

Chapter 4

Differentiation

4.1 Signed measures

Measures, by definition, assign non-negative (possibly infinite) values to sets. We now extend the notion of a measure to a set function—a signed measure—that may return negative values as well. Signed measures will be helpful below, but they are also natural in many applications. For example, electric charge can be viewed as a signed measure, whereas mass can be viewed as a measure.

Definition 4.1 Let (\mathbb{X}, Σ) be a measurable space. A **signed measure** (מידה מסומנת) on (\mathbb{X}, Σ) is a set function $\nu : \Sigma \rightarrow [-\infty, \infty]$ satisfying the following conditions:

1. $\nu(\emptyset) = 0$.
2. ν assumes at most one of the values $\pm\infty$.
3. For every disjoint sequence A_n of measurable sets,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n).$$

(Note that since only one of the values $\pm\infty$ is permitted, the right-hand side converges absolutely in an extended sense.)

Comment: Every measure is a signed measure. Sometimes, to distinguish measures from signed measures we call the former **positive measures**.

Comment: In the literature, one finds two slightly different concepts of a signed measure, depending on whether or not one allows it to take infinite values. Since signed measures are a particular example of complex measures—set functions that may return complex values—one often defines them as set functions returning only finite values. In some places, infinite values are allowed, thus distinguishing between “finite signed measures” (מידות מסומנות סופיות) and “extended signed measures” (מידות מסומנות מוכללות).

Example: Let μ_1 and μ_2 be measures on (\mathbb{X}, Σ) , one of which is finite. Then,

$$\nu = \mu_1 - \mu_2$$

is a signed measure. Clearly, $\nu(\emptyset) = 0$ and ν assumes at most one of the values $\pm\infty$. It is countably-additive because μ_1, μ_2 are countably-additive. ▲▲▲

Example: Let μ be a measure on (\mathbb{X}, Σ) and let $f : \mathbb{X} \rightarrow [-\infty, \infty]$ be a measurable function, such that either

$$\int_{\mathbb{X}} f^+ d\mu < \infty \quad \text{or} \quad \int_{\mathbb{X}} f^- d\mu < \infty.$$

We will call such functions **extended-integrable** (אינטגרבילית באופן מוכלל). The set function,

$$\nu : A \mapsto \int_A f d\mu$$

is a signed measure. Indeed, we have seen that

$$\nu^\pm : A \mapsto \int_A f^\pm d\mu$$

are both measures, one of which is finite; by the previous example, $\nu = \nu^+ - \nu^-$ is a signed measure. ▲▲▲

As we will shortly see, every signed measure is in some sense of one of these two types.

Proposition 4.2 (Semicontinuity of signed measures) Let ν be a signed measure on (\mathbb{X}, Σ) . If A_n is an increasing sequence of measurable sets, then

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \nu(A_n).$$

Likewise, if B_n is a decreasing sequence of measurable sets and $\nu(B_1)$ is finite, then

$$\nu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \nu(B_n).$$

Proof: The proof is the same as for (positive) measures. Take for example the increasing case. Define

$$E_1 = A_1 \quad \text{and} \quad E_n = A_n \setminus A_{n-1}.$$

Then, the E_n are disjoint and satisfy $\bigsqcup_{k=1}^n E_k = A_n$ and $\bigsqcup_{k=1}^{\infty} E_k = \cup_{n=1}^{\infty} A_n$. Now,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \nu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \nu(E_k) = \lim_{n \rightarrow \infty} \nu\left(\bigsqcup_{k=1}^n E_k\right) = \lim_{n \rightarrow \infty} \nu(A_n).$$

In the case where the left-hand side is finite, the convergence is absolute. ■

Let ν be a signed measure and suppose that $A \in \Sigma$ with $\nu(A) > 0$. In principle, A may have subsets that have negative measure. The next definition delineates sets which are either positive or negative in a stronger sense:

Definition 4.3 Let ν be a signed measure on (\mathbb{X}, Σ) . A measurable set E is called **ν -positive** (חיובית בהחלט) if $\nu(F) \geq 0$ for all $\Sigma \ni F \subset E$; we define similarly **ν -negative** (שלילית בהחלט) and **ν -null** (זניחה בהחלט) sets.

Comment: In a positive measure space $(\mathbb{X}, \Sigma, \mu)$, every set is μ -positive and every null set is μ -null.

Lemma 4.4 Every measurable subset of a ν -positive set is ν -positive, and so is any countable union of ν -positive sets.

Proof: The first part is obvious. For the second part, let A_n be a sequence of ν -positive sets. Then,

$$B_1 = A_1 \quad \text{and} \quad B_n = A_n \setminus A_{n-1}$$

are ν -positive. Let $C \subset \bigcup_{n=1}^{\infty} A_n$ be measurable; then,

$$\nu(C) = \nu\left(\bigcup_{n=1}^{\infty} A_n \cap C\right) = \nu\left(\bigcup_{n=1}^{\infty} B_n \cap C\right) = \nu\left(\bigcup_{n=1}^{\infty} (B_n \cap C)\right) = \sum_{n=1}^{\infty} \nu(B_n \cap C) > 0,$$

where we used the countable-additive property of the signed measure. \blacksquare

The following theorem states that every space equipped with a signed measure ν can be partitioned into a ν -positive set and a ν -negative set, and that this partition is in a certain sense unique.

Theorem 4.5 (Hahn decomposition (פרוק האן)) Let ν be a signed measure on (\mathbb{X}, Σ) . Then, there exists a ν -positive set P and a ν -negative set N , such that

$$\mathbb{X} = P \sqcup N.$$

This partition is unique, in the sense that if P' and N' are another such pair of sets, then

$$P \Delta P' = N \Delta N' \quad \text{is } \nu\text{-null},$$

where Δ denotes symmetric difference, $A \Delta B = (A \setminus B) \sqcup (B \setminus A)$.

Comment: Hans Hahn (1879–1934) was an Austrian mathematician; his most celebrated contribution is the Hahn-Banach theorem in analysis.

Proof: Without loss of generality, assume that ν does not assume the value $+\infty$. Let

$$m = \sup\{\nu(A) : A \text{ is } \nu\text{-positive}\}$$

(a priori, m may be infinite). By definition of the supremum, there exists a sequence P_n of positive sets such that

$$\lim_{n \rightarrow \infty} \nu(P_n) = m.$$

Define

$$P = \bigcup_{n=1}^{\infty} P_n.$$

By Lemma 4.4, P is a positive set, and by the lower-semicontinuity of the signed measure, $\nu(P) = m$; in particular $m < \infty$.

Denote $N = P^c$ and assume, by contradiction, that N is not ν -negative; i.e., there exists a subset $A \subset N$ such that $\nu(A) > 0$. This implies that $\nu(P \sqcup A) = \nu(P) + \nu(A) > m$. This is not yet a contradiction, because A may not be a ν -positive set (be careful to distinguish ν -positive sets and sets of positive measure).

Thus, we have to proceed more carefully. First, we note that N cannot contain a ν -positive set which has positive measure, because in this case we would have obtained a contradiction.

Second, suppose that $A \subset N$ and $\nu(A) > 0$. Since A is not ν -positive, there exists a set $C \subset A$ such that $\nu(C) < 0$, hence for $B = A \setminus C$,

$$\nu(B) = \nu(A) - \nu(C) > \nu(A).$$

In other words, every subset of N having positive measure has a subset having larger measure.

So let $A_1 \subset N$ have positive measure and let n_1 be the smallest integer satisfying

$$\nu(A_1) > \frac{1}{n_1}.$$

Then, A_1 has a subset B of larger measure. Let n_2 be the smallest integer for which there exists an $A_2 \subset A_1$ such that

$$\nu(A_2) > \nu(A_1) + \frac{1}{n_2}.$$

We proceed inductively; let n_k be the smallest integer for which there exists an $A_k \subset A_{k-1}$ such that

$$\nu(A_k) > \nu(A_{k-1}) + \frac{1}{n_k}.$$

Set

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Since the A_n are decreasing,


$$\nu(A) = \lim_{n \rightarrow \infty} \nu(A_n) > \sum_{k=1}^{\infty} \frac{1}{n_k},$$

and since $\nu(A) < \infty$, the n_k must be increasing to infinity. But then, by the same argument, there exists $B \subset A$ such

$$\nu(B) \geq \nu(A) + \frac{1}{n}$$

for some n . Since $n < n_k$ for some k , we obtain a contradiction on the construction of the n_k 's.

It remains to prove the uniqueness of the decomposition. Since $P \setminus P' \subset P$ and $P \setminus P' \subset N'$, $P \setminus P'$ is both ν -positive and ν -negative, i.e., it is ν -null; the same applies to all other differences. ■

 **Exercise 4.1** Let ν be a signed measure on the measurable space (\mathbb{X}, Σ) . Let $A \in \Sigma$ such that $\nu(A) > 0$. Show that there exists a ν -positive set $B \subset A$, such that $\nu(B) \geq \nu(A)$.

Definition 4.6 Let μ and ν be signed measures on (\mathbb{X}, Σ) . They are called **mutually singular** (זה ביהם לזה, סינגולרים), denoted $\nu \perp \mu$, if there exist disjoint measurable A and B such that

$$\mathbb{X} = A \sqcup B,$$

A is μ -null and B is ν -null (informally, μ and ν “live” on different parts of \mathbb{X}).

The following theorem states that every signed measure can be expressed as a difference of two (positive) measures that “live on different parts” of \mathbb{X} :

Theorem 4.7 (Jordan decomposition) Let ν be a signed measure. Then, there exist unique (positive) measures, ν^+ and ν^- , such that

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-.$$

Proof: Let $\mathbb{X} = P \sqcup N$ be the Hahn decomposition of \mathbb{X} ; that is, P and N are disjoint ν -positive and ν -negative sets. Define

$$\nu^+ : A \mapsto \nu(A \cap P) \quad \text{and} \quad \nu^- : A \mapsto -\nu(A \cap N).$$

By construction, $\nu^+ \perp \nu^-$ (N is ν^+ -null and P is ν^- -null). Moreover, for every $A \in \Sigma$,

$$\nu(A) = \nu(A \cap P) + \nu(A \cap N) = \nu^+(A) - \nu^-(A).$$

Suppose that $\nu = \mu^+ - \mu^-$ where $\mathbb{X} = P' \sqcup N'$, N' is μ^+ -null and P' is μ^- -null. Then, P', N' is another Hahn-Decomposition for \mathbb{X} , and by uniqueness,

$$\nu(P \Delta P') = 0 \quad \text{and} \quad \nu(N \Delta N') = 0.$$

For every measurable A ,

$$A \cap (P \cup P') = (A \cap P') \sqcup (A \cap (P \setminus P')) = (A \cap P) \sqcup (A \cap (P' \setminus P)),$$

hence

$$\begin{aligned} \mu^+(A) &= \nu(A \cap P') \\ &= \nu(A \cap P) + \nu(A \cap (P \setminus P')) - \nu(A \cap (P' \setminus P)) \\ &= \nu^+(A), \end{aligned}$$

and similarly, $\mu^-(A) = \nu^-(A)$. ■

Definition 4.8 Let ν be a signed measure. Its **total variation** (השחנות כוללת) is the (positive) measure

$$|\nu| = \nu^+ + \nu^-,$$

where ν^\pm is its Jordan decomposition.

Lemma 4.9 Let ν be a signed measure on (\mathbb{X}, Σ) . Then, $A \in \Sigma$ is ν -null if and only if $|\nu|(A) = 0$ (i.e., it is $|\nu|$ -null).

Proof: Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν and suppose that

$$|\nu|(A) = \nu^+(A) + \nu^-(A) = 0.$$


Let $B \subset A$ be measurable. Then, since ν^\pm are positive measures,


$$\nu(B) = \nu^+(B) - \nu^-(B) \leq \nu^+(B) \leq \nu^+(A) = 0,$$

and

$$\nu(B) = \nu^+(B) - \nu^-(B) \geq -\nu^-(B) \geq -\nu^-(A) = 0.$$

The other direction is trivial. ■

 **Exercise 4.2** Let μ, ν be signed measures on (\mathbb{X}, Σ) . Show that $\nu \perp \mu$ if and only if $|\nu| \perp \mu$, if and only if $\nu^+ \perp \mu$ and if and only if $\nu^- \perp \mu$.

 **Exercise 4.3** Let μ, ν be positive measures on (\mathbb{X}, Σ) . Suppose that for every $\varepsilon > 0$ there exists a set $A \in \Sigma$, such that

$$\mu(A) < \varepsilon \quad \text{and} \quad \mu(A^c) < \varepsilon.$$


Prove that $\mu \perp \nu$.

We conclude this section by defining integration with respect to a signed measure. For a signed measure ν , we define

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-).$$

Then, for all $f \in L^1(\nu)$,

$$\int_{\mathbb{X}} f d\nu \stackrel{\text{def}}{=} \int_{\mathbb{X}} f d\nu^+ - \int_{\mathbb{X}} f d\nu^-.$$

 **Exercise 4.4** Let ν be a signed measures on (\mathbb{X}, Σ) . Show that:


(a) $L^1(\nu) = L^1(|\nu|)$.

(b) For all $f \in L^1(\nu)$,

$$\left| \int_{\mathbb{X}} f d\nu \right| \leq \int_{\mathbb{X}} |f| d|\nu|.$$

(c) If A is measurable, then

$$|\nu|(A) = \sup \left\{ \left| \int_A f d\nu \right| : \|f\|_{\infty} \leq 1 \right\}.$$

 **Exercise 4.5** Let ν be a signed measures on (\mathbb{X}, Σ) , and let $A \in \Sigma$. Show that

(a)

$$\nu^+(A) = \sup \{ \nu(B) : \Sigma \ni B \subset A \}.$$

(b)

$$\nu^-(A) = -\inf \{ \nu(B) : \Sigma \ni B \subset A \}.$$

(c)

$$|\nu|(A) = \sup \left\{ \sum_{j=1}^n |\nu(A_j)| : n \in \mathbb{N}, \quad A = \bigsqcup_{j=1}^n A_j \right\}.$$

4.2 The Radon-Nikodym theorem

Definition 4.10 Let ν be a signed measure on (\mathbb{X}, Σ) and let μ be a (positive) measure on (\mathbb{X}, Σ) . ν is said to be **absolutely continuous** (רציפה בהחלט) with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$. We denote this relation by

$$\nu \ll \mu.$$

Comment: Absolute continuity, unlike mutual singularity, is not a symmetric relation. For example, the zero measure is absolutely continuous with respect to any non-zero measure, but the opposite is not true.

Lemma 4.11 Let ν be a signed measure on (\mathbb{X}, Σ) and let μ be a (positive) measure on (\mathbb{X}, Σ) . Then, $\nu \ll \mu$ if and only if every μ -null set is ν -null. (Note that since μ is a positive measure, $\mu(A) = 0$ implies that A is μ -null.)

Proof: Suppose that $\nu \ll \mu$ and $\mu(A) = 0$. For every $B \subset A$, $\mu(B) = 0$ hence $\nu(B) = 0$, i.e., A is ν -null. The other direction is trivial. ■

Lemma 4.12 $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, which in turns holds if and only if $\nu^\pm \ll \mu$.

Proof: This is an immediate consequence of Lemma 4.9 and Lemma 4.11:

$$\begin{aligned} \nu \ll \mu &\iff (\mu\text{-null sets are } \nu\text{-}) \\ &\iff (\mu\text{-null sets are } |\nu|\text{-}) \\ &\iff |\nu| \ll \mu. \end{aligned}$$

■

Lemma 4.13 Absolute continuity is “complementary” to mutual singularity in the following sense: if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

Proof: Let $X = A \sqcup B$, where A is μ -null and B is ν -null. By absolute continuity, A is also ν -null, i.e., X is ν -null. ■

The next proposition is a “quantitative” version of the notion of absolute continuity:

Proposition 4.14 Let μ be a (positive) measure on (\mathbb{X}, Σ) and let ν be a finite signed measure. Then, $\nu \ll \mu$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma, \quad \mu(A) < \delta \quad \text{implies} \quad |\nu(A)| < \varepsilon.$$

Proof: **Reduction:** it suffices to consider the case where ν is a finite positive measure. Indeed, suppose that for positive measures

$$\nu \ll \mu \iff \forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma, \quad \mu(A) < \delta \quad \text{implies} \quad \nu(A) < \varepsilon.$$

Let ν be a finite signed measure. If $\nu \ll \mu$, then $|\nu| \ll \mu$ is a finite positive measure, hence

$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma, \quad \mu(A) < \delta \quad \text{implies} \quad |\nu(A)| \leq |\nu|(A) < \varepsilon,$$

proving one direction. Conversely, suppose that the ε - δ condition holds. Let $P \sqcup N$ be the Hahn-decomposition for ν . Then,

$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma, \quad \mu(A \cap P) < \delta \quad \text{implies} \quad |\nu(A \cap P)| < \varepsilon,$$

which in turn implies that

$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma, \quad \mu(A) < \delta \quad \text{implies} \quad \nu^+(A) < \varepsilon,$$

i.e., $\nu^+ \ll \mu$. Likewise, we prove that $\nu^- \ll \mu$.

proof of the positive case: The easy part: suppose that the ε, δ -condition holds. For $\varepsilon > 0$, let $\delta > 0$ be such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$. Then, $\mu(A) = 0$ implies $\nu(A) < \varepsilon$, and since this holds for every $\varepsilon > 0$, we conclude that $\nu(A) = 0$.

Conversely, suppose that the ε, δ -condition does not hold, i.e.,

$$\exists \varepsilon > 0, \quad \forall \delta > 0 \quad \exists A \in \Sigma, \quad \mu(A) < \delta \quad \text{and} \quad \nu(A) \geq \varepsilon.$$

Setting $\delta = 2^{-n}$,

$$\exists \varepsilon > 0, \quad \forall n \in \mathbb{N} \quad \exists A_n \in \Sigma, \quad \mu(A_n) < 2^{-n} \quad \text{and} \quad \nu(A_n) \geq \varepsilon.$$

Let

$$B_n = \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad B = \bigcap_{n=1}^{\infty} B_n.$$

Then,

$$\mu(B_n) < \sum_{k=n}^{\infty} 2^{-k} = 2^{1-n}.$$

By the upper-semicontinuity of μ (using the fact that $\mu(B_1) < \infty$),

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(B_n) = 0.$$

On the other hand, $\nu(B_n) \geq \varepsilon$, and since ν is finite,

$$\nu(B) = \lim_{n \rightarrow \infty} \nu(B_n) \geq \varepsilon.$$

It follows that $\nu \not\ll \mu$. ■

Given a measure, there exists a standard way of generating a signed measure that it absolutely continuous with respect to it.

Proposition 4.15 Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let f be μ -extended-integrable. Then,

$$\nu : A \mapsto \int_A f d\mu$$

is a signed measure, with $\nu \ll \mu$. In particular, if $f \in L^1(\mu)$, then ν is a finite signed measure.

Proof: We have already seen that ν is a signed measure; it is finite for $f \in L^1(\mu)$ because for every $A \in \Sigma$,

$$|\nu(A)| \leq \int_A |f| d\mu < \infty.$$

It remains to show that $\nu \ll \mu$: let $\mu(A) = 0$. Since integrals are insensitive to sets of measure zero, we may define

$$\tilde{f} = f \chi_{A^c}.$$

Then,

$$\nu(A) = \int_A f d\mu = \int_A \tilde{f} d\mu = 0.$$


■

Comment: It is customary to denote the relation

$$\int_A dv = v(A) = \int_A f d\mu$$


by $dv = f d\mu$.

We will shortly see a powerful theorem stating that every signed measure that is absolutely-continuous with respect to μ is of this form.

 *Exercise 4.6* Let μ, ν be positive measure on (\mathbb{X}, Σ) . Suppose that $\nu \ll \mu$ and that

$$f = \frac{d\nu}{d\mu} > 0$$

a.e., Prove that $\mu \ll \nu$.

 *Exercise 4.7* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Let $f \in L^1(\mu)$. Show that

$$\forall \varepsilon > 0 \quad \exists \delta > 0, \quad \forall A \in \Sigma \quad \mu(A) < \delta \quad \text{implies} \quad \int_A f d\mu < \varepsilon.$$

Lemma 4.16 Let μ, ν be finite (positive) measures on (\mathbb{X}, Σ) . Then, either $\mu \perp \nu$, or,

$$\exists \varepsilon > 0 \quad \text{and} \quad \exists A \in \Sigma \quad \text{such that} \quad \mu(A) > 0 \quad \text{and} \quad A \text{ is } (\nu - \varepsilon\mu)\text{-positive}.$$

That is, for every $\Sigma \ni B \subset A$,

$$\nu(B) \geq \varepsilon\mu(B).$$

Proof: For every $n \in \mathbb{N}$, $\nu - \mu/n$ is a signed measure. Let

$$\mathbb{X} = P_n \sqcup N_n$$

be the Hahn-decomposition for $\nu - \mu/n$. That is, P_n is $(\nu - \mu/n)$ -positive and N_n is $(\nu - \mu/n)$ -negative. Let

$$P = \bigcup_{n=1}^{\infty} P_n \quad \text{and} \quad N = \bigcap_{n=1}^{\infty} N_n = P^c.$$

Note that P_n are increasing and N_n are decreasing. By definition, N is $(\nu - \mu/n)$ -negative for all n . In particular,

$$\nu(N) \leq \frac{\mu(N)}{n}$$

for all n , implying that $\nu(N) = 0$. If $\mu(P) = 0$, then $\mu \perp \nu$. Otherwise, if $\mu(P) > 0$, then since P_n are increasing sets, there exists an n such that $\mu(P_n) > 0$, and P_n is $(\nu - \mu/n)$ -positive. ■

We now prove the main theorem of this section:

Theorem 4.17 (Radon-Nikodym) Let ν be a σ -finite signed measure on (\mathbb{X}, Σ) and let μ be a σ -finite (positive) measure on that same space. Then, ν has a unique decomposition

$$\nu = \lambda + \rho,$$

where

$$\lambda \perp \mu \quad \text{and} \quad \rho \ll \mu.$$

Moreover, there exists an extended-integrable function $f : \mathbb{X} \rightarrow \mathbb{R}$, such that $d\rho = f d\mu$; the function f is unique up to a set of μ -measure zero.

Comment: This theorem was proved by Johann Radon in 1913 in \mathbb{R}^n , and generalized in 1930 by Otto Nikodym.

Comment: In the particular case where $\nu \ll \mu$, then $\lambda = 0$ and there exists an extended-integrable function f , such that $d\nu = f d\mu$. The function f is called the **Radon-Nikodym derivative** of ν with respect to μ , and is denoted

$$f = \frac{d\nu}{d\mu}.$$

Naturally, it is defined up to a set of μ -measure zero.

Proof: Step 1: the finite positive case. We start with the case where both μ and ν are finite positive measures. Let,

$$\mathcal{F} = \left\{ f : X \rightarrow [0, \infty] : \int_A f d\mu \leq \nu(A), \quad \forall A \in \Sigma \right\}.$$

This collection of functions is non-empty, because $0 \in \mathcal{F}$. Moreover, if $f, g \in \mathcal{F}$ then $h = \max(f, g) \in \mathcal{F}$. Indeed, let

$$E = \{x : f(x) \leq g(x)\},$$

then for all $A \in \Sigma$,

$$\begin{aligned} \int_A h d\mu &= \int_{A \cap E} h d\mu + \int_{A \cap E^c} h d\mu \\ &= \int_{A \cap E} g d\mu + \int_{A \cap E^c} f d\mu \\ &\leq \nu(A \cap E) + \nu(A \cap E^c) = \nu(A). \end{aligned}$$

Let

$$M = \sup \left\{ \int_{\mathbb{X}} f d\mu : f \in \mathcal{F} \right\}.$$

By the definition of \mathcal{F} and the finite-measure assumption,

$$M \leq \nu(\mathbb{X}) < \infty.$$

Let (f_n) be a sequence in \mathcal{F} satisfying

$$\lim_{n \rightarrow \infty} \int_{\mathbb{X}} f_n d\mu = M.$$

Let

$$g_n(x) = \max_{j=1}^n f_j(x). \quad \text{and} \quad f(x) = \sup_n f_n(x)$$

then, g_n (which is a sequence in \mathcal{F}) increases to f pointwise, and

$$M \geq \int_{\mathbb{X}} g_n d\mu \geq \int_{\mathbb{X}} f_n d\mu \rightarrow M.$$

It follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{X}} g_n d\mu = M.$$

By monotone convergence,

$$\int_{\mathbb{X}} f d\mu = M,$$

from which we conclude that f is finite a.e., hence, without loss of generality, we may assume that it is real-valued everywhere.

We will show now that

$$d\nu = (d\nu - f d\mu) + f d\mu \stackrel{\text{def}}{=} d\lambda + d\rho$$

satisfies the desired properties. The measure λ is in fact positive, because $g_n \in \mathcal{F}$, hence for every set $A \in \Sigma$,

$$\nu(A) - \int_A g_n d\mu \geq 0.$$

Letting $n \rightarrow \infty$ we obtain that

$$\lambda(A) = \lim_{n \rightarrow \infty} \left(\nu(A) - \int_A g_n d\mu \right) \geq 0.$$

Suppose that λ was *not* singular with respect to μ . By Lemma 4.16,

$$\exists \varepsilon > 0 \text{ and } \exists E \in \Sigma \text{ such that } \mu(E) > 0 \text{ and } E \text{ is } (\lambda - \varepsilon\mu)\text{-positive.}$$

In particular, for every $A \in \Sigma$,

$$\begin{aligned} 0 &\leq \lambda(A \cap E) - \varepsilon\mu(A \cap E) \\ &= \nu(A \cap E) - \int_{A \cap E} f d\mu - \varepsilon\mu(A \cap E) \\ &= \nu(A \cap E) - \int_{A \cap E} (f + \varepsilon) d\mu. \end{aligned}$$

Consider the function $f + \varepsilon\chi_E$. For all $A \in \Sigma$,

$$\begin{aligned} \int_A (f + \varepsilon\chi_E) d\mu &= \int_{A \cap E^c} f d\mu + \int_{A \cap E} (f + \varepsilon) d\mu \\ &\leq \nu(A \cap E^c) + \nu(A \cap E) \\ &= \nu(A). \end{aligned}$$

We deduce that $f + \varepsilon\chi_E \in \mathcal{F}$. On the other hand,

$$\int_{\mathbb{X}} (f + \varepsilon\chi_E) d\mu = M + \varepsilon\mu(E) > M,$$

which contradicts the definition of M as a supremum.

We have thus decomposed ν into a $\lambda \perp \mu$ and $f d\mu$, which is absolutely-continuous with respect to μ . We still need to prove the uniqueness of the decomposition. Suppose that

$$d\nu = d\lambda + f d\mu = d\lambda' + f' d\mu.$$

Then,

$$d\lambda - d\lambda' = (f' - f) d\mu,$$

where $\lambda - \lambda' \perp \mu$ and $\lambda - \lambda' \ll \mu$, from which we conclude (Lemma 4.13) that $\lambda = \lambda'$ and

$$(f' - f) d\mu = 0,$$

i.e., for every set A ,

$$\int_A f d\mu = \int_A f' d\mu,$$

it follows from Proposition 3.35 that $f = f'$ μ -a.e.

Step 2: the σ -finite positive case. Partition \mathbb{X} into a countable collection of disjoint sets, \mathbb{X}_n , that are both μ -finite and ν -finite. Set

$$\mu_j(E) = \mu(E \cap \mathbb{X}_j) \quad \text{and} \quad \nu_j(E) = \nu(E \cap \mathbb{X}_j).$$

These are finite positive measures on \mathbb{X} . Hence, there exist unique measures λ_j and integrable functions f_j , such that

$$d\nu_j = d\lambda_j + f_j d\mu_j = d\lambda_j + f_j \chi_{\mathbb{X}_j} d\mu.$$

Note that $\lambda_j \perp \mu_j$, from which follows that $\lambda_j \perp \mu$. Setting

$$\lambda = \sum_{n=1}^{\infty} \lambda_j \quad \text{and} \quad f = \sum_{n=1}^{\infty} f_n \chi_{A_n},$$

we obtain the desired partition.

Step 3: the σ -finite signed case. Let ν be a signed measure. Then ν^\pm are σ -finite positive measures, and there exist unique decompositions,

$$d\nu^\pm = d\lambda^\pm + f^\pm d\mu.$$

Since ν is a signed measure, either ν^+ or ν^- is a finite measure, hence so are either λ^+ or λ^- ; likewise, either f^+ or f^- is integrable. It follows that

$$d\nu = (d\lambda^+ - d\lambda^-) + (f^+ - f^-) d\mu$$


satisfies the requirements. ■

Example: Suppose that P is a probability measure on $(\mathbb{R}, \mathcal{L})$ which is absolutely continuous with respect to Lebesgue measure m , i.e., $P \ll m$. Then there exists a non-negative function $f \in L^1(m)$, such that


$$dP = f dm.$$

The function f is called the **probability density function** (צפיפות ההסתברות) of the probability measure P . All the continuous random variables encountered in the first Probability course have distributions (which are probability measures on \mathbb{R}) of this form. ▲ ▲ ▲


—27h(2017)—

 **Exercise 4.8** Show that if μ, ν, λ are σ -finite measures with $\mu \ll \nu$ and $\nu \ll \lambda$, then $\mu \ll \lambda$ and the “chain rule” holds:

$$\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \frac{d\nu}{d\lambda} \quad \lambda\text{-a.e.}$$

 **Exercise 4.9** Prove that if $\mu \ll \nu$ and $\nu \ll \mu$, then

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu} \right)^{-1} \quad \text{a.e.}$$


 **Exercise 4.10** Let μ_1 and μ_2 be finite (positive) measures on (\mathbb{X}, Σ) . Show that there exist measurable sets $A \sqcup B = \mathbb{X}$, such that

$$\mu_1 \perp \mu_2$$

on $(A, \Sigma \cap A)$ and

$$\mu_1 \ll \mu_2 \ll \mu_1$$

on $(B, \Sigma \cap B)$. (Hint: show first that $\mu_1, \mu_2 \ll \mu_1 + \mu_2$ and apply the Radon-Nikodym theorem.)

 **Exercise 4.11** Let μ, ν be positive σ -finite measures on (\mathbb{X}, Σ) . Show that there exists a non-negative measurable function φ , and a measurable set S , such that for all $A \in \Sigma$,

$$\nu(A) = \nu(A \cap S) + \int_A \varphi d\mu.$$

4.3 Differentiation in Euclidean space

We saw that there was a correspondence between L^+ functions and measures through integration, and more generally between extended-integrable functions and signed measure (or between L^1 function and finite signed measures). The Radon-Nikodym theorem states that given a reference measure μ , every signed measure can be decomposed into a measure that is singular with respect to μ and a

signed measure induced by an extended-integrable function (its Radon-Nikodym derivative). In this section, we start relating the Radon-Nikodym theorem to the analysis of real functions on \mathbb{R}^n . We consider the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$; the reference measure is the standard Borel measure m .

Suppose that $\nu \ll m$ is a signed measure. By the Radon-Nikodym theorem, there exists an extended-integrable function, f , such that $d\nu = f dm$, i.e.,

$$\nu(A) = \int_A f dm.$$

Fix a point $x \in \mathbb{R}^n$. For every $r > 0$, consider the ratio

$$\frac{\nu(B_r(x))}{m(B_r(x))} = \frac{1}{m(B_r(x))} \int_{B_r(x)} f dm \stackrel{\text{def}}{=} \oint_{B_r(x)} f dm,$$

where \oint denotes a volume-averaged integral. If the limit of this ratio as $r \rightarrow 0$ exists, then we would expect it to coincide with f .

Definition 4.18 A measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **locally-integrable** (אינטגר בילית מקומית) if for every bounded measurable set E ,

$$\int_E |f| dm < \infty.$$

The space of locally-integrable functions on \mathbb{R}^n is denoted $L^1_{loc}(\mathbb{R}^n)$.

Definition 4.19 For $f \in L^1_{loc}(\mathbb{R}^n)$ we define its local average over a ball of radius r ,

$$A_r f(x) = \oint_{B_r(x)} f dm.$$

(Note that $A_r f(x)$ is finite by the local integrability of f .)

Lemma 4.20 If $f \in L^1_{loc}(\mathbb{R}^n)$, then $A_r f(x)$ is continuous, jointly in x and in r for $r > 0$.

Proof: The volume of an n -Ball of radius r is given by

$$m(B_r(x)) = \alpha_n r^n,$$

where $\alpha_n = \pi^{n/2}/\Gamma(n/2 + 1)$; this function is jointly continuous in x and r . Thus, we only need to show that

$$(x, r) \mapsto \int_{B_r(x)} f \, dm = \int_{\mathbb{R}^n} \chi_{B_r(x)} f \, dm$$

is continuous. That is, we need to show that for $x_k \rightarrow x$ and $r_k \rightarrow r$,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \chi_{B_{r_k}(x_k)} f \, dm = \int_{\mathbb{R}^n} \chi_{B_r(x)} f \, dm.$$

This follows from dominated convergence, where for large enough n ,

$$|\chi_{B_{r_k}(x_k)} f| \leq \chi_{B_{r+1}(x)} |f| \in L^1(m).$$

■

The following definition introduces a nonlinear operator on locally-integrable functions, which has many uses in functional and harmonic analysis:

Definition 4.21 Let $f \in L^1_{loc}(\mathbb{R}^n)$. Its **Hardy-Littlewood maximal function** (פונקציית פונקציה) Hf is

$$Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \int_{B_r(x)} |f| \, dm.$$

—27h(2018)—

Proposition 4.22 Hf is measurable.

Proof: For every $t \in \mathbb{R}$,

$$\begin{aligned} Hf^{-1}((t, \infty)) &= \{x \in \mathbb{R}^n : \sup_{r>0} A_r |f|(x) > t\} \\ &= \bigcup_{r>0} (A_r |f|)^{-1}((t, \infty)). \end{aligned}$$

The latter is open (hence measurable) because $A_r |f|$ is continuous. ■

The following lemma is a combinatorial and geometric result commonly used in the measure theory of Euclidean spaces:

Lemma 4.23 (Vitali covering lemma) Let \mathcal{C} be a collection of open balls in \mathbb{R}^n and let U be their union. Then, for every $c < m(U)$ there exists a finite collection of disjoint balls $B_1, \dots, B_k \in \mathcal{C}$, such that

$$\sum_{j=1}^k m(B_j) > \frac{c}{3^n}.$$

Proof: By the regularity of the Borel measure,

$$m(U) = \sup\{m(K) : K \Subset U\},$$

where $K \Subset U$ denotes that $K \subset U$ and K is compact. Hence, there exists for every $c < m(U)$ a compact set $K \Subset U$ such that

$$m(K) > c.$$

By compactness, there are finitely many balls in \mathcal{C} , A_1, \dots, A_m whose union covers K . Let B_1 be the largest of these balls, let B_2 be the largest of the remaining balls that are disjoint of B_1 , and proceed until there are no longer balls A_j that are disjoint of the B_j 's. By construction, if a ball A_i is not one of the B_j 's, then there is a ball B_j intersecting it; take this B_j to be the ball of largest radius intersecting A_i . Then, the radius of A_i is at most the radius of B_j (otherwise this A_i would have been picked). Thus, A_i is contained in a ball B_j^* concentric with B_j and having a radius 3 times as large. It follows that

$$K \subset \bigcup_{j=1}^k B_j^*,$$

hence

$$\sum_{j=1}^k m(B_j) = \frac{1}{3^n} \sum_{j=1}^k m(B_j^*) \geq \frac{m(K)}{3^n} > \frac{c}{3^n},$$

where we use the homogeneity of the volume of a Euclidean ball as function of its radius. ■

TA material 4.1 Vitali's covering theorem for the Lebesgue measure.

Theorem 4.24 (Maximal theorem) For all $f \in L^1(m)$ and all $\alpha > 0$,

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{3^n}{\alpha} \|f\|_{L^1(m)}.$$

Proof: Let

$$E_\alpha = \{x : Hf(x) > \alpha\}.$$

By the definition of the maximal function, for each $x \in E_\alpha$, there exists an $r(x)$ such that

$$A_{r(x)}|f|(x) = \int_{B_{r(x)}(x)} |f| dm > \alpha,$$

or equivalently,

$$m(B_{r(x)}(x)) < \frac{1}{\alpha} \int_{B_{r(x)}(x)} |f| dm.$$

The balls $B_{r(x)}(x)$, $x \in E_\alpha$ cover E_α . By Vitali's lemma, there exist for every $c < m(E_\alpha)$ points x_1, \dots, x_k , such that the balls $B_{r(x_j)}(x_j)$ are disjoint, and

$$\sum_{j=1}^k m(B_{r(x_j)}(x_j)) > \frac{c}{3^n}.$$

That is,

$$\begin{aligned} c &< 3^n \sum_{j=1}^k m(B_{r(x_j)}(x_j)) \\ &< \frac{3^n}{\alpha} \sum_{j=1}^k \int_{B_{r(x_j)}(x_j)} |f| dm \\ &= \frac{3^n}{\alpha} \int_{\bigcup_{j=1}^k B_{r(x_j)}(x_j)} |f| dm \leq \frac{3^n}{\alpha} \|f\|_{L^1(m)}, \end{aligned}$$

where in the last step we used the disjointness of the balls $B_{r(x_j)}(x_j)$. Letting $c \rightarrow m(E_\alpha)$ we obtain the desired result. \blacksquare

With that, we turn to the main theorem of this section:

Theorem 4.25 Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then,

$$\lim_{r \rightarrow 0} A_r f \rightarrow f \quad m\text{-a.e.}$$

Proof: It suffices to prove the theorem for $f \in L^1(m)$. This would imply that for $f \in L^1_{loc}(\mathbb{R}^n)$,

$$\lim_{r \rightarrow 0} A_r(\chi_{B_1(0)} f) \rightarrow \chi_{B_1(0)} f. \quad m\text{-a.e.}$$

Since $B_1(0)$ is open, for every $x \in B_1(0)$, the ball $B_r(x)$ is eventually in $B_1(0)$, i.e., $A_r(\chi_{B_1(0)} f)(x) = A_r f(x)$, which implies that

$$\lim_{r \rightarrow 0} A_r f \rightarrow f \quad \text{for } m\text{-a.e. } x \in B_1(0).$$

Taking $\{x_i\}$ to be a countable $1/2$ -dense net in \mathbb{R}^n ,

$$\lim_{r \rightarrow 0} A_r f \rightarrow f \quad \text{for } m\text{-a.e. } x \in \bigcup_{i=1}^{\infty} B_1(x_i) = \mathbb{R}^n.$$

So let $f \in L^1(m)$ and let $\varepsilon > 0$. Since the continuous functions are dense in $L^1(m)$, there exists a continuous integrable function g , such that

$$\|f - g\|_{L^1(m)} < \varepsilon.$$

Since g is continuous¹, then for all $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0} |A_r g(x) - g(x)| = 0.$$

It follows that,

$$\begin{aligned} \limsup_{r \rightarrow 0} |A_r f(x) - f(x)| &\leq \limsup_{r \rightarrow 0} |A_r f(x) - A_r g(x)| \\ &\quad + \lim_{r \rightarrow 0} |A_r g(x) - g(x)| + |g(x) - f(x)| \\ &= \limsup_{r \rightarrow 0} |A_r(f - g)|(x) + |g(x) - f(x)| \\ &\leq H(f - g)(x) + |g(x) - f(x)|. \end{aligned}$$

Note that we only have control on the L^1 -norm of $g - f$, however here, we need to control it pointwise. Let $\alpha > 0$ be arbitrary and let

$$\begin{aligned} E_\alpha &= \left\{ x : \limsup_{r \rightarrow 0} |A_r f(x) - f(x)| > \alpha \right\} \\ F_\alpha &= \{ x : |g(x) - f(x)| > \alpha \} \\ G_\alpha &= \{ x : H(f - g)(x) > \alpha \}. \end{aligned}$$

Since for $c = a + b$, $c > \alpha$ only if $a > \alpha/2$ or $b > \alpha/2$, it follows that $x \in E_\alpha$ only if $x \in F_{\alpha/2}$ or $x \in G_{\alpha/2}$, i.e.,

$$E_\alpha \subset F_{\alpha/2} \cup G_{\alpha/2}.$$

Hence,

$$m(E_\alpha) \leq m(F_{\alpha/2}) + m(G_{\alpha/2}).$$

Now,

$$m(F_{\alpha/2}) = \int_{F_{\alpha/2}} dm \leq \frac{2}{\alpha} \int_{F_{\alpha/2}} |g - f| dm \leq \frac{2}{\alpha} \|g - f\|_{L^1(m)} \leq \frac{2\varepsilon}{\alpha},$$

which really is Markov's inequality from probability theory. By the maximal theorem (Theorem 4.24),

$$m(G_{\alpha/2}) = m(\{x : H(f - g)(x) > \alpha/2\}) \leq \frac{2 \cdot 3^n}{\alpha} \|g - f\|_{L^1(m)} \leq \frac{2 \cdot 3^n \varepsilon}{\alpha}.$$

Thus, for every $\varepsilon > 0$,

$$m(E_\alpha) \leq \frac{2\varepsilon}{\alpha} + \frac{2 \cdot 3^n \varepsilon}{\alpha},$$

which implies that $m(E_\alpha) = 0$. It follows that for all $k > 0$,

$$m\left(\left\{x : \limsup_{r \rightarrow 0} |A_r f(x) - f(x)| > 1/k\right\}\right) = 0.$$

It follows that

$$m\left(\left\{x : \lim_{r \rightarrow 0} A_r f(x) \neq f(x)\right\}\right) = m\left(\bigcup_{k=1}^{\infty} \left\{x : \limsup_{r \rightarrow 0} |A_r f(x) - f(x)| > 1/k\right\}\right) = 0.$$

■

¹This fact is well-known for Riemann integration; since continuous functions are Riemann-integrable, this fact holds for Lebesgue integration as well.

Comment: This theorem can be rewritten in the following form,

$$\lim_{r \rightarrow 0} \int_{B_r(x)} (f(x) - f) dm = 0 \quad \text{for a.e. } x. \quad (4.1)$$

In fact, we can prove a much stronger result, replacing the integrand in (4.1) with its absolute value.

Definition 4.26 Let $f \in L^1_{loc}(\mathbb{R}^n)$. Its **Lebesgue set** (קבוצת לֵבְנֵה) is

$$L_f \stackrel{\text{def}}{=} \left\{ x : \lim_{r \rightarrow 0} \int_{B_r(x)} |f(x) - f| dm = 0 \right\}.$$

A point in L_f is called a **Lebesgue point** (נקודת לֵבְנֵה) of f .

Theorem 4.27 If $f \in L^1_{loc}(\mathbb{R}^n)$, then almost every point in \mathbb{R}^n is a Lebesgue point.

Proof: For every $c \in \mathbb{R}$, applying Theorem 4.25 with f replaced by $|f - c|$,

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f - c| dm = |f(x) - c|, \quad (4.2)$$

everywhere except for an m -null set E_c . Let D be a countable dense set in \mathbb{R} and let

$$E = \bigcup_{c \in D} E_c,$$

which is an m -null set. Let $x \notin E$, i.e., (4.2) holds for every $c \in D$. For every $\varepsilon > 0$ there exists $c \in D$, such that

$$|f(x) - c| < \varepsilon.$$

Hence,

$$\begin{aligned} \limsup_{r \rightarrow 0} \int_{B_r(x)} |f - f(x)| dm &\leq \limsup_{r \rightarrow 0} \int_{B_r(x)} |f - c| dm + \limsup_{r \rightarrow 0} \int_{B_r(x)} |f(x) - c| dm \\ &= 2|f(x) - c| < 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result. ■

Thus far, we only considered the convergence over averages over shrinking balls. We will need a stronger version of those results.