad f \circ g : \zeta \mapsto \sin \zeta^2,$$

i.e., composition is non-commutative. On the other hand, composition is associative, namely,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Note that for every function f,

$$\mathrm{Id} \circ f = f \circ \mathrm{Id} = f,$$

so that the identity function is the neutral element with respect to function composition. This should not be confused with the fact that $x \mapsto 1$ is the neutral element with respect to function multiplication.

■ **Example 4.1** Consider the function f that assigns the rule

$$f : x \mapsto \frac{x + x^2 + x \sin x^2}{x \sin x + x \sin^2 x}.$$

This function can be written as

$$f = \frac{\mathrm{Id} + \mathrm{Id} \cdot \mathrm{Id} + \mathrm{Id} \cdot \sin \circ (\mathrm{Id} \cdot \mathrm{Id})}{\mathrm{Id} \cdot \sin + \mathrm{Id} \cdot \sin \cdot \sin}.$$

■

4.3 Formal definition a function: graphs

As you know, we can associate with every function a ***graph***. What is a graph? A drawing? A graph has to be thought of as a ***subset of the plane***. For a function $f : A \to B$, we define the graph of f to be the set

$$\mathrm{Graph}\, f = \{(x, y) \ : \ x \in A, y = f(x)\}.$$

This set is a subset of the ***Cartesian product***:

$$A \times B = \{(x,y) \ : \ x \in A, y = B\}.$$

Thus, we can also write,

$$\text{Graph}\, f = \{(x,y) \in A \times B \ : \ y = f(x)\}.$$

■ **Example 4.2** The graph of the function

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2$$

is

$$\{(x,x^2) \ : \ x \in \mathbb{R}\}.$$

■

■ **Example 4.3** The graph of the function

$$f : (0,1] \to \mathbb{R} \qquad f : t \mapsto 1/t$$

is

$$\{(m,1/m) \ : \ 0 < m \le 1\}.$$

■

The defining property of a function is that it is uniquely defined, i.e.,

$$(x,y) \in \text{Graph}\, f \quad \text{and} \quad (x,z) \in \text{Graph}\, f \quad \text{implies} \quad y = z,$$

or written differently,

$$(\forall x \in A)(\exists! y \in B)((x,y) \in \text{Graph}\, f).$$

The function is one-to-one if

$$(x,y) \in \text{Graph}\, f \quad \text{and} \quad (w,y) \in \text{Graph}\, f \quad \text{implies} \quad x = w,$$

or written differently,

$$(\forall y \in \text{Image}\, f)(\exists! x \in A)((x,y) \in \text{Graph}\, f).$$

The function is onto B if

$$(\forall y \in B)(\exists x \in A)((x,y) \in \text{Graph}\, f).$$

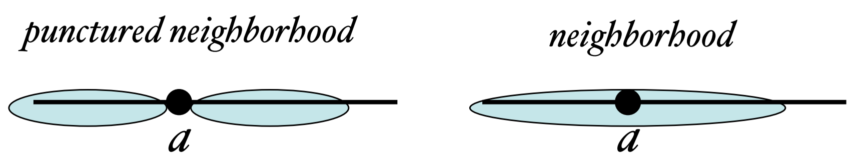
The definition we gave to a function as an assignment rule is, strictly speaking, not a formal one. The standard way to define a function is via its graph. A function *is* a graph—a subset of the Cartesian product space $A \times B$.
Given two sets A and B, any subset $C \subset A \times B$ satisfying

$$(\forall a \in A)(\exists! b \in B) : ((a,b) \in C),$$

defines uniquely a function $A \to B$. The value returned by the function for x is the unique y for which $(x,y) \in C$. Having said that, you should stick to thinking of functions as machines.

4.4 Limits

Definition 4.2 Let $x \in \mathbb{R}$. A ***neighborhood*** of x (סביבה) is an open segment (a,b) that contains the point x (note that since the segment is open, x cannot be a boundary point). A ***punctured neighborhood*** of x (סביבה מנוקבת) is a set $(a,b) \setminus \{x\}$ where $a < x < b$.



Definition 4.3 Let $A \subset \mathbb{R}$. A point $a \in A$ is called an ***interior point*** of A (נקודה פנימית) if it has a neighborhood contained in A.

Notation 4.1 We will mostly deal with ***symmetric*** neighborhoods (whether punctured or not), i.e., neighborhoods of a of the form

$$\{x : |x-a| < \delta\}$$

for some $\delta > 0$. We will introduce the following notations for neighborhoods,

$$B(a,\delta) = (a-\delta, a+\delta)$$
$$B^\circ(a,\delta) = \{x : 0 < |x-a| < \delta\}.$$

We will also define ***one-sided neighborhoods***,

$$B_+(a,\delta) = [a, a+\delta)$$
$$B_+^\circ(a,\delta) = (a, a+\delta)$$
$$B_-(a,\delta) = (a-\delta, a]$$
$$B_-^\circ(a,\delta) = (a-\delta, a).$$

■ **Example 4.4** For $A = [0,1]$, the $1/2$ is an interior point, but 0 is not an interior point. ■

■ **Example 4.5** The set $\mathbb{Q} \subset \mathbb{R}$ has no interior points, and neither does its complement, $\mathbb{R} \setminus \mathbb{Q}$. ■

Lemma 4.1 Let $a \in \mathbb{R}$. Then,

1. The intersection of two neighborhoods of a is a neighborhood of a.
2. The intersection of two punctured neighborhoods of a is a punctured neighborhood of a.
3. Every neighborhood of a contains a symmetric neighborhood $B(a,\delta)$.

4. Every punctured neighborhood of a contains a symmetric punctured neighborhood $B^\circ(a,\delta)$.

Proof. Immediate. ■

In this section we define the concept of *the limit of a function at a point*. Informally, we say that:

> The limit of a function f at a point a is ℓ, if we can make $f(x)$ assign values as close to ℓ as we wish, by making its argument x sufficiently close to a (excluding the value a itself).

Note that the function does not need to be equal to ℓ at a; in fact, it does not even need to be defined at a.

■ **Example 4.6** Consider the function

$$f:\mathbb{R}\to\mathbb{R} \qquad f:x\mapsto 3x.$$

We claim that the limit of f at 5 is 15. This means that we can make $f(x)$ be as close to 15 as we wish, by making x sufficiently close to 5, with 5 itself being excluded. Suppose you want $f(x)$ to differ from 15 by less than $1/100$. This means that you want

$$15-\frac{1}{100}<f(x)=3x<15+\frac{1}{100}.$$

This requirement is guaranteed if

$$5-\frac{1}{300}<x<5+\frac{1}{300}.$$

Thus, if we take x to differ from 5 by less than $1/300$ (but more than zero!), then we are guaranteed to have $f(x)$ within the desired range. Since we can repeat this construction for any number other than $1/100$, we conclude that the limit of f at 5 is 15.

We can be more precise. Suppose you want $f(x)$ to differ from 15 by less than ε, for some $\varepsilon>0$ of your choice. In other words, you want

$$|f(x)-15|=|3x-15|<\varepsilon.$$

This is guaranteed if $|x-5|<\varepsilon/3$, thus given $\varepsilon>0$, choosing x within a symmetric punctured neighborhood of 5 of radius $\varepsilon/3$ guarantees that $f(x)$ is within a distance of ε from 15.

To express the fact that the limit of f at 5 is 15 we write,

$$\lim_5 f = 15.$$

The more common notation is

$$\lim_{x\to 5} f(x) = 15.$$

Note however, that we could have as well written

$$\lim_{\aleph\to 5} f(\aleph) = 15 \qquad \text{or} \qquad \lim_{\xi\to 5} f(\xi) = 15.$$

■

This example insinuates what would be a formal definition of the limit of a function at a point:

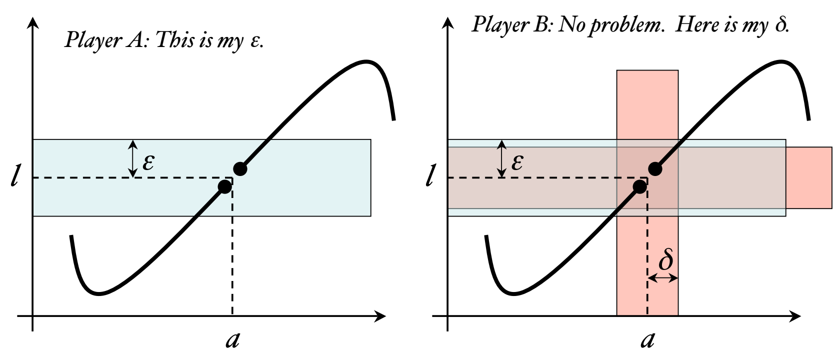
Definition 4.4 Let $f : A \to B$ with $a \in A$ an interior point. We say that the limit of f at a is ℓ, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$f(x) \in B(\ell, \varepsilon) \qquad \text{for all} \qquad x \in B^\circ(a, \delta).$$

In formal notation,

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in B^\circ(a, \delta))(f(x) \in B(\ell, \varepsilon)).$$

(The game is “you give me ε and I give you δ in return".)



Since Player B can find a δ for every choice of ε made by Player A, it follows that the limit of f at a is l.

This definition of the limit of a function at a point is known as the “epsilon-delta version". An alternative definition in terms of neighborhoods would be:

Proposition 4.2 — Alternative characterization of the limit. Let $f : A \to B$ with $a \in A$ an interior point. Then,

$$\lim_a f = \ell$$

if and only if for every neighborhood V of ℓ there exists a punctured neighborhood U of a, such that $f(x) \in V$ for all $x \in U$.

Proof. Suppose first that the limit of f at a is ℓ. Let V be a neighborhood of ℓ. It contains a symmetric neighborhood of ℓ,

$$B(\ell,\varepsilon) \subset V.$$

By the definition of the limit, there exists a symmetric punctured neighborhood $U = B^\circ(a,\delta)$ of a, such that $f(x) \in B(\ell,\varepsilon)$ (and in particular, $f(x) \in V$) for every $x \in U$.

Conversely, suppose that for every neighborhood V of ℓ there exists a punctured neighborhood U of a, such that $f(x) \in V$ for all $x \in U$. Let $\varepsilon > 0$ be given. Then, there exists a punctured neighborhood U of a, such that $f(x) \in B(\ell,\varepsilon)$ for all $x \in U$. Also, there exists a $\delta > 0$ such that $B^\circ(a,\delta) \subset U$. Then, $f(x) \in B(\ell,\varepsilon)$ for all $x \in B^\circ(a,\delta)$, which completes the proof. ■

Comment 4.2 For the limit of a function f at a point a to exist, f must be defined in some punctured neighborhood of a. (This is a necessary condition—not a sufficient condition.)

■ **Example 4.7** Consider the square function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto x^2$. We will show that

$$\lim_3 f = 9.$$

By definition, we need to show that for every $\varepsilon > 0$ we can find a $\delta > 0$, such that

$$f(x) \in B(9,\varepsilon) \qquad \text{whenever} \qquad x \in B^\circ(3,\delta).$$

Thus, the game it to respond with an appropriate δ for every ε.

To find the appropriate δ, we examine the condition that needs to be satisfied:

$$|f(x) - 9| = |x-3|\,|x+3| < \varepsilon.$$

If we impose that $|x-3| < \delta$ then

$$|f(x) - 9| < |x+3|\,\delta.$$

If the $|x+3|$ wasn't there we would have set $\delta = \varepsilon$ and we would be done. Instead, we note that by the triangle inequality,

$$|x+3| = |x-3+6| \le |x-3| + 6,$$

so that for all $|x-3| < \delta$,

$$|f(x) - 9| \le |x-3|\,(|x-3| + 6) < (6+\delta)\delta.$$

Now recall: given ε we have the freedom to choose $\delta > 0$ such to make the right-hand side less or equal than ε. We may freely require for example that $\delta \le 1$, in which case

$$|f(x) - 9| < (6+\delta)\delta \le 7\delta.$$

If we furthermore take $\delta \le \varepsilon/7$, then we reach the desired goal.
To summarize, given $\varepsilon > 0$ we take $\delta = \min(1, \varepsilon/7)$, in which case $x \in B^\circ(3,\delta)$,

$$|f(x) - 9| = |x-3||x+3| \le |x-3|(|x-3|+6) < (6+\delta)\delta \le (6+1)\frac{\varepsilon}{7} = \varepsilon.$$

This proves (by definition) that $\lim_3 f = 9$. Equivalently, we may say that for all $\varepsilon > 0$,

$$f(x) \in B(9,\varepsilon) \qquad \text{for all} \qquad x \in B^\circ(3,1) \cap B^\circ(3,\varepsilon/7).$$

In the similar way, we can show that

$$\lim_a f = a^2$$

for all $a \in \mathbb{R}$. [do it!] ■

■ **Example 4.8** The next example concerns the function $f : (0,\infty) \to \mathbb{R}$ that assigns to every number its multiplicative inverse,

$$f : x \mapsto 1/x.$$

We are going to show that for every $a > 0$,

$$\lim_a f = \frac{1}{a}.$$

First fix a; it is no longer a variable. Let $\varepsilon > 0$ be given. We first observe that

$$\left| f(x) - \frac{1}{a} \right| = \left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|x-a|}{ax}.$$

We need to be careful that the domain of x does not include zero. We start by requiring that $|x-a| < a/2$ (or equivalently, that $x \in B^\circ(a, a/2)$), which at once implies that $x > a/2$, hence

$$\left| f(x) - \frac{1}{a} \right| < \frac{2|x-a|}{a^2}.$$

If we further require that $|x-a| < a^2\varepsilon/2$, then $|f(x) - 1/a| < \varepsilon$. To conclude,

$$\left| f(x) - \frac{1}{a} \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-a| < \delta,$$

for $\delta = \min(a/2, a^2\varepsilon/2)$. Equivalently, for all $\varepsilon > 0$,

$$f(x) \in B(1/a, \varepsilon) \qquad \text{for all} \qquad x \in B^\circ(a, a/2) \cap B^\circ(a, a^2\varepsilon^2/2).$$

■

■ **Example 4.9** Consider next the Dirichlet function, $f:\mathbb{R}\to\mathbb{R}$,

$$f : x \mapsto \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational.} \end{cases}$$

We will show that f does not have a limit at zero. To show that for all ℓ,

$$\lim_0 f \neq \ell,$$

we need to show that

$$\neg\left((\exists \ell \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in B^\circ(0,\delta))(f(x) \in B(\ell,\varepsilon))\right),$$

i.e.,

$$(\forall \ell \in \mathbb{R})(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in B^\circ(0,\delta))(|f(x)-\ell| \geq \varepsilon).$$

So let $\ell \in \mathbb{R}$ be given. We will take $\varepsilon = 1/4$. By the density of the rationals in the reals, for all $\delta > 0$, we can find $x_1, x_2 \in B^\circ(0,\delta)$, such that

$$f(x_1) = 1 \qquad \text{and} \qquad f(x_2) = 0.$$

Now either

$$|f(x_1)-\ell| \geq \frac{1}{4} \qquad \text{or} \qquad |f(x_2)-\ell| \geq \frac{1}{4},$$

for otherwise, by the triangle inequality

$$1 = |f(x_1) - f(x_2)| = |(f(x_1)-\ell) - (f(x_2)-\ell)| \leq |f(x_1)-\ell| + |f(x_2)-\ell| \leq \frac{1}{2},$$

which is a contradiction. ■

Comment 4.3 It is important to stress what is the negation that the limit f at a is ℓ:

> There *exists* an $\varepsilon > 0$, such that *for all* $\delta > 0$, there *exists* an x, which satisfies $0 < |x-a| < \delta$, but not $|f(x)-\ell| < \varepsilon$.

■ **Example 4.10** Consider, in contrast, the function $f:\mathbb{R}\to\mathbb{R}$,

$$f : x \mapsto \begin{cases} 0 & x \text{ is irrational} \\ x & x \text{ is rational.} \end{cases}$$

Here we show that

$$\lim_0 f = 0.$$

Indeed, let $\varepsilon > 0$ be given, then

$$f(x) \in B(0,\varepsilon) \qquad \text{for all} \qquad x \in B^\circ(0,\varepsilon).$$

■

Having a formal definition of a limit, and having seen a number of example, we are in measure to prove general theorems about limits. The first theorem states that a limit, if it exists, is unique.

Theorem 4.3 — Uniqueness of the limit. A function $f : A \to B$ has at most one limit at any interior point $a \in A$.

Proof. Suppose, by contradiction, that

$$\lim_a f = \ell \qquad \text{and} \qquad \lim_a f = m,$$

with $\ell \neq m$. Then there exist disjoint neighborhoods V_1 of ℓ and V_2 of m. By the definition of the limit, there exists a punctured neighborhood U_1 of a such that

$$f(x) \in V_1 \qquad \forall x \in U_1,$$

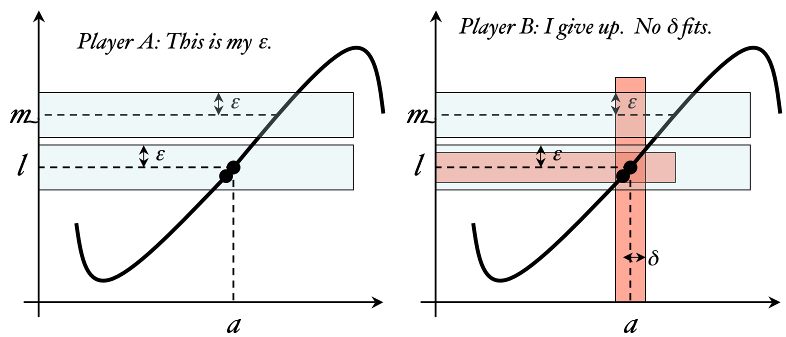
and there exists a punctured neighborhood U_2 of a such that

$$f(x) \in V_2 \qquad \forall x \in U_2.$$

Then, $U = U_1 \cap U_2$ is a punctured neighborhood of a, and for all $x \in U$,

$$f(x) \in V_1 \qquad \text{and} \qquad f(x) \in V_2,$$

which is impossible since $V_1 \cap V_2 = \varnothing$.



Since Player B cannot find a δ for a particular choice of ε made by Player A, it follows that the limit of f at a cannot be both l and m.

■

4.5 The Heine characterization of the limit

The epsilon-delta characterization of the limit of a function at a point is sometimes cumbersome to work with. The following theorem provides us with an alternative definition:

Theorem 4.4 — Heine's characterization of the limit. Let the function f be defined in a punctured neighborhood U of a. Then,

$$\lim_a f = \ell$$

if and only if,

$$\lim_{n\to\infty} f(x_n) = \ell$$

for every sequence (x_n) in U converging to a.

Proof. Once again, there are two statements to prove: suppose first that

$$\lim_a f = \ell,$$

and let (x_n) be a sequence in U converging to a. By the definition of the limit of f at a, given $\varepsilon > 0$ there exists a punctured neighborhood $U_1 \subset U$ of a, such that

$$f(x) \in B(\ell, \varepsilon) \qquad \text{for all} \qquad x \in U_1.$$

By the definition of the limit of (x_n), there exists an $N \in \mathbb{N}$ such that

$$x_n \in U_1 \qquad \text{for all} \qquad n > N.$$

Thus, for all $n > N$,

$$f(x_n) \in B(\ell, \varepsilon),$$

which proves that

$$\lim_{n\to\infty} f(x_n) = \ell$$

Conversely, suppose that

$$\lim_{n\to\infty} f(x_n) = \ell$$

for every sequence (x_n) in U converging to a. We will prove that the limit of f at a is ℓ. Suppose this was not the case. Then,

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in B^\circ(a, \delta))(f(x) \notin B(\ell, \varepsilon)).$$

Thus, for every $n \in \mathbb{N}$ set $\delta_n = 1/n$. Then, there exists an $x_n \in B^\circ(a, \delta_n)$ for which $f(x_n) \notin B(\ell, \varepsilon)$.
Consider the sequence (x_n). Since

$$|x_n - a| < \frac{1}{n},$$

it follows that

$$\lim_{n\to\infty} x_n = a.$$

But since for all n, $f(x_n) \notin B(\ell, \varepsilon)$, we conclude that $f(x_n)$ does not converge to ℓ, which contradicts the assumption. ∎

Comment 4.4 Note that in order to prove that the limit of f at a is not ℓ, it suffices to find one sequence (x_n) converging to a (but never equal to a) for which $f(x_n)$ does not converge to ℓ. In particular, if there exist two sequences (x_n) and (y_n), both converging to a, satisfying that $f(x_n)$ and $f(y_n)$ converge to different limits, then f does not have a limit at a.

■ **Example 4.11** Using Heine's characterization it is easy to prove that Dirichlet's function has no limit at any point. Note however that if we take

$$x_n = 1/n,$$

then (x_n) converges to zero and $(f(x_n))$ converges to zero as well. Yes, it is not true that the limit of the Dirichlet function at zero is zero. ■

■ **Example 4.12** The same idea proves that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$,

$$f(x) = \sin \frac{1}{x}$$

doesn't have a limit at zero, because take

$$x_n = \frac{1}{(2n+1/2)\pi} \qquad \text{and} \qquad y_n = \frac{1}{(2n-1/2)\pi}.$$

Then,

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} y_n = 0,$$

however

$$\lim_{n\to\infty} f(x_n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = -1.$$

■

Corollary 4.5 A function $f : A \to B$ has a limit at $a \in A$ if and only if it is defined in a punctured neighborhood of a, and for every sequence (x_n) converging to a (but not equal to a), the sequence $(f(x_n))$ converges.

Proof. Suppose first that f has a limit at a—call it ℓ. By Heine's characterization, for every sequence (x_n) converging to a, the sequence $(f(x_n))$ converges to ℓ, hence converges.

Conversely, suppose that for every sequence (x_n) converging to a the sequence $(f(x_n))$ converges. Take one such sequence (x_n) and denote

$$\lim_{n\to\infty} f(x_n) = \ell.$$

We will prove that ℓ is the limit of f at a. For that, it suffices, once again by Heine's characterization, to prove that for every sequence (y_n) converging to a, the sequence $(f(y_n))$ converges to ℓ. If this weren't the case, we would have

$$\lim_{n\to\infty} f(y_n) \equiv m \neq \ell,$$

where we used the fact that $(f(y_n))$ is convergent. Consider now the sequence

$$(z_n) = x_1, y_1, x_2, y_2, \dots.$$

This sequence converges to a (why), however the sequence $(f(z_n))$ does not converge (why), contradicting the given property of f. ■

4.6 Limit arithmetic

We have seen above various examples of limits. In each case, we had to "work hard" to prove what the limit was, by showing that the definition was satisfied. This becomes impractical when the functions are more complex. Thus, like for sequences, we need to develop theorems that will make our task easier.

Theorem 4.6 — Limits arithmetic. Let f and g be two functions defined in a neighborhood of a, such that

$$\lim_a f = \ell \qquad \text{and} \qquad \lim_a g = m.$$

Then,

$$\lim_a (f+g) = \ell + m \qquad \text{and} \qquad \lim_a (f \cdot g) = \ell \cdot m.$$

If, moreover, $\ell \neq 0$, then

$$\lim_a \left(\frac{1}{f}\right) = \frac{1}{\ell}.$$

Proof. There are two basic way to prove this theorem. The first way follows exactly the same lines as the proof for the limit arithmetic for sequences.

By definition of the limit, for all $e > 0$,

$$(\exists \delta_1 > 0)(\forall x \in B^\circ(a, \delta_1))(f(x) \in B(\ell, \varepsilon/2)),$$

and

$$(\exists \delta_2 > 0)(\forall x \in B^\circ(a, \delta_2))(g(x) \in B(m, \varepsilon/2)).$$

Since $f(x) \in B(\ell, \varepsilon/2)$ and $g(x) \in B(m, \varepsilon/2)$ implies that $(f+g)(x) \in B(\ell+m, \varepsilon)$ it follows that if we take $\delta = \min(\delta_1, \delta_2)$, then $B^\circ(a,\delta) = B^\circ(a,\delta_1) \cap B^\circ(a,\delta_2)$, and

$$(\forall x \in B^\circ(a,\delta))(f(x) \in B(\ell, \varepsilon/2) \qquad \text{and} \qquad g(x) \in B(m, \varepsilon/2)),$$

from which follows that

$$(\forall x \in B^\circ(a,\delta))((f+g)(x) \in B(\ell+m, \varepsilon)),$$

namely,

$$\lim_a (f+g) = \ell + m.$$

We proceed similarly for products and ratios.

The second way uses Heine's characterization. By Heine, for every sequence (x_n) in a punctured neighborhood converging to a,

$$\lim_{n\to\infty} f(x_n) \to \ell \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) \to m.$$

By limit arithmetic for sequences,

$$\lim_{n\to\infty} (f+g)(x_n) = \ell + m,$$

and since this holds for every sequence x_n, it follows from Heine's characterization that

$$\lim_a (f+g) = \ell + m.$$

We proceed similarly for products and ratios. ■

Comment 4.5 It is important to point out that the fact that $f+g$ has a limit at a does not imply that either f or g has a limit at that point; take for example the functions $f(x) = 5 + 1/x$ and $g(x) = 6 - 1/x$ at $x = 0$.

■ **Example 4.13** What does it take now to show that[1]

$$\lim_{x\to a} \frac{x^3+7x^5}{x^2+1} = \frac{a^3+7a^5}{a^2+1}.$$

We need to show that for all $c \in \mathbb{R}$

$$\lim_a c = c,$$

and that for all $k \in \mathbb{N}$,

$$\lim_{x\to a} x^k = a^k,$$

which follows by induction once we show that for $k = 1$. ■

We conclude this section by extending the notion of a limit to that of a ***one-sided limit***.

Definition 4.5 Let $f : A \to B$ with a having a right neighborhood in A. We say that the ***limit on the right*** (גבול מימין) of f at a is ℓ, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$f(x) \in B(\ell,\varepsilon) \qquad \text{whenever} \qquad x \in B^\circ_+(a,\delta).$$

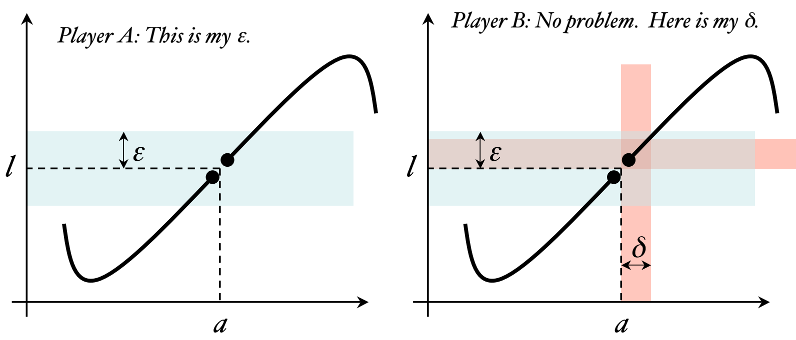
We write,

$$\lim_{a^+} f = \ell.$$

An analogous definition is given for the ***limit on the left*** (גבול משמאל).

[1]This is the standard notation to what we write in these notes as

$$f : \xi \mapsto \frac{\xi^3+7\xi^5}{\xi^2+1} \qquad \text{and} \qquad \lim_a f = \frac{a^3+7a^5}{a^2+1}.$$



Since Player B can find a δ for every choice of ε made by Player A, it follows that the right-limit of f at a is l.

The following is easily proved:

Theorem 4.7 — Heine's characterization for one-sided limits. Let $f : A \to B$ with a having a right neighborhood in A. Then,

$$\lim_{a^+} f = \ell$$

if and only if for every sequence (x_n) in a right neighborhood of a that converges to a,

$$\lim_{n\to\infty} f(x_n) = \ell.$$

4.7 Limits and order

In this section we will prove a number of properties pertinent to limits and order.

Proposition 4.8 Let f and g be two functions defined in a punctured neighborhood U of a point a. Suppose that

$$f(x) < g(x) \qquad \forall x \in U,$$

and that

$$\lim_a f = \ell \qquad \text{and} \qquad \lim_a g = m.$$

Then $\ell \le m$.

Comment 4.6 Note that even though $f(x) < g(x)$ is a strict inequality, the resulting inequality of the limits is in a weak sense. To see what this must be the case, consider the example

$$f(x) = |x| \qquad \text{and} \qquad g(x) = 2|x|.$$

Even though $f(x) < g(x)$ in an open neighborhood of 0,

$$\lim_0 f = \lim_0 g = 0.$$

Proof. By Heine's characterization, for every sequence $U \supset (x_n) \to a$,

$$\forall n \in N \qquad f(x_n) < g(x_n),$$

and

$$\lim_{n\to\infty} f(x_n) = \ell \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = m.$$

It follows from the parallel proposition for sequences that $\ell \le m$. ■

Proposition 4.9 Let f and g be two functions defined in a punctured neighborhood U of a point a. Suppose that

$$\lim_a f = \ell \qquad \text{and} \qquad \lim_a g = m,$$

with $\ell < m$. Then there exists a $\delta > 0$ such that

$$(\forall x \in B^\circ(a,\delta)) : (f(x) < g(x)).$$

Comment 4.7 This time both inequalities are strong.

Proof. Take again disjoint neighborhoods V_1 of ℓ and V_2 of m; every element in V_2 is greater than every element in V_1. By definition of the limit, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(\forall x \in B^\circ(a,\delta_1))(f(x) \in V_1) \qquad \text{and} \qquad (\forall x \in B^\circ(a,\delta_2))(g(x) \in V_2).$$

Letting $\delta = \min(\delta_1, \delta_2)$,

$$(\forall x \in B^\circ(a,\delta))(f(x) \in V_1 \quad \text{and} \quad g(x) \in V_2),$$

which implies that

$$(\forall x \in B^\circ(a,\delta))(f(x) < g(x)).$$

■

Proposition 4.10 — "Sandwich". Let f, g, h be defined in a punctured neighborhood U of a. Suppose that

$$(\forall x \in U) : (f(x) \le g(x) \le h(x)),$$

and

$$\lim_a f = \lim_a h = \ell.$$

Then

$$\lim_a g = \ell.$$

Proof. For every $U \supset (x_n) \to a$,

$$f(x_n) \leq g(x_n) \leq h(x_n),$$

and by Heine's characterization,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} h(x_n) = \ell.$$

It follows from the sandwich theorem for sequences that

$$\lim_{n\to\infty} g(x_n) = \ell.$$

Since this holds for every such sequence (x_n), it follows from Heine's characterization that

$$\lim_a g = \ell.$$

■

Definition 4.6 Let $f : A \to B$ and let $U \subset A$. We say that f is (upper/lower) ***bounded*** in U if the set

$$\{f(x) \,:\, x \in U\}$$

is (upper/lower) bounded.

Definition 4.7 Let $f : A \to B$ and $a \in A$ an interior point. We say that f is ***locally bounded*** (חסומה מקומית) near a if there exists a punctured neighborhood U of a, such that f is bounded in U. Equivalently, there exists a $\delta > 0$ such that the set

$$\{f(x) : x \in B^\circ(a,\delta)\}$$

is bounded.

Comment 4.8 As in the previous section, we consider punctured neighborhoods of a point, and we don't even care if the function is defined at that point.

Proposition 4.11 Let f be defined in a punctured neighborhood U of a. If the limit

$$\lim_a f = \ell$$

exists, then f is locally bounded near a.

Proof. Immediate from the definition of the limit. ■

Proposition 4.12 Let f be defined in a punctured neighborhood of a point a. Then,

$$\lim_a f = \ell \qquad \text{if and only if} \qquad \lim_a (f - \ell) = 0.$$

Proof. Very easy. ■

Proposition 4.13 Let f and g be defined in a punctured neighborhood of a point a. Suppose that

$$\lim_a f = 0$$

whereas g is locally bounded near a. Then,

$$\lim_a (fg) = 0.$$

Proof. Very easy. ■

Comment 4.9 Note that we do not require g to have a limit at a.

With the above tools, here is another way of proving the product property of the arithmetic of limits (this time without ε and δ).

Proposition 4.14 — Arithmetic of limits, product. Let f, g be defined in a punctured neighborhood of a point a, and

$$\lim_a f = \ell \qquad \text{and} \qquad \lim_a g = m.$$

Then

$$\lim_a (fg) = \ell m.$$

Proof. Write

$$fg - \ell m = \underbrace{\underbrace{(f-\ell)}_{\text{limit zero}} \underbrace{g}_{\text{loc. bdd}}}_{\text{limit zero}} \underbrace{-\ell \underbrace{(g-m)}_{\text{limit zero}}}_{\text{limit zero}}.$$

■

4.8 Continuity

Intuitively, we think of a function as continuous if "we can draw its graph without lifting the pencil". This is a very naive approach, as we will see lots of delicate examples of continuity.

Definition 4.8 A function $f : A \to \mathbb{R}$ is said to be ***continuous*** (רציפה) at an inner point $a \in A$, if it has a limit at a, and

$$\lim_a f = f(a).$$

Comment 4.10 If f has a limit at a and the limit differs from $f(a)$ (or that $f(a)$ is undefined) , then we say that f has a ***removable discontinuity*** (אי רציפות סליקה) at a.

■ **Examples 4.2**

1. We saw that the function $f(x) = x^2$ has a limit at 3, and that this limit was equal 9. Hence, f is continuous at $x = 3$.
2. We saw that the function $f(x) = 1/x$ has a limit at any $a > 0$ and that this limit equals $1/a$. By a similar calculation we could have shown that it has a limit at any $a < 0$ and that this limit also equals $1/a$. Hence, f is continuous at all $x \neq 0$. Since f is not even defined at $x = 0$, then it is not continuous at that point.
3. The function $f(x) = x \sin 1/x$ is continuous at all $x \neq 0$. At zero it is not defined, but if we rather define

$$f(x) = \begin{cases} x \sin 1/x & x \neq 0 \\ x & x = 0, \end{cases}$$

then f is continuous for all $x \in \mathbb{R}$ (by the bounded times limit zero argument).
4. The function

$$f : x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

is continuous at $x = 0$, however it is not continuous at any other point, because the limit of f at $a \neq 0$ does not exist.
5. The elementary functions sin and cos are continuous everywhere. To prove that sin is continuous, we need to prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\sin x - \sin a| < \varepsilon \qquad \text{whenever} \qquad x \in B^\circ(a, \delta).$$

We will take it for granted at the moment.

■

Our theorems on limit arithmetic imply right away similar theorems for continuity:

Theorem 4.15 If $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $a \in A$ then $f + g$ and $f \cdot g$ are continuous at a. Moreover, if $f(a) \neq 0$ then $1/f$ is continuous at a.

Proof. Obvious. ■

Comment 4.11 Recall that in the definition of the limit, we said that the limit of f at a is ℓ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \qquad \text{whenever} \qquad x \in B^\circ(a, \delta).$$

There was an emphasis on the fact that the point x itself is excluded. Continuity is defined as that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \qquad x \in B^\circ(a, \delta).$$

Note that we may require that this be true whenever $x \in B(a, \delta)$. There is no need to exclude the point $x = a$. In other words,

$$f \text{ is continuous at } a \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in B(a,\delta))(f(x) \in B(f(a),\varepsilon)).$$

With that, we have all the tools to show that a function like, say,

$$f(x) = \frac{\sin^2 x + x^2 + x^4 \sin x}{1 + \sin^2 x}$$

is continuous everywhere in $\mathbb{R}$. But what about a function like $\sin x^2$. Do we have the tools to show that it is continuous. No. We don't yet have a theorem for the composition of continuous functions.

Theorem 4.16 Let $g : A \to B$ and $f : B \to C$. Suppose that g is continuous at an inner point $a \in A$, and that f is continuous at $g(a) \in B$, which is an inner point. Then, $f \circ g$ is continuous at a.

Proof. Since g is continuous at a and f is continuous at $g(a)$, for all $\varepsilon > 0$

$$(\exists \delta_1 > 0)(\forall y \in B(g(a), \delta_1))(f(y) \in B(f(g(a)), \varepsilon)),$$

and

$$(\exists \delta_2 > 0)(\forall x \in B(a, \delta_2))(g(x) \in B(g(a), \delta_1)).$$

It follows that

$$(\forall x \in B(a, \delta_2))(f(g(x)) \in B(f(g(a)), \varepsilon)),$$

i.e., $f \circ g$ is continuous at a. ■

Definition 4.9 A function f is said to be ***right-continuous*** (רציפה מימין) at a if

$$\lim_{a^+} f = f(a).$$

A similar definition is given for ***left-continuity*** (רציפה משמאל).

So far, we have only dealt with continuity at points. Usually, we are interested in continuity on intervals.

Definition 4.10 A function f is said to be continuous on an open interval (a,b) if it is continuous at all $x \in (a,b)$. It is said to be continuous on a closed interval $[a,b]$ if it is continuous on the open interval, and in addition it is right-continuous at a and left-continuous at b.

Comment 4.12 We usually think of continuous function as "well-behaved". One should be careful with such interpretations; see for example

$$f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

■ **Example 4.14** Here is another "crazy function" due to Johannes Karl Thomae (1840-1921):

$$r(x) = \begin{cases} 1/q & x = p/q \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

where $x = p/q$ assumes that x is rational in reduced form. This function has the wonderful property of being continuous at all $x \notin \mathbb{Q}$ and discontinuous at all $x \in \mathbb{Q}$ (this is because its limit is everywhere zero). It took some more time until Vito Volterra proved in 1881 that there can be no function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. ■

4.9 Theorems about continuous functions

Theorem 4.17 Suppose that f is continuous at a and $f(a) > 0$. Then there exists a neighborhood of a in which $f(x) > 0$. That is,

$$(\exists\delta > 0)(\forall x \in B(a,\delta))(f(x) > 0).$$

Comment 4.13 An analogous theorem holds if $f(a) < 0$. Also, a one-sided version can be proved, whereby if f is right-continuous at a and $f(a) > 0$, then

$$(\exists\delta > 0)(\forall x \in B_+(a,\delta))(f(x) > 0).$$

Comment 4.14 Note that continuity is only required at the point a.

Proof. Let V be a neighborhood of $f(a)$ that does not contain zero, i.e., all its elements are positive. Since f is continuous at a, there exists a $\delta > 0$ such that

$$(\forall x \in B(a,\delta))(f(x) \in V).$$

■

This theorem can be viewed as a lemma for the following important theorem:

Theorem 4.18 — Intermediate value theorem משפט ערך הביניים. Suppose f is continuous on an interval $[a,b]$, with $f(a) < 0$ and $f(b) > 0$. Then, there exists a point $c \in (a,b)$, such that $f(c) = 0$.

Comment 4.15 Continuity is required on the whole intervals, for consider $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} -1 & 0 \le x < \frac{1}{2} \\ +1 & \frac{1}{2} \le x \le 1. \end{cases}$$

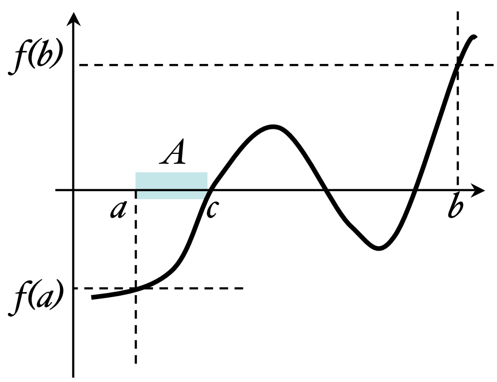
Comment 4.16 If $f(a) > 0$ and $f(b) < 0$ the same conclusion holds for just replace f by $(-f)$.

Proof. Consider the set

$$A = \{x \in [a,b] \ : \ f(y) < 0 \text{ for all } y \in [a,x]\}.$$

The set is non empty for it contains a. The previous theorems guarantees the existence of a $\delta > 0$ such that $[a, a+\delta) \subseteq A$. This set is also upper bounded by b. The previous theorem guarantees the existence of a $\delta > 0$ such that $b - \delta$ is an upper bound for A. By the axiom of completeness, there exists a number $c \in (a,b)$ such that

$$c = \sup A.$$



Suppose it were true that $f(c) < 0$. By the previous theorem, there exists a $\delta > 0$ such that

$$f(x) < 0 \qquad \text{whenever} \qquad c \le x \le c + \delta,$$

Because c is the least upper bound of A it follows that also

$$f(x) < 0 \qquad \text{whenever} \qquad x < c,$$

i.e., $f(x) < 0$ for all $x \in [a, c+\delta]$, which means that $c + \delta \in A$, contradicting the fact that c is an upper bound for A.

Suppose it were true that $f(c) > 0$. By the previous theorem, there exists a $\delta > 0$ such that

$$f(x) > 0 \qquad \text{whenever} \qquad c - \delta \le x \le c,$$

which implies that $c - \delta$ is an upper bound for A, i.e., c cannot be the least upper bound of A. By the trichotomy property, we conclude that $f(c) = 0$. ∎

Alternative proof. Suppose, by contradiction, that f does not vanish in $[a,b]$. Construct a sequence of segments $[a_n,b_n]$ as follows: first,

$$[a_1,b_1]=[a,b].$$

Then,

$$[a_{n+1},b_{n+1}]=\begin{cases}[a_n,(a_n+b_n)/2] & f((a_n+b_n)/2)>0\\ [(a_n+b_n)/2,b_n] & f((a_n+b_n)/2)<0.\end{cases}$$

By construction, for all $n\in\mathbb{N}$,

$$f(a_n)<0 \qquad \text{and} \qquad f(b_n)>0.$$

By Cantor's lemma, the sequence a_n and b_n converge to the same limit, say, c. Since f is continuous,

$$\lim_{n\to\infty} f(a_n)=\lim_{n\to\infty} f(b_n)=f(c).$$

However,

$$\lim_{n\to\infty} f(a_n)\le 0 \qquad \text{and} \qquad \lim_{n\to\infty} f(b_n)\ge 0,$$

from which we obtain that $f(c)=0$, contradicting our assumption. ■

Corollary 4.19 If f is continuous on a closed interval $[a,b]$, and $\alpha\in\mathbb{R}$ satisfies

$$f(a)<\alpha<f(b),$$

then there exists a point $c\in(a,b)$ at which $f(c)=\alpha$.

Proof. Apply the previous theorem for $g(x)=f(x)-\alpha$. ■

Lemma 4.20 If f is continuous at a, then there is a $\delta>0$ such that f has an upper bound on the interval $(a-\delta,a+\delta)$.

Proof. This is obvious, for by the definition of continuity, there exists a $\delta>0$, such that

$$f(x)\in B(f(a),1) \qquad \text{whenever} \qquad x\in B(a,\delta),$$

i.e., $f(x)<f(a)+1$ on this interval. ■

Theorem 4.21 — Karl Theodor Wilhelm Weierstraß. If f is continuous on $[a,b]$ then it is bounded from above of that interval, that is, there exists a number M, such that $f(x)<M$ for all $x\in[a,b]$.

Comment 4.17 Here too, continuity is needed on the whole interval, for look at the "counter example" $f:[-1,1]\to\mathbb{R}$,

$$f(x)=\begin{cases}1/x & x\ne 0\\ 0 & x=0.\end{cases}$$

Comment 4.18 It is crucial that the interval $[a,b]$ be closed. The function $f : (0,1] \to \mathbb{R}$, $f : x \mapsto 1/x$ is continuous on the semi-open interval, but it is not bounded from above.

Proof. Let

$$A = \{x \in [a,b] \ : \ f \text{ is bounded from above on } [a,x]\}.$$

By the previous lemma, there exists a $\delta > 0$ such that $a + \delta \in A$. Also A is upper bounded by b, hence there exists a $c \in (a,b]$, such that

$$c = \sup A.$$

We claim that $c = b$, for if $c < b$, then by the previous lemma, there exists a neighborhood of c in which f is upper bounded, and c cannot be an upper bound for A.

We have thus shown that for every $\delta > 0$, f is bounded on the interval $[a, b-\delta]$. It remains to show that f is bounded on $[a,b]$. Indeed, there exists an $\eta > 0$ such that f is bounded on $(b-\eta, b]$. In addition, f is bounded on $[a, b-\eta]$, which concludes the proof. ■

Alternative proof. Suppose that f were not upper bounded on $[a,b]$. This means that there exists a sequence $(x_n) \subset [a,b]$ such that $(f(x_n))$ tends to infinity. By Bolzano-Weierestrass, there exists a converging subsequence, $y_k = x_{n_k}$,

$$\lim_{k\to\infty} y_k = c.$$

We first argue that $c \in [a,b]$. Indeed, since $a \le y_k \le b$, then by the properties of limits and order $a \le c \le b$. Since f is continuous on $[a,b]$ it follows that

$$\infty = \lim_{k\to\infty} f(y_k) = f(c),$$

which is a contradiction. ■

Comment 4.19 In essence, this theorem is based on the fact that if a continuous function is bounded up to a point, then it is bounded up to a little farther. To be able to take such increments up to b we need the axiom of completeness.

Comment 4.20 With very little adaptation, this proof holds also for functions $\mathbb{R}^n \to \mathbb{R}$.

Theorem 4.22 — Weierstraß, Maximum principle (עקרון המקסימום). If f is continuous on $[a,b]$, then there exists a point $c \in [a,b]$ such that

$$f(x) \le f(c) \quad \text{for all} \quad x \in [a,b].$$

(Of course, there is a corresponding ***minimum principle***.)

Proof. We have just proved that f is upper bounded on $[a,b]$, i.e., the set

$$A = \{f(x) \ : \ a \le x \le b\}$$

is upper bounded. This set is non-empty for it contains the point $f(a)$. By the axiom of completeness it has a least upper bound, which we denote by

$$\alpha = \sup A.$$

We need to show that this supremum is in fact a maximum; that there exists a point $c \in [a,b]$, for which $f(c) = \alpha$.
Suppose, by contradiction, that this were not the case, i.e., that $f(x) < \alpha$ for all $x \in [a,b]$. We define then a new function $g : [a,b] \to \mathbb{R}$,

$$g = \frac{1}{\alpha - f}.$$

This function is defined everywhere on $[a,b]$ (since we assumed that the denominator does not vanish), it is continuous and positive. We will show that g is not upper bounded on $[a,b]$, contradicting thus the previous theorem. Indeed, since $\alpha = \sup A$:

For all $M > 0$ there exists a $y \in [a,b]$ such that $f(y) > \alpha - 1/M$.

For this y,

$$g(y) > \frac{1}{\alpha - (\alpha - 1/M)} = M,$$

i.e., for every $M > 0$ there exists a point in $[a,b]$ at which f takes a value greater than M. ∎

Alternative proof. By the property of the supremum, there exists a sequence $(x_n) \subset [a,b]$ such that $(f(x_n))$ tends to α. By Bolzano-Weierestrass, there exists a converging subsequence, $y_k = x_{n_k}$,

$$\lim_{k\to\infty} y_k = c.$$

Once again, we claim that $c \in [a,b]$. Since f is continuous on $[a,b]$ it follows that

$$\alpha = \lim_{k\to\infty} f(y_k) = f(c),$$

which proves that c is a maximum point. ∎

Comment 4.21 Here too we needed continuity on a closed interval, for consider the “counter examples"

$$f : [0,1) \to \mathbb{R} \qquad f(x) = x^2,$$

and

$$f : [0,1] \to \mathbb{R} \qquad f(x) = \begin{cases} x^2 & x < 1 \\ 0 & x = 1 \end{cases}.$$

These functions do not attain a maximum in $[0,1]$.

Theorem 4.23 If $n \in \mathbb{N}$ is odd, then the equation

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = 0$$

has a root (a solution) for any set of constants $a_0, \ldots, a_{n-1}$.

Proof. The idea it to show that existence of points a, b for which $f(b) > 0$ and $f(a) < 0$, and apply the intermediate value theorem, based on the fact that polynomials are continuous. The only technical issue is to find such points a, b in a way that works for all choices of $a_0, \ldots, a_{n-1}$.

Let

$$M = \max(1, 2n|a_0|, \ldots, 2n|a_{n-1}|).$$

Then for $|x| > M$,

$$\begin{aligned}
\frac{f(x)}{x^n} &= 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \\
(u + v \geq u - |v|) \quad &\geq 1 - \frac{|a_{n-1}|}{|x|} - \cdots - \frac{|a_0|}{|x^n|} \\
(|x^n| > |x|) \quad &\geq 1 - \frac{|a_{n-1}|}{|x|} - \cdots - \frac{|a_0|}{|x|} \\
(|x| > |a_i|) \quad &\geq 1 - \frac{|a_{n-1}|}{2n|a_{n-1}|} - \cdots - \frac{|a_0|}{2n|a_0|} \\
&= 1 - n \cdot \frac{1}{2n} > 0.
\end{aligned}$$

Since the sign of x^n is the same as the sign of x, it follows that $f(x)$ is positive for $x \geq M$ and negative for $x \leq -M$, hence there exists a $a < c < b$ such that $f(c) = 0$. ■

The next theorem deals with the case where n is even:

Theorem 4.24 Let n be even. Then the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = 0$$

has a minimum. Namely, there exists a $c \in \mathbb{R}$, such that $f(c) \leq f(x)$ for all $x \in \mathbb{R}$.

Proof. The idea is simple. We are going to show that there exists a $b > 0$ for which $f(x) > f(0)$ for all $|x| \geq b$. Then, looking at the function f in the interval $[-b, b]$, we know that it assumes a minimum in this interval, i.e., that there exists a $c \in [-b, b]$ such that $f(c) \leq f(x)$ for all $x \in [-b, b]$. In particular, $f(c) \leq f(0)$, which in turn is less than $f(x)$ for all $x \notin [-b, b]$. It follows that $f(c) \leq f(x)$ for all $x \in \mathbb{R}$.

It remains to find such a b. Let M be defined as in the previous theorem. Then, for all $|x| > M$, using the fact that n is even,

$$\begin{aligned} f(x) &= x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) \\ &\geq x^n \left(1 - \frac{|a_{n-1}|}{|x|} - \cdots - \frac{|a_0|}{|x^n|}\right) \\ &\geq x^n \left(1 - \frac{|a_{n-1}|}{|x|} - \cdots - \frac{|a_0|}{|x|}\right) \\ &\geq x^n \left(1 - \frac{|a_{n-1}|}{2n|a_{n-1}|} - \cdots - \frac{|a_0|}{2n|a_0|}\right) \\ &= \frac{1}{2}x^n. \end{aligned}$$

Let then $b > \max(M, \sqrt[n]{2|f(0)|})$, from which follows that $f(x) > f(0)$ for $|x| > b$. This concludes the proof. ■

Corollary 4.25 Consider the equation

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = \alpha$$

with n even. Then there exists a number m such that this equation has a solution for all $\alpha \geq m$ but has no solution for $\alpha < m$.

Proof. According to the previous theorem there exists a $c \in \mathbb{R}$ such that $f(c)$ is the minimum of f. If $\alpha < f(c)$ then for all x, $f(x) \geq f(c) > \alpha$, i.e., there is no solution. For $\alpha = f(c)$, c is a solution. For $\alpha > f(c)$ we have $f(c) < \alpha$ and for large enough x, $f(x) > \alpha$, hence, by the intermediate value theorem a root exists. ■

4.10 Infinite limits and limits at infinity

In this section we extend the notions of limits discussed in previous sections to two cases: (i) the limit of a function at a point is infinite, and (ii) the limit of a function at infinity is either finite or infinite. Before we start recall: ***infinity is not a real number!***.

Definition 4.11 — The limit at a point is infinite. Let $f : A \to B$ be a function. We say that the limit of f at a is infinity, denoted

$$\lim_a f = \infty,$$

if

$$(\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in B^\circ(a, \delta))(f(x) > M).$$

Similarly,

$$\lim_a f = -\infty$$

if

$$(\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in B^\circ(a,\delta))(f(x) < M).$$

■ **Example 4.15** Consider the function

$$f : x \mapsto \begin{cases} 1/|x| & x \neq 0 \\ 17 & x = 0. \end{cases}$$

The limit of f at zero is infinity. Indeed, for every $M > 0$ let $\delta = 1/M$. Then

$$(\forall x \, : \, 0 < |x| < \delta)(f(x) = 1/|x| > 1/\delta = M).$$

■

Definition 4.12 — The limit at infinity is finite. Let $f : \mathbb{R} \to \mathbb{R}$ and let $\ell \in \mathbb{R}$. We say that

$$\lim_\infty f = \ell,$$

if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{R})(\forall x > M)(f(x) \in B(\ell,\varepsilon)).$$

Similarly,

$$\lim_{-\infty} f = \ell$$

if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{R})(\forall x < M)(f(x) \in B(\ell,\varepsilon))$$

■ **Example 4.16** Consider the function $f : x \mapsto 3 + 1/x^2$. Then the limit of f at infinity is 3. Indeed, for all $\varepsilon > 0$ let $M = 1/\sqrt{\varepsilon}$. Then,

$$(\forall x > M)(|f(x) - 3| = 1/x^2 < 1/M^2 = \varepsilon).$$

■

Comment 4.22 These definitions are in full agreement with all previous definitions of limits, if we adopt the idea that "neighborhoods of infinity" are sets of the form (M,∞).

Finally,

Definition 4.13 — The limit at infinity is infinite. Let $f : \mathbb{R} \to \mathbb{R}$. We say that

$$\lim_\infty f = \infty,$$

if

$$(\forall K \in \mathbb{R})(\exists M \in \mathbb{R})(\forall x > M)(f(x) > K).$$

Comment 4.23 All those definitions have their natural analog with ∞ replaced by $(-\infty)$.

4.11 Monotonic functions

(To be addressed in tutoring session.)

4.12 Inverse functions

Suppose that $f : A \to B$ is one-to-one and onto. This means that for every $b \in B$ there exists a unique $a \in A$, such that $f(a) = b$. This property defines a function from B to A. In fact, this function is also one-to-one and onto. We call this function the ***function inverse to*** f (פונקציה הופכית), and denote it by f^{-1} (not to be mistaken with $1/f$). Thus, $f^{-1} : B \to A$,

$$f^{-1}(y) = (!x \in A \,:\, f(x) = y).$$

Differently stated,

$$f^{-1}(y) = x \iff f(x) = y.$$

Definition 4.14 If a function $f : A \to B$ has an inverse $f^{-1} : B \to A$ then it is called ***invertible*** (הפיכה).

Proposition 4.26 Let $f : A \to B$ be one-to-one and onto. Then,

$$f^{-1} \circ f = \mathrm{Id}_A \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{Id}_B .$$

Proof. This is immediate from the definition. Let $x \in A$, then

$$f^{-1}(f(x))$$

is the number if A, which is mapped by f to $f(x)$. This number is of course x. ■

Comment 4.24 Recall that a function $f : A \to B$ can be defined as a subset of $A \times B$,

$$\mathrm{Graph}\, f = \{(a,b) \,:\, f(a) = b\}.$$

The inverse function can be defined as a subset of $B \times A$: it is all the pairs (b,a) for which $(a,b) \in \mathrm{Graph}\, f$, or

$$\mathrm{Graph}\, f^{-1} = \{(b,a) \,:\, f(a) = b\} = \{(b,a) \,:\, (a.b) \in \mathrm{Graph}\, f\}.$$

▪ **Example 4.17** Let $f : [0,1] \to \mathbb{R}$ be given by $f : x \mapsto x^2$. This function is one-to-one and its image is the segment $[0,1]$. Thus we can define an inverse function $f^{-1} : [0,1] \to [0,1]$ by

$$f^{-1}(y) = (!x \in [0,1] \,:\, x^2 = y),$$

i.e., it is the (positive) square root. ▪

Definition 4.15 A set $I \subset \mathbb{R}$ is called an ***interval*** (מקטע) if $a,b \in I$ implies that $(a,b) \subset I$.

Theorem 4.27 Let I be an interval. If $f : I \to \mathbb{R}$ is continuous and one-to-one, then f is either strictly monotonically increasing, or strictly monotonically decreasing.

Comment 4.25 Continuity is crucial. The function $f : [0,2] \to \mathbb{R}$,

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 6-x & 1 < x \le 2 \end{cases}$$

is one-to-one with image $[0,1] \cup [4,5)$, but it is not monotonic. The fact that I is an interval is also crucial. The function $g : [0,1] \cup [2,3] \to \mathbb{R}$,

$$g(x) = \begin{cases} x & x \in [0,1] \\ -x & x \in [2,3] \end{cases}$$

is continuous, but is not monotonic.

Proof. First, note that if f is one-to-one, then $x \ne y$ implies $f(x) \ne f(y)$. We first prove that for every three points $a < b < c$, either

$$f(a) < f(b) < f(c), \qquad \text{or} \qquad f(a) > f(b) > f(c).$$

Suppose that $f(a) < f(c)$ and that, by contradiction, $f(b) < f(a)$ (see Figure 4.1). By the intermediate value theorem, there exists a point $x \in (b,c)$ such that $f(x) = f(a)$, contradicting the fact that f is one-to-one. Similarly, if $f(b) > f(c)$, then there exists a point $y \in [a,b]$ such that $f(y) = f(c)$. Thus, $f(a) < f(c)$ implies that $f(a) < f(b) < f(c)$. We proceed similarly if $f(a) > f(c)$.
It follows as once that for every four points $a < b < c < d$, either

$$f(a) < f(b) < f(c) < f(d) \qquad \text{or} \qquad f(a) > f(b) > f(c) > f(d).$$

Fix now a and b, and suppose w.l.o.g that $f(a) < f(b)$ (they can't be equal). Then, for every x,y such that $x < y$, we apply the above arguments to the four points a,b,x,y (whatever their order is) and conclude that $f(x) < f(y)$.

∎

Proposition 4.28 Let $f : [a,b] \to \mathbb{R}$ be one-to-one and monotonically increasing (a similar proposition holds if f is monotonically decreasing). Then,

$$\text{Image } f = [f(a), f(b)].$$

In particular, f maps closed segments into closed segments[a].

[a] More generally, a continuous one-to-one function maps connected sets into connected sets,

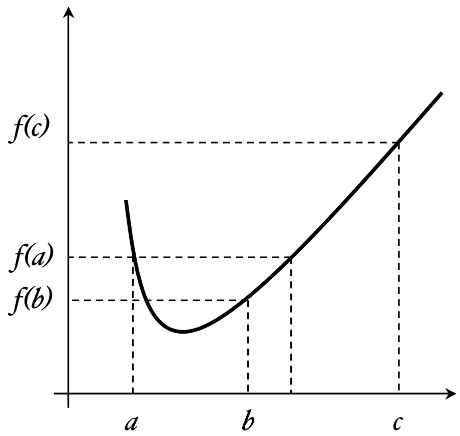


Figure 4.1: Illustration of proof

and retains the open/closed properties.

Proof. Since f is monotonically increasing, it assumes its minimum at a and its maximum at b, i.e.,

$$\text{Image}\, f \subset [f(a), f(b)].$$

It remains to prove the reverse inclusion $[f(a), f(b)] \subset \text{Image}\, f$. This follows from the intermediate-value theorem. ■

Theorem 4.29 If $f : [a,b] \to \mathbb{R}$ is invertible and continuous then so is f^{-1}.

Proof. Let $y \in \text{Image}\, f$ and let $x = f^{-1}(y)$. We need to show that for every neighborhood V of y there exists a neighborhood U of x such that

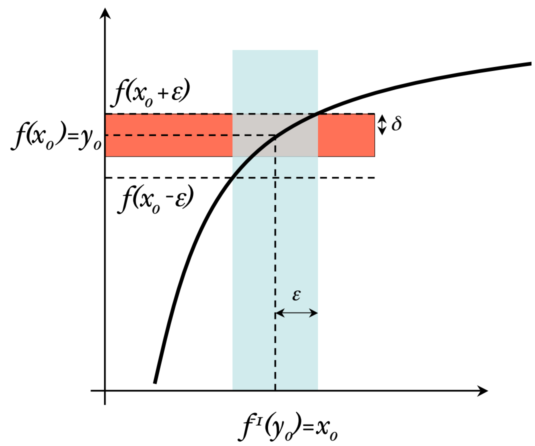
$$(\forall y' \in V)(f^{-1}(y') \in U).$$

Let $(y - \delta, y + \delta)$ be a neighborhood y. f is monotonic, and without loss of generality we can assume it is increasing. Consider then the set

$$U = (f^{-1}(y - \delta), f^{-1}(y + \delta)).$$

Since monotonic continuous functions map open segments to open segments, U is an open neighborhood of x. By monotonicity,

$$(\forall y' \in V)(f^{-1}(y') \in U).$$



■

■ **Example 4.18** The function $f:\mathbb{R}\to(0,\infty)$, $f(x)=e^x=\exp x$, is continuous and invertible (it is monotonically increasing). Its inverse,

$$\exp^{-1}:(0,\infty)\to\mathbb{R},$$

denoted log (or ln) is therefore also continuous. ■

■ **Example 4.19** The function $f:(-\pi/2,\pi/2)\to\mathbb{R}$, $f(x)=\tan x$, is continuous and invertible (it is monotonically increasing). Its inverse,

$$\tan^{-1}:\mathbb{R}\to(-\pi/2,\pi/2),$$

denoted also arctan is therefore also continuous. ■

4.13 Uniform continuity

Recall the definition of a continuous function: a function $f:(a,b)\to\mathbb{R}$ is continuous on the interval (a,b), if it is continuous at every point in the interval. That is, for every $x\in(a,b)$ and every $\varepsilon>0$, there exists a $\delta<0$, such that

$$(\forall y\in(a,b)\cap B(x,\delta))(|f(y)-f(x)|<\varepsilon).$$

In general, we expect δ to depend both on x and on ε. In fact, the way we set it here, this definition applies for continuity on a closed interval as well (i.e, it includes one-sided continuity as well). Let's follows all the quantifiers: f is continuous on $[a,b]$ if

$$(\forall x\in[a,b])(\forall\varepsilon>0)(\exists\delta>0)(\forall y\in[a,b]\cap B(x,\delta))(|f(y)-f(x)|<\varepsilon).$$

Let us re-examine two examples:

■ **Example 4.20** Consider the function $f:(0,1)\to\mathbb{R}$, $f:x\mapsto x^2$. This function is continuous on $(0,1)$. Why? Because for every $x\in(0,1)$, and every $y\in(0,1)$,

$$|f(y)-f(x)| = |y^2-x^2| = |y-x||y+x| \le (x+1)|y-x|.$$

Thus, given x and $\varepsilon>0$, if we choose $\delta=\delta(\varepsilon,x)=\varepsilon/(x+1)$, then

$$|f(y)-f(x)|<\varepsilon \qquad \text{whenever} \qquad |x-y|<\delta(\varepsilon,x) \quad \text{and} \quad y\in(0,1).$$

Thus, δ depends both on ε and x. However, it is always legitimate to replace δ by a smaller number. If we take $\delta=\varepsilon/2$, then the same δ fits all points x. ■

■ **Example 4.21** Consider next the function $f:(0,1)\to\mathbb{R}$, $f:x\mapsto 1/x$. This function is also continuous on $(0,1)$. Why? Given $x\in(0,1)$ and $y\in(0,1)$,

$$|f(x)-f(y)| = \frac{|x-y|}{xy} = \frac{|x-y|}{x[x+(y-x)]}.$$

If we take $\delta(\varepsilon,x)=\min(x/2,\varepsilon x^2/2)$, then

$$|x-y|<\delta \qquad \text{implies} \qquad |f(x)-f(y)| < \frac{\delta}{x(x-\delta)} < \varepsilon.$$

Here, given ε, the closer x is to zero, the smaller is $\delta(\varepsilon,x)$. There is no way we can find a $\delta=\delta(\varepsilon)$ that would fit all x. ■

Thus, there is a fundamental difference between these two examples. In the first, we could choose δ independently of x. If this is the case, then we can set x after having chosen δ and get

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in[a,b])(\forall y\in[a,b]\cap B(x,\delta))(|f(y)-f(x)|<\varepsilon),$$

which amounts to

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x,y\in[a,b] \,:\, |x-y|<\delta)(|f(y)-f(x)|<\varepsilon).$$

These two examples motivate the following definition:

Definition 4.16 $f:A\to B$ is said to be ***uniformly continuous*** on A (**רציפה במידה שווה**) if for every $\varepsilon>0$ corresponds a $\delta>0$, such that

$$(\forall x,y\in[a,b] \,:\, |x-y|<\delta)(|f(y)-f(x)|<\varepsilon)$$

Note that x and y play here symmetric roles. (The adjective "uniform" means that the same number can be used for all points.)

What is the essential difference between the two above examples? The function $x\mapsto x^2$ could have been defined on the closed interval $[0,1]$ and it would have been continuous there too. In contrast, there is no way we could have defined the function

$x \mapsto 1/x$ as a continuous function on $[0,1]$, even if we took care of its value at zero. We have already seen examples where continuity on a closed interval had strong implications (ensures boundedness and the existence of a maximum). This is also the case here. We will prove that *continuity on a closed interval implies uniform continuity*:

Theorem 4.30 If f is continuous on $[a,b]$ then it is uniformly continuous on that interval.

Proof. Let's proceed by contradiction. Suppose that f were not uniformly continuous on $[a,b]$. Then,

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x, y \in [a,b] \,:\, |x-y| < \delta)(|f(y) - f(x)| \geq \varepsilon).$$

Let $\varepsilon > 0$ be that number. Then, for every $n \in \mathbb{N}$ there exist $x_n, y_n \in [a,b]$ such that

$$|x_n - y_n| < \frac{1}{n} \qquad \text{and} \qquad |f(x_n) - f(y_n)| \geq \varepsilon.$$

Since x_n is a bounded sequence, it has a converging subsequence (x_{n_k}). Consider now the subsequence (y_{n_k}). Since it is bounded, it has a converging sub-subsequence $(y_{n_{k_\ell}})$. Since $(x_{n_{k_\ell}})$ is a subsequence of a convergent sequence, it converges as well. Denote,

$$\alpha = \lim_{\ell\to\infty} x_{n_{k_\ell}} \qquad \text{and} \qquad \beta = \lim_{\ell\to\infty} y_{n_{k_\ell}}.$$

Since

$$|x_{n_{k_\ell}} - y_{n_{k_\ell}}| < \frac{1}{n_{k_\ell}},$$

it follows from limit arithmetic that $\alpha = \beta$. Thus, $x_{n_{k_\ell}}$ and $y_{n_{k_\ell}}$ both converge to the same limit. On the other hand, since for all ℓ

$$|f(x_{n_{k_\ell}}) - f(y_{n_{k_\ell}})| > 0,$$

the sequences $f(x_{n_{k_\ell}})$ and $f(y_{n_{k_\ell}})$ cannot both converge to the same limit, contradicting the continuity of f. ■

Comment 4.26 In this course we will make use of uniform continuity only once, when we study integration.

■ Examples 4.3

1. Consider the function $f(x) = \sin(1/x)$ defined on $(0,1)$. Even though it is continuous, it is not uniformly continuous.
2. Consider the function $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$. It is easy to see that it is uniformly continuous. The function $\mathrm{Id}\cdot\mathrm{Id}$, however, is continuous on $\mathbb{R}$, but not uniformly continuous. Thus, the product of uniformly continuous functions is not necessarily uniformly continuous.
3. Finally the function $f(x) = \sin x^2$ is continuous and bounded on $\mathbb{R}$, but it is not uniformly continuous.

■