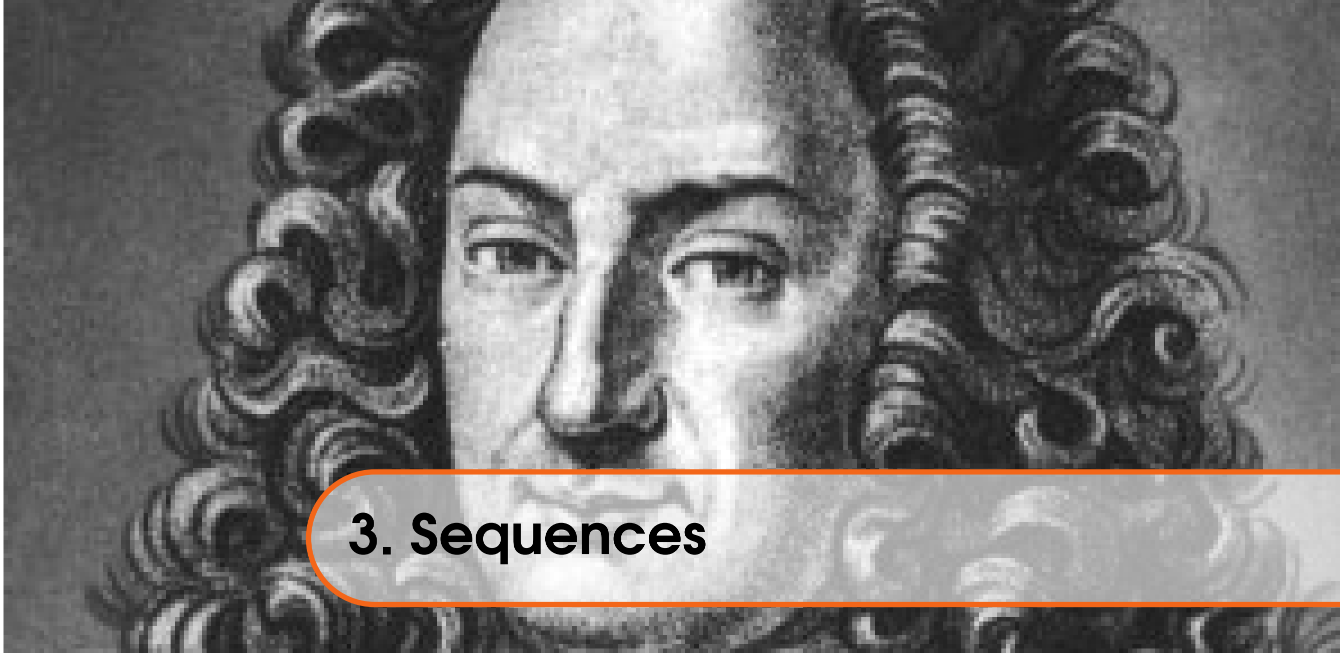


3. Sequences

3.1 Basic definitions

Definition 3.1 An (infinite) ***sequence*** is a function from the naturals to the real numbers. That is, it is an assignment of a real number to every natural number.

Comment 3.1 This is the first time we meet the notion of a ***function***, which will be the central concept of the next chapter. As for now, we take this (very nontrivial) notion as evident.

Notation: Sequences, like any other functions, are labeled by letters. We may refer, for example, to the sequence a. The value that a returns for, say, the input 3 is denoted by a_3, rather than $a(3)$. More generally, we denote by a_n the value that the function a returns for the input n. The subscript n in a_n is called the ***index*** (**אינדקס**) of that element.

But sequences are very special functions as their domain of definition (**תחום הגדרה**) is an inductive set. Thus, we can refer to the first element and to a ***successor*** or a ***predecessor*** of a certain element.

A more common notation for the sequence a is as follows,

$$(a_n)_{n=1}^{\infty}.$$

Note, however, that the index n in this notation is a ***dummy variable*** (**משתנה סרק**). We could have as well written

$$(a_k)_{k=1}^{\infty}.$$

When there is no risk of confusion, we will denote the sequence simply by (a_n) (which should not be confused with its n-th element a_n).

Sequences can be defined in various ways. The most common way of defining a function is by providing a ***formula*** for the n-th element of that sequence (i.e., a rule for calculating a_n given n). Another way of defining a function is based on the inductive property of $\mathbb{N}$: the first element is specified along with the formula for calculating a_{n+1} given a_n (or, more generally, given $a_1, a_2, \dots, a_n$). Such a definition is called ***recursive***.

■ **Examples 3.1**

(a) The constant sequence: $a_n = 5$.
(b) The sequence of naturals: $a_n = n$.
(c) An alternating sequence: $b_n = (-1)^n$.
(d) The harmonic sequence: $c_n = 1/n$.
(e) The sequence of primes $(d_n) = (2,3,5,7,11,\dots)$. Note that we do not have an explicit formula for d_n.
(f) The sequence of digits of π: $(e_n) = (3,1,4,1,5,9,\dots)$.
(g) The Fibonacci sequence: $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$.

■

Comment 3.2 It is very common to refer, say, to the harmonic sequence as "the sequence $1/n$". While very intuitive, this way of reference is problematic. How does it differ, for example, from the sequence $1/k$ or the sequence $1/m$? On the other hand, the notation

$$(1/n)_{n=1}^{\infty}$$

is unambiguous (here n is again a dummy variable).

Comment 3.3 We have to distinguish a sequence $(a_n)_{n=1}^{\infty}$ from the set of values that the sequence assumes,

$$\{a_n : n \in \mathbb{N}\}.$$

For the example, if

$$(a_n)_{n=1}^{\infty} = (0,1,0,0,1,1,0,0,0,1,1,1,\dots),$$

then the set of values it assumes is

$$\{a_n : n \in \mathbb{N}\} = \{0,1\}.$$

In a set, every element appears once and there is no order among elements.

3.2 Limits of sequences

Consider the harmonic sequence,

$$a_n = 1/n.$$

Its elements are positive and decreasing (every element is smaller than its successor). While no element equals zero, we understand on an intuitive level that "the sequence tends toward zero". In this section we will define formally what it means for a sequence to tend to some real number (there is nothing special about tending to zero).

Let's start to construct a definition to the statement "the sequence (a_n) tends to the real number α". Very informally, we would say that this means that "when n is very large, a_n is very close to α". This is, of course, not a mathematical statement. What does "n very large" mean? And what does "very close to α" mean?

Let's start by making the "very close to α" clause more rigorous. How can we measure a distance from α? The distance of a number x from α is the absolute value $|x-\alpha|$. When we say that the distance of x from α is less than some $r>0$, we mean that

$$|x-\alpha|<r.$$

Definition 3.2 Given $\alpha\in\mathbb{R}$ and $r>0$, we define the ***open ball*** (כדור פתוח) of radius r about α by

$$B(\alpha,r)=\{x\in\mathbb{R}\ :\ |x-\alpha|<r\}.$$

The term "ball" is natural when we think about the same definition in three-dimensional space.

Equipped with this new definition, we will try to refine our definition of a sequence tending to a number. (a_n) tends to α if for every $r>0$ and sufficiently large n, $a_n\in B(\alpha,r)$. This is still not good enough. What do we mean now by sufficiently large n? Think of the harmonic sequence: no matter how small r is, from some n onwards, *all* the elements of the sequence are in $B(0,r)$.

This observation motivates the following definition:

Definition 3.3 Let P_n be a sequence of logical propositions, which can be either True or False. We say that the propositions hold for sufficiently large n, if there exists an index $N\in\mathbb{N}$ such that for all $n>N$, $P_n=$ True. In formal notation,

$$(\exists N\in\mathbb{N})(\forall n>N)(P_n=\text{True}).$$

With that we can finally define what it means for a sequence to tend to a number:

Definition 3.4 A sequence (a_n) ***converges*** (מתכנסת) to $\alpha\in\mathbb{R}$ if for every $\varepsilon>0$, the elements of the sequence are in $B(\alpha,\varepsilon)$ for sufficiently large n. Formally,

$$(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n>N)(a_n\in B(\alpha,\varepsilon)),$$

or

$$(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n>N)(|a_n-\alpha|<\varepsilon).$$

We call the real number α a ***limit*** (גבול) of the sequence, and denote the fact that

α is a limit of the sequence (a_n) by

$$\lim_{n\to\infty} a_n = \alpha.$$

Other popular notation are

$$a_n \to \alpha \qquad \text{or} \qquad a_n \xrightarrow{n\to\infty} \alpha.$$

Comment 3.4 Note the *a limit* rather than *the limit*. We don't yet know that a limit of a sequence, if it exists, is unique.

Definition 3.5 A sequence is called ***convergent*** (מתכנסת) if it has a limit; otherwise it is called ***divergent*** (מתבדרת).

■ **Example 3.1** The simplest example to start with is the constant sequence

$$a_n = \alpha.$$

It seems obvious that this sequence tends to α. We have to be careful, and make sure that α is the limit *according to the definition*.
That is, we have to prove that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - \alpha| < \varepsilon).$$

Substituting the value of a_n, we have to prove that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|\underbrace{\alpha - \alpha}_{0}| < \varepsilon).$$

This is trivially true. Given any $\varepsilon > 0$ we may take $N = 1$. Indeed, for every $n > N$, $|a_n - \alpha| = 0 < \varepsilon$. ■

■ **Example 3.2** Consider next the harmonic sequence $a_n = 1/n$. We want to show that

$$\lim_{n\to\infty} a_n = 0,$$

namely that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(1/n < \varepsilon).$$

Take for example $\varepsilon = 0.01$. All the elements a_n of the sequence are in $B(0, 0.01)$ from $n = 101$. More generally, let $\varepsilon > 0$ be given. Take $N = \lceil 1/\varepsilon \rceil$. Then, for all $n > N$,

$$|a_n = 0| = \frac{1}{n} < \frac{1}{N} = \frac{1}{\lceil 1/\varepsilon \rceil} < \varepsilon,$$

which completes the proof. ■

■ **Example 3.3** Let $a_n = \sqrt{n+1} - \sqrt{n}$, or

$$(a_n) = (\sqrt{2} - \sqrt{1}, \sqrt{3} - \sqrt{2}, \sqrt{4} - \sqrt{3}, \dots).$$

If you evaluate the elements of this sequence you'll quickly guess that

$$\lim_{n\to\infty} a_n = 0.$$

The question is whether we can *prove* that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(\sqrt{n+1} - \sqrt{n} < \varepsilon)\ ?$$

We will use the following algebraic trick,

$$a_n = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

In order to have $|a_n - 0| < \varepsilon$ we can take n to be greater than $(1/2\varepsilon)^2$. Therefore, given $\varepsilon > 0$, we take

$$N = \left\lceil \left(\frac{1}{2\varepsilon}\right)^2 \right\rceil.$$

Then for all $n > N$,

$$|a_n - 0| < \frac{1}{2\sqrt{n}} < \frac{2\varepsilon}{2} = \varepsilon,$$

which completes the proof. ∎

■ **Example 3.4** Consider next the sequence

$$a_n = \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63}.$$

Start with intuition. As n becomes very large, the numerator is dominated by the $3n^3$ term, whereas the denominator is dominated by the $4n^3$ term. It makes sense to guess that as n becomes larger and larger, the sequence approaches a constant,

$$\lim_{n\to\infty} a_n = \frac{3}{4}.$$

To prove it we need to show that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - 3/4| < \varepsilon).$$

This requires some work. Consider the difference,

$$\begin{aligned} a_n - \frac{3}{4} &= \frac{4(3n^3 + 7n^2 + 1) - 3(4n^3 - 8n + 63)}{4(4n^3 - 8n + 63)} \\ &= \frac{28n^2 - 24n - 185}{4(4n^3 - 8n + 63)} < \frac{28n^2}{16n^3 - 32n} = \frac{7n}{7n^2 - 8}. \end{aligned}$$

For $n > 3$, $n^2 > 8$, which implies that $7n^2 - 8 > 6n^2$, hence for all $n > 3$,

$$\left| a_n - \frac{3}{4} \right| < \frac{7n}{6n^2} = \frac{7}{6n}.$$

We can now close the proof. Given $\varepsilon > 0$, let

$$N = \max(3, 6\varepsilon/7).$$

Then, for every $n > N$,

$$\left| a_n - \frac{3}{4} \right| < \frac{7}{6n} < \varepsilon.$$

■

■ **Example 3.5** Let $\alpha \in \mathbb{R}$. We will show that there exists a sequence (r_n) of rational numbers that converges to α. The idea is very simple. For every n consider the open ball $B(\alpha, 1/n)$. By the density of the rationals, there exists a rational number $r_n \in B(\alpha, 1/n)$. Pick one. This constructs a sequence (which we don't care to know explicitly).

This sequence converges to α, because given $\varepsilon > 0$, let $N = \lceil 1/\varepsilon \rceil$. Then, for every $n > N$,

$$r_n \in B(\alpha, 1/n) \subset B(\alpha, \varepsilon).$$

■

3.3 Uniqueness of the limit

A converging sequence has a limit. The question is whether it is possible to converge to two different limits. We will show that the limit is unique, thus justifying the reference to *the* limit of a converging sequence. The rationale behind the proof is very simple. If a sequence (a_n) converges to α, then for any (small) interval around α, the sequence must eventually be within this interval. If the sequence also converges to β, then for any (small) interval around β, the sequence must eventually be within this interval. We can take those intervals sufficiently small so that they are disjoint (זרים), leading to a contradiction.

Let's proceed step by step:

Lemma 3.1 Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$. Then there exists an $\varepsilon > 0$ such that $B(\alpha, \varepsilon)$ and $B(\beta, \varepsilon)$ are disjoint.

Proof. Suppose, without loss of generality (a notion we have to discuss), that $\alpha < \beta$. Let $\varepsilon = (\beta - \alpha)/2$. Then,

$$B(\alpha, \varepsilon) = \left(\tfrac{3}{2}\alpha - \tfrac{1}{2}\beta, \tfrac{1}{2}(\alpha + \beta)\right)$$

and

$$B(\beta, \varepsilon) = \left(\tfrac{1}{2}(\alpha + \beta), \tfrac{3}{2}\beta - \tfrac{1}{2}\alpha\right),$$

and these two open segments are indeed disjoint. ■

Lemma 3.2 Let P_n and Q_n be two sequences of propositions (assuming values True and False). If P_n holds for large enough n and Q_n holds for large enough n, then $P_n \wedge Q_n$ holds for large enough n.

Proof. It is given that

$$(\exists N_1)(\forall n > N_1)(P_n = \text{True}),$$

and

$$(\exists N_2)(\forall n > N_2)(Q_n = \text{True}).$$

Let $N = \max(N_1, N_2)$. Then, for all $n > N$, both $n > N_1$ and $n > N_2$, hence

$$P_n = \text{True} \qquad \text{and} \qquad Q_n = \text{True}.$$

■

Theorem 3.3 — Uniqueness of the limit. Let (a_n) be a convergent sequence. If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are limits of (a_n), then $\alpha = \beta$.

Proof. Assume, by contradiction that $\alpha \neq \beta$. By Lemma 3.1 there exists an $\varepsilon > 0$ such that

$$B(\alpha, \varepsilon) \cap B(\beta, \varepsilon) = \varnothing.$$

By the definition of the limit, $a_n \in B(\alpha, \varepsilon)$ for large enough n and $a_n \in B(\beta, \varepsilon)$ for large enough n. It follows from Lemma 3.2 that $a_n \in B(\alpha, \varepsilon) \cap B(\beta, \varepsilon)$ for large enough n, which is impossible. ■

We conclude this section by discussing ***divergent*** sequences. A sequence is divergent if it does not have a limit. In other words,

$$\forall \alpha \in \mathbb{R} \quad \alpha \text{ is not a limit of } (a_n).$$

This requires some elaboration. Since

$$\alpha \text{ is a limit of } (a_n) \iff (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - \alpha| < \varepsilon),$$

it follows that

$$\alpha \text{ is a not limit of } (a_n) \iff (\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n > N)(|a_n - \alpha| \geq \varepsilon).$$

Thus,

$$(a_n) \text{ is divergent} \iff (\forall \alpha \in \mathbb{R})(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n > N)(|a_n - \alpha| \geq \varepsilon).$$

■ **Example 3.6** Consider the sequence of natural, $a_n = n$. This sequence is divergent, for let $\alpha \in \mathbb{R}$. Take $\varepsilon = 1$. For all $N \in \mathbb{N}$, there exists an $n > N$ such that

$$a_n - \alpha \geq 1.$$

■

■ **Example 3.7** Consider the alternating sequence $a_n = (-1)^n$. We first claim that 1 is not a limit is this sequence, take $\varepsilon = 2$. For every $N \in \mathbb{N}$ there exists an $n > N$, such that $a_n = (-1)$, i.e.,

$$|a_n - 1| \geq 2 \qquad \text{or} \qquad a_n \notin B(2,1).$$

We next claim that no $\alpha \neq 1$ can be a limit of (a_n), for let $\varepsilon > 0$ be such that

$$B(\alpha,\varepsilon) \cap B(1,\varepsilon) = \varnothing.$$

For every $N \in \mathbb{N}$ there exists an $n > N$ such that $a_n = 1$, namely, $a_n \notin B(\alpha,\varepsilon)$. ■

3.4 Bounds and order

Definition 3.6 A sequence (a_n) is said to be ***upper bounded*** (חסומה מלעיל) if there exists an $M \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})(a_n \leq M).$$

If is said to be ***lower bounded*** (חסומה מלרע) if there exists an $m \in \mathbb{R}$ such that

$$(\forall n \in \mathbb{N})(m \leq a_n).$$

It is said to be ***bounded*** if it is both upper bounded and lower bounded.

Comment 3.5 The sequence (a_n) is upper (resp. lower) bounded if and only if the set of values it assumes,

$$\{a_n \,:\, n \in \mathbb{N}\}$$

is upper (resp. lower) bounded. The property of being bounded does not “see" the order within the sequence.

■ **Example 3.8**

1. The sequence of naturals, $a_n = n$, is lower bounded by not upper bounded.
2. The harmonic sequence is bounded.
3. The sequence $a_n = (-1)^n n$ is neither upper nor lower bounded.

■

Theorem 3.4 A convergent sequence is bounded.

Proof. Let (a_n) be a sequence that converges to a limit α. We need to show that there exist L_1, L_2 such that

$$L_1 \leq a_n \leq L_2 \qquad \forall n \in \mathbb{N}.$$

By definition, setting $\varepsilon = 1$,

$$(\exists N \in \mathbb{N})(\forall n > N)(a_n \in B(\alpha,1)),$$

or,

$$(\exists N \in \mathbb{N})(\forall n > N)(\alpha - 1 < a_n < \alpha + 1).$$

Since

$$\{a_n : 1 \le n \le N\}$$

is a finite set, there exist

$$M = \max\{a_n : 1 \le n \le N\}$$
$$m = \min\{a_n : 1 \le n \le N\}.$$

Then for all n,

$$\min(m, \alpha - 1) \le a_n \le \max(M, \alpha + 1).$$

■

Proposition 3.5 Suppose that (a_n) and (b_n) are convergent sequences,

$$\lim_{n\to\infty} a_n = \alpha \qquad \text{and} \qquad \lim_{n\to\infty} b_n = \beta.$$

Suppose that $\alpha > \beta$. Then there exists an $N \in \mathbb{N}$, such that

$$b_n > a_n \qquad \text{for all} \qquad n > N,$$

i.e., the sequence (b_n) is eventually greater (term-by-term) than the sequence (a_n).

Proof. By Lemma 3.1, there exists an $\varepsilon > 0$ such that

$$A = B(\alpha, \varepsilon) \qquad \text{and} \qquad B = B(\beta, \varepsilon)$$

are disjoint. In fact, every element in A is smaller than every element in B. By Lemma 3.2 there exists an $N \in \mathbb{N}$ such that for all $n > N$, $a_n \in A$ and $b_n \in B$, which implies that $a_n < b_n$. ■

Corollary 3.6 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Let (a_n) be a convergent sequence with limit α. Then, $a_n < \beta$ for large enough n.

Proof. Apply the previous proposition with the constant sequence $b_n = \beta$. ■

Proposition 3.7 Suppose that (a_n) and (b_n) are convergent sequences,

$$\lim_{n\to\infty} a_n = \alpha \qquad \text{and} \qquad \lim_{n\to\infty} b_n = \beta,$$

and there exists an $N \in \mathbb{N}$, such that $a_n \le b_n$ for all $n > N$. Then $\alpha \le \beta$.

Proof. This is an immediate corollary of the previous proposition (reverse implication of the negations). ∎

Comment 3.6 If instead, $a_n < b_n$ for all $n > N$, then we still only have $\alpha \le \beta$. Take for example the sequences $a_n = 1/n$ and $b_n = 2/n$. Even though $a_n < b_n$ for all n, both converge to the same limit.

Theorem 3.8 — Sandwich. Suppose that (a_n) and (b_n) are sequences that converge to the same limit ℓ. Let (c_n) be a sequence for which there exists an $N \in \mathbb{N}$ such that

$$a_n \le c_n \le b_n \qquad \text{for all} \qquad n > N.$$

Then

$$\lim_{n\to\infty} c_n = \ell.$$

Proof. By the given assumptions and Lemma 3.2,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(-\varepsilon < a_n - \ell \quad \text{and} \quad b_n - \ell < \varepsilon \quad \text{and} \quad a_n \le c_n \le b_n).$$

Since,

$$-\varepsilon < a_n - \ell \quad \text{and} \quad b_n - \ell < \varepsilon \quad \text{and} \quad a_n \le c_n \le b_n \implies -\varepsilon < c_n - \ell < \varepsilon$$

It follows that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|c_n - \ell| < \varepsilon).$$

∎

▪ **Example 3.9** Since for all n,

$$1 < \sqrt{1 + 1/n} < \sqrt{1 + 2/n + 1/n^2} = 1 + 1/n,$$

it follows that

$$\lim_{n\to\infty} \sqrt{1 + 1/n} = 1.$$

▪

3.5 Limit arithmetic

Suppose we have two sequences (a_n) and (b_n). We can form new sequences, such as (c_n) given by

$$c_n = a_n + b_n,$$

a (d_n) given by

$$d_n = a_n b_n.$$

If the elements of b_n are non-zero, then we can also form a sequence (e_n), given by

$$e_n = \frac{1}{b_n}.$$

Suppose that (a_n) and (b_n) are both convergent sequences with limits α and β. Can we infer the convergence and limits of the sequences $(a_n + b_n)$, $(a_n b_n)$ and $(1/b_n)$?

Lemma 3.9 Let (a_n) be a sequence. Then the following statements are equivalent:

1. (a_n) converges to α.
2. $(a_n - \alpha)$ converges to zero.
3. $(|a_n - \alpha|)$ converges to zero.

Proof. Since

$$|a_n - \alpha| = |(a_n - \alpha) - 0| = ||a_n - \alpha| - 0|,$$

it follows that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - \alpha| < \varepsilon)$$
$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|(a_n - \alpha) - 0| < \varepsilon)$$
$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(||a_n - \alpha| - 0| < \varepsilon)$$

are equivalent statements. ■

Proposition 3.10 If (a_n) converge to α, then $(|a_n|)$ converges to $|\alpha|$.

Proof. From the reverse triangle inequality,

$$0 \le ||a_n| - |\alpha|| \le |a_n - \alpha|.$$

Since $(|a_n - \alpha|)$ converges to zero, we can invoke the sandwich theorem. ■

Comment 3.7 Note that the converse is not true. Set $a_n = (-1)^n$, then $(|a_n|)$ converges to 1, but (a_n) is divergent.

Lemma 3.11 If $x \in B(\alpha, r)$ and $y \in B(\beta, r)$ then $x + y \in B(\alpha + \beta, 2r)$.

Proof. This is immediate. We know that

$$\alpha - r < x < \alpha + r$$
$$\beta - r < y < \beta + r.$$

It only remain to “add" the two inequalities. ■

Theorem 3.12 — Limits of sums of sequences. Let (a_n) and (b_n) be convergent sequences. Then the sequence $c_n = a_n + b_n$ is also convergent, and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof. Denote the limits of (a_n) and (b_n) by α and β. Let $\varepsilon > 0$ be given. From the definition of the limit and Lemma 3.2,

$$a_n \in B(\alpha, \varepsilon/2) \qquad \text{and} \qquad b_n \in B(\beta, \varepsilon/2)$$

for n large enough. Invoking Lemma 3.11, we obtain that

$$c_n \in B(\alpha + \beta, \varepsilon)$$

for n large enough, which implies that

$$\lim_{n\to\infty} c_n = \alpha + \beta.$$

■

Comment 3.8 Note that the converse is not true. $(a_n + b_n)$ may be convergent, whereas (a_n) and (b_n) are not.

Comment 3.9 The theorem about the limit of a sum of two sequences can be readily extended to any *finite* sum of sequences.

Comment 3.10 By a similar argument we may show that

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n,$$

provided that the right-hand side exists.

Lemma 3.13 Let $\varepsilon > 0$ and let $\alpha, \beta \in \mathbb{R}$. If

$$x \in B\left(\alpha, \min\left(1, \tfrac{\varepsilon}{2(|\beta|+1)}\right)\right) \qquad \text{and} \qquad y \in B\left(\beta, \min\left(1, \tfrac{\varepsilon}{2(|\alpha|+1)}\right)\right),$$

then

$$xy \in B(\alpha\beta, \varepsilon).$$

Proof. Start with

$$xy - \alpha\beta = (x - \alpha)y + \alpha(y - \beta).$$

Using the triangle inequality,

$$|xy - \alpha\beta| \le |x - \alpha||y| + |\alpha||y - \beta|.$$

Since $|y| < |\beta| + 1$. $|x - \alpha| < \varepsilon/2(|\beta| + 1)$ and $|y - \beta| < \varepsilon/2(|\alpha| + 1)$, it follows that

$$|xy - \alpha\beta| < (|\beta| + 1)\frac{\varepsilon}{2(|\beta| + 1)} + |\alpha|\frac{\varepsilon}{2(|\alpha| + 1)} < \varepsilon,$$

which concludes the proof. ■

Theorem 3.14 — Limits of products of sequences. Let (a_n) and (b_n) be convergent sequences. Then the sequence $c_n = a_n b_n$ is also convergent, and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

Proof. Denote the limits of (a_n) and (b_n) by α and β. Let $\varepsilon > 0$ be given. From the definition of the limit and Lemma 3.2,

$$a_n \in B\left(\alpha, \min\left(1, \tfrac{\varepsilon}{2(|\beta|+1)}\right)\right) \qquad \text{and} \qquad b_n \in B\left(\beta, \min\left(1, \tfrac{\varepsilon}{2(|\alpha|+1)}\right)\right)$$

for n large enough. Invoking Lemma 3.13, we obtain that

$$c_n \in B(\alpha\beta, \varepsilon)$$

for n large enough, which implies that

$$\lim_{n\to\infty} c_n = \alpha\beta.$$

■

Corollary 3.15 Let (a_n) be a convergence sequence and let $b \in \mathbb{R}$. Then, the sequence $(b a_n)$ is convergent with

$$\lim_{n\to\infty} (b a_n) = b \lim_{n\to\infty} a_n.$$

Proof. Apply Theorem 3.14 with the constant sequence $b_n = b$. ■

In remains to prove a sequence arithmetic theorem regarding the ratio of sequences.

Lemma 3.16 Let $\beta \neq 0$ and

$$y \in B\left(\beta, \min\left(\tfrac{|\beta|}{2}, \tfrac{|\beta|^2 \varepsilon}{2}\right)\right).$$

Then, $y \neq 0$ and

$$\frac{1}{y} \in B\left(\frac{1}{\beta}, \varepsilon\right).$$

Proof. It is given that

$$|y - \beta| < \frac{|\beta|}{2}.$$

Since $y = \beta - (\beta - y)$, it follows from the triangle inequality that

$$|y| \geq |\beta| - |\beta - y| > \frac{|\beta|}{2},$$

which proves that $y \neq 0$. Then,

$$\left|\frac{1}{y} - \frac{1}{\beta}\right| = \frac{|\beta - y|}{|y||\beta|} < \frac{|\beta|^2\varepsilon/2}{|\beta|/2|\beta|} = \varepsilon,$$

which concludes the proof. ■

Theorem 3.17 Let (b_n) be a convergent sequence whose limit is not zero. Then, the sequence $c_n = 1/b_n$ is well-defined for n large enough. Furthermore, it is convergent, and

$$\lim_{n\to\infty} c_n = \frac{1}{\lim_{n\to\infty} b_n}.$$

Proof. Denote the limit of b_n by β. Let $\varepsilon > 0$ be given. From the definition of the limit

$$b_n \in B\left(\beta, \min\left(\tfrac{|\beta|}{2}, \tfrac{|\beta|^2\varepsilon}{2}\right)\right)$$

for n large enough. It follows from Lemma 3.16 that $b_n \neq 0$ and

$$c_n \in B\left(\frac{1}{\beta}, \varepsilon\right)$$

for n large enough, which concludes the proof. ■

Corollary 3.18 Let (a_n) be a convergent sequence, and let (b_n) be a convergent sequence whose limit is not zero. Then, the sequence $c_n = a_n/b_n$ is well-defined for n large enough. Furthermore, it is convergent, and

$$\lim_{n\to\infty} c_n = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

■ **Example 3.10** Use limit arithmetic to calculate the limit of

$$a_n = \frac{n^3 + 6n^2 - 6}{3n^3 + 5n + 10}.$$

■

Theorem 3.19 Let (a_n) be a bounded sequence and let (b_n) be a sequence that converges to zero. Then

$$\lim_{n\to\infty} a_n b_n = 0.$$

Proof. Let M be a bound for (a_n), namely,

$$(\forall n \in \mathbb{N})(|a_n| \leq M).$$

Since (b_n) converges to zero,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)\left(|b_n| < \frac{\varepsilon}{M}\right).$$

Thus,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)\left(|a_n b_n| \le M|b_n| < \varepsilon\right),$$

which implies that the sequence $(a_n b_n)$ converges to zero. ■

■ **Example 3.11** The sequence

$$a_n = \frac{\sin n}{n}$$

converges to zero. ■

3.6 Convergence of means

Let (a_n) be a sequence. We define a new sequence (s_n) as follows,

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= \tfrac{1}{2}(a_1 + a_2) \\ s_3 &= \tfrac{1}{3}(a_1 + a_2 + a_3) \\ &\text{etc.} \end{aligned}$$

For the general term,

$$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Theorem 3.20 — Cezaro. If (a_n) is convergent, then so is (s_n) and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_n.$$

Proof. Denote by α the limit of (a_n). Note that

$$s_n - \alpha = \frac{1}{n}\sum_{k=0}^{n}(a_k - \alpha),$$

and by the triangle inequality,

$$|s_n - \alpha| \le \frac{1}{n}\sum_{k=0}^{n}|a_k - \alpha|.$$

Recall that a convergent sequence is bounded, let M be a bound for $\{a_n : n \in \mathbb{N}\}$. By the triangle inequality, for all $n \in \mathbb{N}$,

$$|a_n - \alpha| \le |a_n| + |\alpha| \le M + |\alpha|.$$

Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that for every $n > N$,

$$|a_n - \alpha| < \frac{\varepsilon}{2}.$$

Then, for every $n > N$,

$$\begin{aligned} |s_n - \alpha| &\le \frac{1}{n}\sum_{k=0}^{N}|a_k - \alpha| + \frac{1}{n}\sum_{k=N+1}^{n}|a_k - \alpha| \\ &< \frac{N}{n}(M + |\alpha|) + \frac{n-N}{n}\frac{\varepsilon}{2} \\ &\le \frac{N}{n}(M + |\alpha|) + \frac{\varepsilon}{2}. \end{aligned}$$

Let

$$N' = \max\left(N, \frac{\varepsilon}{2N(M + |\alpha|)}\right).$$

Then for every $n > N'$,

$$|s_n - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

3.7 Generalized limits

A sequence is divergent if it does not have a limit. There are two types of divergent sequences: some “just don’t have a limit", whereas other “grow indefinitely without bounds", or “decrease indefinitely without bounds".

Definition 3.7 Let (a_n) be a sequence. We say that it ***tends to infinity*** (שואפת לאינסוף) if

$$(\forall M \in \mathbb{R})(\exists N \in \mathbb{N})(\forall n > N)(a_n > M).$$

We write

$$\lim_{n\to\infty} a_n = \infty.$$

Comment 3.11 Recall that infinity is not a real number.

Likewise:

Definition 3.8 Let (a_n) be a sequence. We say that it ***tends to minus infinity*** if

$$(\forall M \in \mathbb{R})(\exists N \in \mathbb{N})(\forall n > N)(a_n < M).$$

We write

$$\lim_{n\to\infty} a_n = -\infty.$$

Comment 3.12 If a sequence tends to plus or minus infinity we say that it ***converges in a wide sense*** (במובן הרחב). A sequence that tends to plus or minus infinity is still divergent.

We now start to investigate properties of sequences that converge in a wide sense.

Proposition 3.21 A sequence that tends to infinity is not bounded from above. Similarly, a sequence that tends to minus infinity is not bounded from below.

Proof. If (a_n) is bounded from above,

$$(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})(a_n < M).$$

It is then not true that

$$(\forall M \in \mathbb{R})(\exists n \in \mathbb{N})(a_n \geq M).$$

A fortiori, it is not true that

$$(\forall M \in \mathbb{R})(\exists N \in \mathbb{N})(\forall n > N)(a_n > M).$$

■

Proposition 3.22 Let (a_n) and (b_n) be sequences satisfying

$$a_n \leq b_n$$

for sufficiently large n. If

$$\lim_{n\to\infty} a_n = \infty,$$

then

$$\lim_{n\to\infty} b_n = \infty.$$

Proof. Let $M > 0$ be given. By definition, and using Lemma 3.2, there exists an $N \in \mathbb{N}$, such that for all $n > N$,

$$a_n > M \qquad \text{and} \qquad a_n \leq b_n.$$

If follows that for all $n > N$,

$$b_n > M,$$

which concludes the proof. ■

Proposition 3.23 Let (a_n) be a sequence of non-zero elements, satisfying

$$\lim_{n\to\infty} a_n = 0.$$

Then

$$\lim_{n\to\infty} \frac{1}{|a_n|} = \infty.$$

Proof. By definition,

$$(\forall M > 0)(\exists N \in \mathbb{N})(\forall n > N)(0 < |a_n| < 1/M),$$

hence

$$(\forall M > 0)(\exists N \in \mathbb{N})(\forall n > N)(1/|a_n| > M),$$

which concludes the proof. ■

Comment 3.13 Note that

$$\lim_{n\to\infty} a_n = 0.$$

does not implies that $(1/a_n)$ converges a wide sense.

Proposition 3.24 Let (a_n) be a sequence satisfying

$$\lim_{n\to\infty} |a_n| = \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof. By definition,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n| > 1/\varepsilon),$$

hence

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(0 < 1/|a_n| < \varepsilon),$$

which concludes the proof. ■

3.8 Monotone sequences

Definition 3.9 A sequence a is called ***increasing*** (עולה) if $a_{n+1} \geq a_n$ for all n. It is called ***strictly increasing*** (עולה ממש) if $a_{n+1} > a_n$ for all n. We define similarly ***decreasing*** (יורדת) and ***strictly decreasing*** (יורדת ממש) sequences. Any one of those sequences is called ***monotone***.

■ **Example 3.12**

1. The sequence $(a_n) = n$ is strictly increasing.
2. The sequence $(b_n) = 1/n$ is strictly decreasing.
3. The sequence $(c_n) = (-1)^n$ is not monotone.
4. The sequence $(d_n) = \alpha$ is both increasing and decreasing.

■

Theorem 3.25 Let (a_n) be an increasing sequence. If it is bounded from above, then it is convergent. Otherwise, it tends to infinity.

Proof. The second statement is easier to prove. Suppose that (a_n) is increasing and not bounded from above. Then, for every $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that

$$a_N > M.$$

Since the sequence is increasing,

$$(\forall n > N)(a_n > M),$$

which proves that the sequence tends to infinity.
Suppose now that (a_n) is bounded from above. This implies the existence of a least upper bound. Set

$$\alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

(Note that a supremum is a property of a set, i.e., the order in the set does not matter.) By the definition of the supremum,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\alpha - \varepsilon < a_N \le \alpha).$$

Since the sequence is non-decreasing,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(\alpha - \varepsilon < a_N \le a_n \le \alpha),$$

and in particular,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - \alpha| < \varepsilon).$$

■

Similarly,

Theorem 3.26 Let (a_n) be a decreasing sequence. If it is bounded from below, then it is convergent. Otherwise, it tends to minus infinity.

Corollary 3.27 Every monotone sequence converges in a wide sense.

■ **Example 3.13** Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

We will first show that $a_n < 3$ for all n. From the binomial formula,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

follows that

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{1}{k!}\left(\frac{1}{n}\right)^k\prod_{j=0}^{k-1}(n-j) = \sum_{k=0}^{n}\frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right).$$

This sequence is increasing as the larger n, the more terms there are, and each grows. Moreover,

$$\left(1+\frac{1}{n}\right)^n \le \sum_{k=0}^{n}\frac{1}{k!} = 1+1+\frac{1}{2}+\sum_{k=3}^{n}\frac{1}{k!} \le 1+1+\frac{1}{2}+\sum_{k=3}^{n}\frac{1}{2^k} < 3.$$

It follows that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{exists}$$

(and equals to $2.718\dots \equiv e$). ■

3.9 Cantor's lemma

Consider the sequence of segments,

$$I_n = [0, 1/n].$$

For every $n \in \mathbb{N}$, $I_{n+1} \subset I_n$. Moreover, the length of the segments tends to zero. If we look at the intersection of all the I_n's, we find out that it contains a single point,

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

If we rather used open, or semi-open segments,

$$J_n = (0, 1/n],$$

It still holds that $J_{n+1} \subset J_n$, and that the length of the segments tends to zero. Yet,

$$\bigcap_{n=1}^{\infty} J_n = \varnothing.$$

Theorem 3.28 — Cantor's lemma. Let (I_n) be a sequence of closed segments satisfying

$$I_{n+1} \subset I_n \qquad \text{and} \qquad \lim_{n\to\infty}|I_n| = 0,$$

where $|I_n|$ denotes the segment's length. Then there exists a unique real number c such that

$$A = \bigcap_{n=1}^{\infty} I_n = \{c\}.$$

Proof. Let $I_n = [a_n, b_n]$. Since $I_{n+1} \subset I_n$, it follows that (a_n) is increasing and (b_n) is decreasing. Since $a_1 \le a_n < b_n \le b_1$, it follows that (a_n) is bounded from above and (b_n) is bounded from below. By Theorem 3.25, both sequences are convergent. Denote,

$$\alpha = \lim_{n\to\infty} a_n \qquad \text{and} \qquad \beta = \lim_{n\to\infty} b_n.$$

Recall that for monotone sequence,

$$\alpha = \sup\{a_n \,:\, n \in \mathbb{N}\} \qquad \text{and} \qquad \beta = \inf\{b_n \,:\, n \in \mathbb{N}\}.$$

Since the length of the segments tends to zero, if follows from limit arithmetic that

$$0 = \lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (b_n - a_n) = \beta - \alpha,$$

hence $\alpha = \beta$. Furthermore, since

$$\alpha = \sup\{a_n \,:\, n \in \mathbb{N}\} = \inf\{b_n \,:\, n \in \mathbb{N}\},$$

it follows that $\alpha \in I_n$ for all n, namely $\alpha \in A$.

It remains to prove that A contains a unique point. Let $\gamma \in A$. Then, for every n,

$$0 \le |\gamma - \alpha| \le (b_n - a_n),$$

and by the sandwich theorem, $\gamma = \alpha$. ■

3.10 Subsequences and partial limits

Definition 3.10 Let (a_n) be a sequence. A ***subsequence*** (תת סדרה) of (a_n) is any sequence

$$a_{n_1}, a_{n_2}, \dots,$$

such that

$$n_1 < n_2 < \cdots.$$

More formally, (b_n) is a subsequence of (a_n) if there exists a strictly increasing sequence of natural numbers $(n_k)_{k=1}^\infty$, such that

$$b_k = a_{n_k}.$$

Comment 3.14 Every sequence is its own subsequence for $n_k = k$.

Comment 3.15 The sequence $(a_n)_{n=1}^\infty$ is the same as $(a_k)_{k=1}^\infty$, but unless $n_k = k$, it is not the same as the sequence $(a_{n_k})_{k=1}^\infty$.

■ **Example 3.14** The sequence $b_n = 1/2n$ is a subsequence of the harmonic sequence $a_n = 1/n$, for the choice $n_k = 2k$. Indeed,

$$b_k = \frac{1}{2k} = a_{2k}.$$

■

■ **Example 3.15** Let (a_n) be the sequence of natural numbers, namely $a_n = n$. The subsequence (b_n) of all even numbers is

$$b_k = a_{2k},$$

i.e., $n_k = 2k$. ■

The following lemma makes a number of obvious statements:

Lemma 3.29

1. If (n_k) is an increasing sequence of indexes then $n_k \geq k$.
2. Let (n_k) be an increasing sequence of integers. Let (P_n) be a sequence of propositions. If P_n holds for sufficiently large n, i.e.,

   $$(\exists N \in \mathbb{N})(\forall n > N)(P_n = \text{True}),$$

   then (P_{n_k}) holds for sufficiently large k. i.e.,

   $$(\exists K \in \mathbb{N})(\forall k > K)(P_{n_k} = \text{True}).$$

3. Let (n_k) be an increasing sequence of integers. Let (P_n) be a sequence of propositions. If (P_{n_k}) holds for sufficiently large k, i.e.,

   $$(\exists K \in \mathbb{N})(\forall k > K)(P_{n_k} = \text{True}).$$

   then (P_n) holds for infinitely many n's, i.e.,

   $$(\forall N \in \mathbb{N})(\exists n > N)(P_n = \text{True}).$$

4. Every sub-subsequence is a subsequence.
5. If $A \subset \mathbb{N}$ is an infinite set, then there exists a sequence (n_k) of indexes such that $n_k \in A$ for all k.

Definition 3.11 Let (a_n) be a sequence. A real number α is called a ***partial limit*** (גבול חלקי) of (a_n) if it is the limit of a subsequence of (a_n). That is, if there exists a strictly increasing sequence of integers (n_k), such that

$$\alpha = \lim_{k\to\infty} a_{n_k}.$$

Similarly, we define partial limits in the wide sense.

■ **Example 3.16** A constant sequence $a_n = c$ only has one subsequence, and only one partial limit, c. More generally, every limit is also a partial limit. ■

■ **Example 3.17** The sequence $a_n = (-1)^n$ has two partial limits, 1 and -1. It is easy to show that these are its only partial limits. ■

■ **Example 3.18** Every natural number is a partial limit of the sequence,

$$(1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,\dots).$$

■

Proposition 3.30 If (a_n) is convergent with limit α, then every subsequence of (a_n) converges to α, and in particular, α is the only partial limit.

Proof. Let (n_k) be an increasing sequence of indexes. Given $\varepsilon > 0$, let

$$P_n = (|a_n - \alpha| < \varepsilon).$$

This clause holds for sufficiently large n, hence by Lemma 3.29(2),

$$P_{n_k} = (|a_{n_k} - \alpha| < \varepsilon).$$

holds for sufficiently large k. ■

Corollary 3.31 If a sequence has two partial limits then it is not convergent.

Partial limits can be characterized with no reference to a particular subsequence:

Proposition 3.32 A real number α is a partial limit of a sequence (a_n) if and only if every neighborhood of α contains infinitely many elements of that sequence.

Proof. Suppose first that α is a partial limit of (a_n). If follows that there exists an increasing sequence of indexes (n_k) such that

$$\lim_{k\to\infty} a_{n_k} = \alpha.$$

Let $\varepsilon > 0$ be given, and let

$$P_n = (a_n \in B(\alpha, \varepsilon)).$$

Then, P_{n_k} holds for sufficiently large k, and by Lemma 3.29(3), P_n holds for infinitely many n's.

Suppose next that every neighborhood of α contains infinitely many elements of (a_n). Consider the set

$$I_1 = \{n \in \mathbb{N} \,:\, a_n \in B(\alpha, 1)\}$$

Since this set is not empty, there exists an $n_1 \in I_1$, i.e., $a_{n_1} \in B(\alpha, 1)$.

Consider next the set

$$I_2 = \{n \in \mathbb{N} \,:\, a_n \in B(\alpha, 1/2)\} \setminus \{n \in \mathbb{N} \mid n \le n_1\}.$$

This set is not empty, hence it contains an element n_2, which, by definition, satisfies

$$n_2 > n_1 \qquad \text{and} \qquad a_{n_1} \in B(\alpha, 1/2).$$

We proceed inductively, setting

$$I_{k+1} = \{n \in \mathbb{N} \, : \, a_n \in B(\alpha, 1/(k+1))\} \setminus \{n \in \mathbb{N} \mid n \le n_k\}.$$

This set is not empty, hence it contains an element n_{k+1}, which, by definition, satisfies

$$n_{k+1} > n_k \qquad \text{and} \qquad a_{n_{k+1}} \in B(\alpha, 1/(k+1)).$$

We have thus constructed a subsequence a_{n_k}. Since

$$0 \le |a_{n_k} - \alpha| < \frac{1}{k},$$

it follows from the "sandwich theorem" that (a_{n_k}) converges to α. ∎

Proposition 3.33 ∞ is a partial limit of (a_n) if and only if (a_n) is not bounded from above. Similarly, $-\infty$ is a partial limit of (a_n) if and only if (a_n) is not bounded from below.

Proof. The proof is essentially the same. ∎

We next prove this very important theorem:

Theorem 3.34 — Bolzano-Weierstrass. Every bounded sequence has a converging subsequence.

Proof. Suppose that $M > 0$ is a bound for the sequence, namely,

$$(\forall n \in \mathbb{N})(a_n \in [-M, M]).$$

We construct recursively a sequence of segments (I_n) satisfying:

1. $I_{n+1} \subset I_n$.
2. $|I_{n+1}| = \frac{1}{2}|I_n|$.
3. I_n contains infinitely many elements of (a_n).

This sequence is constructed using ***bisection*** (**חצייה**).
Specifically, let I_n contain infinitely many elements of (a_n), which means that

$$A_n = \{k \in \mathbb{N} \, : \, a_n \in I_n\}$$

is an infinite set of indexes. Partition I_n into two closed segments of equal size, which only intersect at one point,

$$I_n = I_n^R \cup I_n^L,$$

and define

$$A_n^R = \{k \in \mathbb{N} \, : \, a_n \in I_n^R\} \qquad \text{and} \qquad A_n^L = \{k \in \mathbb{N} \, : \, a_n \in I_n^L\}.$$

Since $A_n = A_n^R \cup A_n^L$ is an infinite set, either A_n^R or A_n^L must be infinite. Then, set

$$I_{n+1} = \begin{cases} I_n^R & |I_n^R| = \infty \\ I_n^L & \text{otherwise.} \end{cases}$$

By Cantor's lemma, there exists a unique number α in the intersection of all the I_n. We will prove that α is a partial limit of (a_n). Indeed, given $\varepsilon > 0$, let n be large enough such that $|I_n| < \varepsilon$. Then,

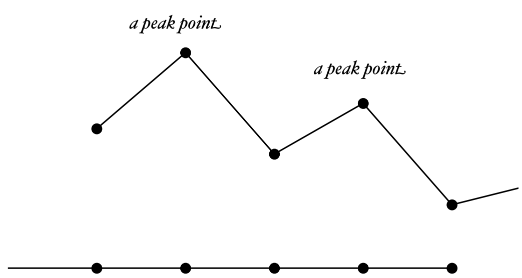
$$B(\alpha, \varepsilon) \supset I_n.$$

Since I_n contains infinitely many elements of (a_n) so does $B(\alpha, \varepsilon)$, and by Proposition 3.32, α is a partial limit of (a_n). ■

The Bolzano-Weierestrass can be proved in a completely different way: it is an immediate corollary of the following lemma:

Lemma 3.35 Any sequence contains a subsequence which is either decreasing or increasing.

Proof. Let (a_n) be a sequence. Let's call a number n a peak point (נקודת שיא) of the sequence a if $a_m < a_n$ for all $m > n$.



There are now two possibilities.

There are infinitely many peak points: If $n_1 < n_2 < \cdots$ are a sequence of peak points, then the subsequence a_{n_k} is decreasing.

There are finitely many peak points: Then let n_1 be greater than all the peak points. Since it is not a peak point, there exists an n_2, such that $a_{n_2} \geq a_{n_1}$. Continuing this way, we obtain a non-decreasing subsequence. ■

Comment 3.16 There is a fundamental difference between the two proofs. The first proof can be generalized with little modification to bounded sequences in $\mathbb{R}^n$. The second proof relies on the fact that $\mathbb{R}$ is an ordered set, hence the possiblity to define monotone sequences.

Corollary 3.36 Every sequence has a subsequence that converges in the wide sense.

Proof. Either the sequence of bounded, in which case this is a consequence of the Bolzano-Weierstrass theorem, or it is not bounded, and this is a consequence of Proposition 3.33. ■

Proposition 3.37 Let (a_n) be a sequence that does not converge in the wide sense. Then, it has at least two partial limits (in the wide sense).

Proof. Let α be a partial limit of (a_n) in the wide sense. Suppose first that $\alpha \in \mathbb{R}$. Since, by assumption, α is not a limit of (a_n) there exists an $\varepsilon > 0$ such that

$$a_n \notin B(\alpha, \varepsilon)$$

for infinitely many n's. Thus, we can construct a subsequence a_{n_k} such that

$$(\forall k \in \mathbb{N})(a_{n_k} \notin B(\alpha, \varepsilon)).$$

By Corollary 3.36, this subsequence has a partial limit (in the wide sense) β. Since $\beta \notin B(\alpha, \varepsilon)$, it differs from α. The proof is similar if $\alpha = \pm\infty$. ■

Corollary 3.38 A sequence (a_n) converges in the wide sense to α if and only if α is its only partial limit.

3.11 The exponential function

We have seen that the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

is bounded and monotonically increasing, hence converging. The limit, which we denoted by

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

is a number between 2 and 3.

Likewise, for every $x \in \mathbb{R}$ we may define the sequence

$$a_n = \left(1 + \frac{x}{n}\right)^n.$$

As for the case $x = 1$,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{x^k}{n^k} \prod_{j=1}^{k-1} (n - j) = \sum_{k=0}^{n} \frac{x^k}{k!} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right).$$

This sequence is increasing as the larger n the more terms there are, and the k-th term is larger. Also,

$$a_n \le \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Let $N = \lceil 2x \rceil$, i.e., $x/N < 1/2$. Then, for $n > N$,

$$\begin{aligned}
a_n &\le \sum_{k=0}^{N} \frac{x^k}{k!} + \sum_{k=N+1}^{n} \frac{x^k}{k!} \\
&= \sum_{k=0}^{N} \frac{x^k}{k!} + \sum_{k=N+1}^{n} \frac{x^N}{N!} \frac{x^{n-N}}{(N+1)\dots n} \\
&= \sum_{k=0}^{N} \frac{x^k}{k!} + \frac{x^N}{N!} \sum_{k=N+1}^{n} \frac{x^{n-N}}{(N+1)\dots n} \\
&\le \sum_{k=0}^{N} \frac{x^k}{k!} + \frac{x^N}{N!} \sum_{k=N+1}^{n} \frac{1}{2^{n-N}} \\
&\le \sum_{k=0}^{N} \frac{x^k}{k!} + \frac{x^N}{2N!}.
\end{aligned}$$

The right-hand side is independent of n, which means that (a_n) is a bounded sequence, hence converges. Since the sequence depends on x, so does the limit. We define

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

In particular,

$$\exp(1) = e.$$

In the previous chapter, we define the notion of powers with real-valued exponents. Thus, for every $x \in \mathbb{R}$, we can define a number e^x, whose definition, we recall, is

$$e^x = \sup\{e^r \mid \mathbb{Q} \ni r \le x\}.$$

We now claim that

Theorem 3.39 For every $x \in \mathbb{R}$,

$$\exp(x) = e^x.$$

That is,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \sup\{e^r \mid \mathbb{Q} \ni r \le x\},$$

where

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. We will first show that this identity holds for every $x \in \mathbb{N}$. Set $x = m \in \mathbb{N}$, and consider the sequence

$$a_n = \left(1 + \frac{m}{n}\right)^n.$$

Since it converges to $\exp(m)$, every subsequence converges to $\exp(m)$ as well. Set $n_k = mk$. Then,

$$a_{n_k} = \left(1 + \frac{m}{mk}\right)^{mk} = \left[\left(1 + \frac{1}{k}\right)^k\right]^m.$$

Since

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e,$$

it follows from limit arithmetic that

$$\lim_{k\to\infty} a_{n_k} = \left[\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k\right]^m = e^m,$$

i.e.,

$$\lim_{n\to\infty} \left(1 + \frac{m}{n}\right)^n = e^m.$$

Next suppose that $x = p/q$, $p, q \in \mathbb{N}$, and consider the sequence

$$a_n = \left(1 + \frac{p}{qn}\right)^n.$$

Then, consider the sequence

$$a_n^q = \left(1 + \frac{p}{qn}\right)^{qn}.$$

This sequence is a subsequence (every q-th term) of a sequence that converges to e^p, i.e.,

$$\lim_{n\to\infty} a_n^q = e^p.$$

Again, by limit arithmetic,

$$\lim_{n\to\infty} a_n^q = \left(\lim_{n\to\infty} a_n\right)^q,$$

which implies that

$$(\exp(p/q))^q = e^p,$$

or equivalently,

$$\exp(p/q) = e^{p/q}.$$

It remains to deal with the case $x \in \mathbb{R}$. Note that both e^x and $\exp(x)$ are increasing functions of x. Consider the sets,

$$A = \{\exp(r) \mid \mathbb{Q} \ni r \le x\} \qquad \text{and} \qquad B = \{\exp(r) \mid \mathbb{Q} \ni r \ge x\}.$$

We already know that

$$A = \{e^r \mid \mathbb{Q} \ni r \le x\} \qquad \text{and} \qquad B = \{e^r \mid \mathbb{Q} \ni r \ge x\}.$$

For the latter case, we know that e^x is the unique number separating the sets A and B. Since $\exp(x)$ also separates A and B, it follows that

$$\exp(x) = e^x.$$

■

3.12 Limit inferior and limit superior

Not taught this year.

3.13 Cauchy sequences

In many cases, we would like to know whether a sequence is convergent even if we do not know what the limit is. We will now provide such a convergence criterion.

Definition 3.12 A sequence (a_n) is called a ***Cauchy sequence*** if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n > N)(|a_n - a_m| < \varepsilon).$$

Comment 3.17 A common notation for the condition satisfied by a Cauchy sequence is

$$\lim_{n,m\to\infty} |a_n - a_m| = 0.$$

Theorem 3.40 A sequence converges if and only if it is a Cauchy sequence.

Proof. One direction is easy[1]. If a sequence (a_n) converges to a limit α, then

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|a_n - \alpha| < \varepsilon/2).$$

By the triangle inequality,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n > N)(|a_n - a_m| \le |a_n - \alpha| + |a_m - \alpha| < \varepsilon),$$

i.e., the sequence is a Cauchy sequence.

Suppose next that (a_n) is a Cauchy sequence. We first show that the sequence is bounded. Taking $\varepsilon = 1$,

$$(\exists N \in \mathbb{N})(\forall n > N)(|a_n - a_{N+1}| < 1).$$

[1] There is something amusing about calling sequences satisfying this property a Cauchy sequence. Cauchy assumed that sequences that get eventually arbitrarily close converge, without being aware that this is something that ought to be proved.

Then, for every $n > N$,

$$|a_n| < |a_{N+1}| + 1,$$

whereas for $n \le N$,

$$|a_n| \le \max_{k \le N} |a_k|,$$

which proves that the sequence is bounded.

By the Bolzano-Weierstrass theorem, it follows that (a_n) has a converging subsequence. Denote this subsequence by $b_k = a_{n_k}$ and its limit by β. We will show that the whole sequence converges to β.

By the Cauchy property

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n > N)(|a_n - a_m| < \varepsilon/2),$$

whereas by the convergence of the sequence (a_{n_k}),

$$(\forall N \in \mathbb{N})(\exists K \in \mathbb{N})(\forall k > K)(|b_k - \beta| < \varepsilon/2).$$

Combining the two,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\exists k \in \mathbb{N} \,:\, n_k > N)(\forall n > N)(|a_n - \beta| \le |a_n - b_k| + |b_k - \beta| < \varepsilon).$$

This concludes the proof. ■

Comment 3.18 Limits of sequences can be defined for only for sequences in $\mathbb{R}$. Limits can be defined for sequences in any ***metric space***, which is a set S on which a ***distance function*** d is defined. A sequence (a_n) in S converges to $\alpha \in S$ if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(d(a_n, \alpha) < \varepsilon).$$

In any metric space we can define a Cauchy sequence: (a_n) is a Cauchy sequence if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m > N)(d(a_n, a_m) < \varepsilon).$$

It is not generally true that a Cauchy sequence in a metric space converges. Metric spaces in which every cauchy sequence converges are called ***complete***. This is the fact the more general definition of completeness for an ordered field.