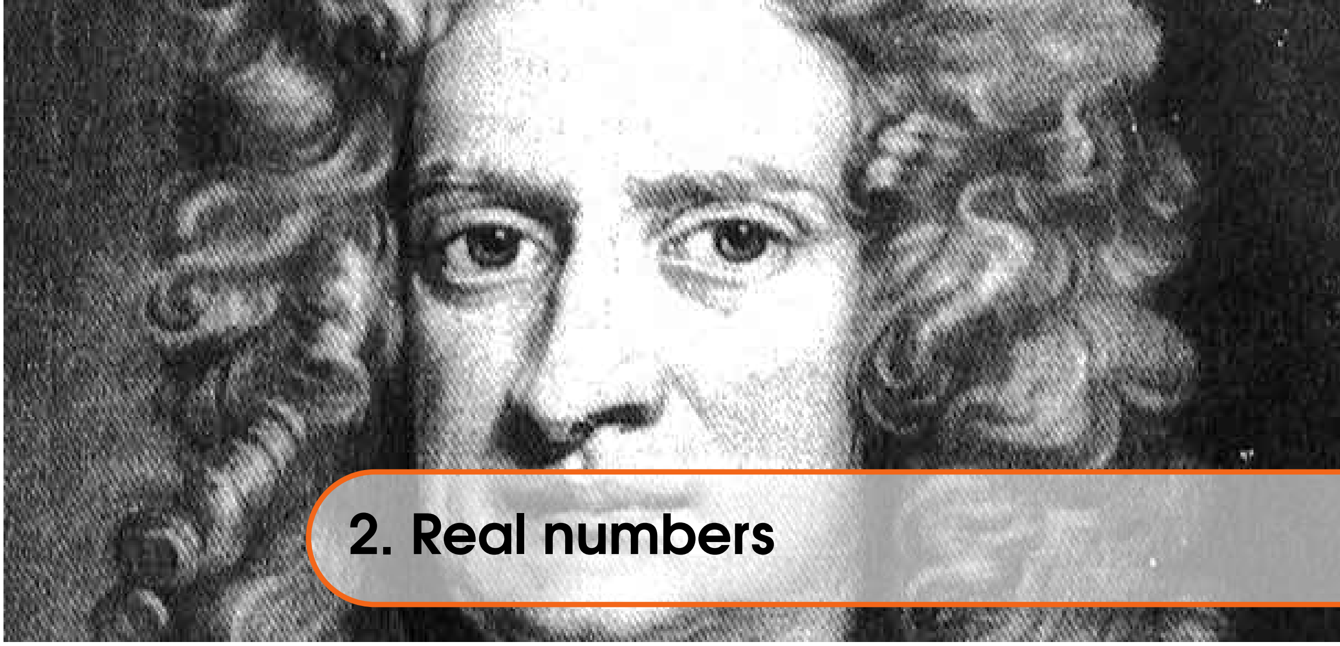


2. Real numbers

2.1 Axioms of field

Even though we have been using “numbers" as an elementary notion since first grade, a rigorous course of calculus has to start by putting even such basic concept on axiomatic grounds.

The set of real numbers will be defined as an instance of a ***complete, ordered field*** (שדה סדור שלם).

Definition 2.1 A ***field*** $\mathbb{F}$ (שָׂדֶה) is a non-empty set on which two ***binary operations*** are defined: an operation which we call ***addition*** and denote by $+$, and an operation which we call ***multiplication*** and denote by $\cdot$ (or by nothing, as in $a \cdot b = ab$). The operations on elements of a field satisfy nine defining properties, which we list now.

Comment 2.1 A binary operation in $\mathbb{F}$ is a function that “takes" an ordered pair of elements in $\mathbb{F}$ and “returns" an element in $\mathbb{F}$. That is, to every $a, b \in \mathbb{F}$ corresponds one and only one $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$. In formal notation,

$$(\forall a, b \in \mathbb{F})(\exists! c \in \mathbb{F})(c = a + b).$$

Also, note that we refer to *elements* of a field, and not to *numbers*. Fields whose elements are numbers are a special instance.

Comment 2.2 A few words about the equal sign; we will take it literally to mean that “the expressions on both sides are the same". More precisely, equality is an ***equivalence relation*** (יחס שקילות), a concept which will be explained further below.

1. ***Addition is associative*** (קיבוצי): for all $a, b, c \in \mathbb{F}$,

$$(a + b) + c = a + (b + c). \tag{F1}$$

It should be noted that addition was defined as a *binary* operation. As such, there is no a-priori meaning to $a+b+c$. In fact, the meaning is ambiguous, as we could first add $a+b$ and then add c to the sum, or conversely, add $b+c$, and then add the sum to a. The use of brackets removes the ambiguity. The first axiom states that in either case, the value is the same, therefore we may write expressions such as $a+b+c$.

What about the addition of four elements, $a+b+c+d$? Does it require a separate axiom of equivalence? We would like the following additions

$$((a+b)+c)+d$$
$$(a+(b+c))+d$$
$$a+((b+c)+d)$$
$$a+(b+(c+d))$$
$$(a+b)+(c+d)$$

to be equivalent. It is easy to see that the equivalence follows from Property (F1). Likewise (although it requires some non-trivial work), it can be shown that Property (F1) implies the n-fold addition

$$a_1+a_2+\cdots+a_n$$

is defined unambiguously.

2. ***Existence of an additive-neutral element*** (איבר נייטרלי לחיבור): there exists an element $0 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$

$$a+0=0+a=a. \tag{F2}$$

3. ***Existence of an additive inverse*** (איבר נגדי): for all $a \in \mathbb{F}$ there exists an element $b \in \mathbb{F}$, such that

$$a+b=b+a=0. \tag{F3}$$

We would like to denote the additive inverse of a, as customary, by $(-a)$. There is however one problem. The axiom postulates the *existence* of an additive inverse, but it does not postulate its *uniqueness*. Suppose there were two additive inverses: which one would be denoted $(-a)$?

The uniqueness of the additive inverse turns out to be something we can *prove* based on the first three axioms. In fact, we will prove something even stronger:

Proposition 2.1 — Elimination rule. Let $a,b,c \in \mathbb{F}$ satisfy

$$a+b=a+c.$$

Then,

$$b=c.$$

Similarly, if
$$b + a = c + a.$$
Then,
$$b = c.$$

Proof. The proof requires all three axioms. By the third axiom, a has an additive inverse; let's denote it by d. Then, each line below implies its successor:

$$\begin{aligned} a + b &= a + c \\ d + (a + b) &= d + (a + c) \\ (d + a) + b &= (d + a) + c && \text{(F1)} \\ 0 + b &= 0 + c && \text{(F3)} \\ b &= c. && \text{(F2)} \end{aligned}$$

The second statement is proved similarly. ■

Corollary 2.2 — Uniqueness of zero. The additive-neutral element is unique: if $a, b \in \mathbb{F}$ satisfy
$$a + b = a,$$
then $b = 0$.

Proof. By (F2),
$$a + b = a = a + 0,$$
and it follows from Proposition 2.1 that $b = 0$. ■

Corollary 2.3 — Uniqueness of additive inverse. Every element $a \in \mathbb{F}$ has a unique additive inverse, namely, if
$$a + b = 0 \qquad \text{and} \qquad a + c = 0,$$
then $b = c$.

Proof. This is an immediate consequence of Proposition 2.1. ■

Thus, we can refer to *the* additive inverse of a, which justifies the notation $(-a)$.

Comment 2.3 We only postulate the operations of addition and multiplication. ***Subtraction*** is a short-hand notation for the addition of the additive inverse,
$$a - b \stackrel{\text{def}}{=} a + (-b).$$

Comment 2.4 A set satisfying the first three axioms is called a ***group*** (חבורה). By themselves, those three axioms have many implications, which can fill an entire course (and they do: see Algebraic Structures 1).

4. ***Addition is commutative*** (חילופי): for all $a,b \in \mathbb{F}$

$$a + b = b + a. \tag{F4}$$

With this law we finally obtain that any (finite!) summation of elements of $\mathbb{F}$ can be re-arranged in any order.

■ **Example 2.1** Our experience with numbers tells us that $a - b = b - a$ implies that $a = b$. There is no way we can prove it using only the first four axioms! Indeed, all we obtain from it that

$$\begin{aligned}
a + (-b) &= b + (-a) && \text{(given)}\\
(a + (-b)) + (b + a) &= (b + (-a)) + (a + b) && \text{(F4)}\\
a + ((-b) + b) + a &= b + ((-a) + a) + b && \text{(F1)}\\
a + 0 + a &= b + 0 + b && \text{(F3)}\\
a + a &= b + b, && \text{(F2)}
\end{aligned}$$

but how can we deduce from that that $a = b$? ■

With very few comments, we state the corresponding laws for multiplication:

5. ***Multiplication is associative***: for all $a,b,c \in \mathbb{F}$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c. \tag{F5}$$

6. ***Existence of a multiplicative-neutral element***: there exists an element $1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$,

$$a \cdot 1 = 1 \cdot a = a. \tag{F6}$$

7. ***Existence of a multiplicative inverse*** (איבר הופכי): for every $a \neq 0$ there exists an element $b \in \mathbb{F}$ such that,

$$a \cdot b = b \cdot a = 1. \tag{F7}$$

Like for the additive inverse, we can show that the multiplicative inverse is unique; we denote it by a^{-1}.

The condition that $a \neq 0$ has strong implications. For example, if

$$a \cdot b = a \cdot c,$$

then only if $a \neq 0$ we can deduce that $b = c$.

Comment 2.5 ***Division*** is defined as multiplication by the multiplicative inverse,

$$a/b \stackrel{\text{def}}{=} a \cdot b^{-1} \qquad \forall a,b \in \mathbb{F},\ b \neq 0.$$

8. ***Multiplication is commutative***: for every $a,b \in \mathbb{F}$,

$$a \cdot b = b \cdot a. \tag{F8}$$

The last axiom relates addition and multiplication:

9. ***Distributive law*** (חוק הקיבוץ): for all $a,b,c \in \mathbb{F}$,

$$a \cdot (b+c) = a \cdot b + a \cdot c. \tag{F9}$$

This concludes the nine axioms of field.

With 9 axioms at hand, we can start proving theorems that are satisfied by all fields.

Proposition 2.4 For every $a \in \mathbb{F}$, $a \cdot 0 = 0$.

Proof. Using sequentially the property of the neutral additive element (F2) and the distributive law (F9):

$$a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0.$$

Adding to both sides $-(a \cdot 0)$ we obtain

$$a \cdot 0 = 0.$$

■

Comment 2.6 Suppose it were the case that $1 = 0$, i.e., that the same element of $\mathbb{F}$ is both the additive neutral and the multiplicative neutral. It would follows that for every $a \in \mathbb{F}$,

$$a = a \cdot 1 = a \cdot 0 = 0,$$

which means that 0 is the only element of $\mathbb{F}$. This is indeed a field according to the axioms, however a very boring one. Thus, we rule it out and require the field to satisfy $0 \neq 1$, i.e., comprise at least two elements.

Comment 2.7 In principle, every algebraic identity should be proved from the axioms of field. In practice, we will assume henceforth that all known algebraic identities are valid. For example, we will not bother to prove that

$$a - b = -(b - a), \qquad \text{or that} \qquad (-a)(-b) = ab.$$

(even though the proof is very easy). The only exception is (of course) if you get to prove an algebraic identity as an assignment.

We can revisit now the $a - b = b - a$ example. We proceed,

$$\begin{aligned} a + a &= b + b \\ 1 \cdot a + 1 \cdot a &= 1 \cdot b + 1 \cdot b && \text{(F6)} \\ (1+1) \cdot a &= (1+1) \cdot b && \text{(F5)}. \end{aligned}$$

We would be done if we knew that $1 + 1 \neq 0$, for we would multiply both sides by $(1+1)^{-1}$. However, this does not follow from the axioms of field!

■ **Example 2.2** Consider a set comprising two elements, $\mathbb{F} = \{e, f\}$, satisfying the following properties,

$+$	e	f
e	e	f
f	f	e

$\cdot$	e	f
e	e	e
f	e	f

It takes some explicit verification to check that this is indeed a field (in fact the smallest possible field), with e being the additive neutral and f being the multiplicative neutral (do you recognize this field?).
Note that

$$e - f = e + (-f) = e + f = f + e = f + (-e) = f - e,$$

and yet $e \neq f$, which is then no wonder that we can't prove, based only on the field axioms, that $f - e = e - f$ implies that $e = f$. ∎

2.2 Axioms of order

The real numbers contain much more structure than just being a field. They also form an ***ordered set*** (קבוצה סדורה). The property of being ordered can be formalized by four axioms:

Definition 2.2 — (שָׂדֶה סָדוּר). A field $\mathbb{F}$ is said to be ***ordered*** if there exists a ***relation*** (יחס) $<$ (a relation is a property that every ordered pair of elements either satisfies or not, i.e., for every $a, b \in \mathbb{F}$ either $a < b$ is True or it is False), such that the following four properties are satisfied:

1. ***Trichotomy*** (טריכוטומיה): every pair of elements $a, b \in \mathbb{F}$ satisfies one and only one of the following properties,

$$a < b \quad \text{or} \quad b < a \quad \text{or} \quad a = b. \tag{O1}$$

2. ***Transitivity***:

$$\text{if} \quad a < b \quad \text{and} \quad b < c \quad \text{then} \quad a < c. \tag{O2}$$

3. ***Invariance under addition***:

$$\text{if} \quad a < b \quad \text{then} \quad \forall c \in \mathbb{F} \quad a + c < b + c. \tag{O3}$$

4. ***Invariance under multiplication by positive elements***:

$$\text{if} \quad a < b \quad \text{then} \quad \forall 0 < c \quad ac < bc. \tag{O4}$$

Comment 2.8 The set of elements a satisfying $0 < a$ are called the ***positive elements***. The set of elements a satisfying $a < 0$ are called the ***negative elements***. Trichotomy implies that every element in an ordered field is either positive, negative or zero.

Comment 2.9 Transitivity implies that it makes sense to write chains of inequalities, such that

$$a < b < c < d.$$

We supplement these axioms with the following definitions:

$$\begin{array}{lll} a > b & \text{means} & b < a \\ a \le b & \text{means} & a < b \text{ or } a = b \\ a \ge b & \text{means} & a > b \text{ or } a = b. \end{array}$$

As in the previous section, we proceed to deduce properties of ordered fields based on a set of, now, 9 plus 4 defining properties.

Proposition 2.5 The set of positive elements is closed under addition and multiplication, namely

$$\text{If } \; 0 < a, b \quad \text{then} \quad 0 < a + b \quad \text{and} \quad 0 < ab.$$

Proof. Closure under addition follows from

$$\begin{array}{ll} 0 < b & \text{(given)} \\ \quad = 0 + b & \text{(F2)} \\ \quad < a + b & \text{(O3) and } a < 0. \end{array}$$

Closure under multiplication follows from

$$\begin{array}{ll} 0 = 0 \cdot a & \text{(proved)} \\ \quad < b \cdot a & \text{(O4) and } 0 < a, b. \end{array}$$

■

Proposition 2.6 $0 < a$ if and only if $(-a) < 0$, i.e., a number is positive if and only if its additive inverse is negative.

Comment 2.10 The phrase *if and only if* will appear repeatedly throughout this course. It is worth clarifying what it means.

When we say that Property A holds *if* Property B holds, we mean that if Property B holds, we can infer that Property A holds as well. Each property being either True or False, when we say that Property A holds *if* Property B holds, we mean that the only possible scenarios are:

Property A	Property B
True	True
True	False
False	False

When we say that Property A holds *only if* Property B holds, we mean that Property A cannot holds without Property B holding. Thus, if Property B does not holds, we can infer that Property A does not holds either. When we say that Property A holds *only if* Property B holds, we mean that the only possible scenarios are:

Property A	Property B
True	True
False	True
False	False

When we say that Property A holds *of and only if* Property B holds, it means that if Property B holds, then so does Property A, but if Property B does not holds then so does Property A. When we say that Property A holds *if and only if* Property B holds, we mean that the only possible scenarios are:

Property A	Property B
True	True
False	False

Proof. Suppose $0 < a$, then

$$\begin{aligned} (-a) &= 0 + (-a) && \text{(F2)} \\ &< a + (-a) && \text{(O3) and } 0 < a \\ &= 0 && \text{(F3)}, \end{aligned}$$

i.e., $(-a) < 0$. Conversely, if $(-a) < 0$ then

$$\begin{aligned} 0 &= (-a) + a && \text{(F3)} \\ &< 0 + a && \text{(O3) and } (-a) < 0 \\ &= a && \text{(F2)}, \end{aligned}$$

i.e., $0 < a$. ■

Corollary 2.7 The set of positive numbers is not empty.

Proof. Since by assumption $1 \neq 0$, it follows by (O1) that either $0 < 1$ or $1 < 0$. In the latter case $0 < (-1)$, hence either 1 or (-1) is positive. ■

Proposition 2.8 $0 < 1$.

Proof. By trichotomy, either $0 < 1$, or $1 < 0$, or $0 = 1$. The third possibility is ruled out by assumption. Suppose then, by contradiction, that $1 < 0$. Then for every $0 < a$ (and we know that at least one such exists),

$$a = a \cdot 1 < a \cdot 0 = 0,$$

i.e., $a < 0$ which violates the trichotomy axiom. Hence $0 < 1$. ■

Proposition 2.9 $0 < a$ if and only if $0 < a^{-1}$, i.e., a number is positive if and only if its multiplicative inverse is also positive.

Proof. Let $0 < a$ and suppose by contradiction that $a^{-1} < 0$ (it can't be zero because $a^{-1}a = 1$). Then,

$$1 = a \cdot a^{-1} < a \cdot 0 = 0,$$

which is a contradiction. Since $a = (a^{-1})^{-1}$, it follows that $0 < a^{-1}$ implies that $0 < a$. ■

Proposition 2.10 If $a < 0$ and $b < 0$ then $ab > 0$.

Proof. We have seen above that $a, b < 0$ implies that $0 < (-a), (-b)$. By the closure under multiplication, it follows that $(-a)(-b) > 0$. It remains to check that the axioms of field imply that $ab = (-a)(-b)$. ■

Corollary 2.11 If $a \neq 0$ then $a \cdot a \equiv a^2 > 0$.

Proof. If $a \neq 0$, then by the trichotomy either $a > 0$, in which case $a^2 > 0$ follows from the closure under multiplication, or $a < 0$ in which case $a^2 > 0$ follows from the previous proposition, with $a = b$. ■

Comment 2.11 An interesting corollary is that the field of ***complex numbers*** (מספרים מרוכבים) (which is not within the scope of the present course) cannot be ordered, since $i^2 = (-1)$, which implies that (-1) has to be positive, hence 1 has to be negative, which is a violation.

2.3 Absolute value

Definition 2.3 For every element a in an ordered field $\mathbb{F}$ we define its ***absolute value*** (ערך מוחלט),

$$|a| = \begin{cases} a & a \geq 0 \\ (-a) & a < 0. \end{cases}$$

We see right away that $|a| = 0$ if and only if $a = 0$, and otherwise $|a| > 0$.

Lemma 2.12 For every $a \in \mathbb{F}$ and $b \geq 0$,

$$|a| \leq b \qquad \text{if and only if} \qquad -b \leq a \leq b.$$

Proof. Recall that there are two statements to prove. Suppose first that $-b \leq a \leq b$. Equivalently,

$$a \leq b \qquad \text{and} \qquad -a \leq b.$$

By trichotomy, either $a \ge 0$ or $a < 0$. If $a \ge 0$, then

$$|a| = a \le b.$$

If, on the other hand, $a < 0$, then

$$|a| = -a \le b,$$

which completes the first part.
Conversely, suppose that $|a| \le b$. If $a \ge 0$, then

$$-b \le 0 \le a = |a| \le b.$$

If, on the other hand, $a < 0$,

$$-b \le -|a| = a < 0 \le b.$$

■

Corollary 2.13 For every $a \in \mathbb{F}$,

$$-|a| \le a \le |a|.$$

Proof. Apply Proposition 2.12 for $b = |a|$, using the fact that $|a| \le |a|$ for every $a \in \mathbb{F}$. ■

Proposition 2.14 — Triangle inequality (אי שיוויון המשולש). For every $a, b \in \mathbb{F}$,

$$|a+b| \le |a| + |b|.$$

Proof. By the previous lemma,

$$-|a| \le a \le |a|$$
$$-|b| \le b \le |b|.$$

Adding the two inequalities, we obtain using again the above lemma,

$$-(|a| + |b|) \le a + b \le |a| + |b|,$$

which, by that same lemma, implies the triangle inequality.

■

Comment 2.12 By replacing b by $(-b)$ we obtain that

$$|a-b| \le |a| + |b|.$$

The following version of the triangle inequality is also useful:

Proposition 2.15 — Reverse triangle inequality. For every $a,b \in \mathbb{F}$,

$$||a| - |b|| \le |a+b|.$$

Proof. By the triangle inequality,

$$\forall a,c \in \mathbb{F}, \qquad |a-c| \le |a| + |c|.$$

Set $c = b+a$, in which case $|b| \le |a| + |b+a|$, or

$$|b| - |a| \le |b+a|.$$

By interchanging the roles of a and b,

$$|a| - |b| \le |a+b|,$$

namely,

$$-|a+b| \le |b| - |a|.$$

It follows from Lemma 2.12 that

$$||a| - |b|| \le |a+b|,$$

■

Proposition 2.16 For every $a,b,c \in \mathbb{F}$,

$$|a+b+c| \le |a| + |b| + |c|.$$

Proof. Use the triangle inequality twice,

$$|a+b+c| = |a+(b+c)| \le |a| + |b+c| \le |a| + |b| + |c|.$$

■

2.4 Special subsets of an ordered field

2.4.1 The naturals

For all we know, a field may contain just two elements 0 and 1. An ordered field, however, cannot be that small, for $1+1$ must differ from both 0 and 1. This follows from the axioms of order: $0 < 1$ implies that

$$1 = 1 + 0 < 1 + 1.$$

We can proceed, and determine that $1+1+1$ is greater than $1+1$, and by transitivity also greater than 1 and 0.

Thus, we can form the set of elements

$$1, \quad 1+1, \quad 1+1+1, \quad 1+1+1+1, \quad \text{etc.}$$

which we'll call the ***naturals*** (הטבעיים). We denote this set by the letter $\mathbb{N}$. The set of naturals must be contained in every ordered field $\mathbb{F}$. It is clearly not a field as all its elements are positive, and it does not contain, say, 0. Note also that the fact that we may name those elements (two, three, four, ...) and may represent them using a decimal system is immaterial.

This way of defining the naturals is very intuitive, but does not constitute a rigorous definition (what does "and so on" mean?). A sound definition of the naturals is based on the notion of an ***inductive set***:

Definition 2.4 Let $\mathbb{F}$ be an ordered field. A set $I \subset \mathbb{F}$ is called ***inductive*** if $1 \in I$ and $x \in I$ implies $x+1 \in I$, namely

$$(1 \in I) \quad \text{and} \quad (\forall x \in \mathbb{F})((x \in I) \Rightarrow (x+1 \in I)).$$

Comment 2.13 Every ordered field contains inductive sets. For example, the entire field is an inductive set. Likewise,

$$I = \{a \in \mathbb{F} \,:\, a \geq 1\}$$

is an inductive set.

Proposition 2.17 Let $\mathscr{A}$ be a collection of inductive sets (that is, the elements of $\mathscr{A}$ are subsets of $\mathbb{F}$ that satisfy the inductive property). Then,

$$I = \bigcap_{A \in \mathscr{A}} A$$

is an inductive set.

Proof. First recall that

$$x \in I \quad \Longleftrightarrow \quad (\forall A \in \mathscr{A})(x \in A).$$

Since 1 is an element of every inductive set,

$$(\forall A \in \mathscr{A})(1 \in A).$$

It follows that $1 \in I$.

Next, let $n \in I$. By definition,

$$(\forall A \in \mathscr{A})(n \in A).$$

Since every such A is an inductive set, i.e.,

$$(\forall A \in \mathscr{A})(\forall x \in A)((x \in A) \Rightarrow (x+1 \in A)),$$

it follows that

$$(\forall A \in \mathscr{A})(n+1 \in A),$$

which implies that $n+1 \in I$, i.e., I is an inductive set. ■

Definition 2.5 Let $\mathbb{F}$ be an ordered field. Let $\mathscr{A}$ be the collection of all inductive subsets of $\mathbb{F}$. We define the set of naturals as the intersection of all the inductive sets

$$\mathbb{N} = \bigcap_{A \in \mathscr{A}} A,$$

that is, the naturals are those field elements contained in every inductive set.

Proposition 2.18 $\mathbb{N}$ is an inductive set.

Proof. This follows from the fact that any intersection of inductive sets is an inductive set. ■

Proposition 2.19 If I is an inductive set, then $\mathbb{N} \subset I$.

Proof. Since $\mathbb{N}$ is the intersection of *all* the inductive set, it cannot include elements that do not belong to I. ■

Corollary 2.20 If $I \subset \mathbb{N}$ is an inductive set, then $I = \mathbb{N}$.

Proof. This follows from the fact that $I \subset \mathbb{N}$ (given) and $\mathbb{N} \subset I$ (because I is inductive). ■

This last proposition is the basis of proof by induction:

Proposition 2.21 — Proof by induction. For every $n \in \mathbb{N}$ let $P(n)$ an assertion (פסוק לוגי). If $P(1) = \text{True}$ and if $P(n) = \text{True}$ implies that $P(n+1) = \text{True}$, then $P(n) = \text{True}$ for all $n \in \mathbb{N}$.

Proof. Let

$$I = \{n \in \mathbb{N} \,:\, P(n) = \text{True}\}.$$

Then $I \subset \mathbb{N}$ is an inductive set, and therefore coincides with $\mathbb{N}$. ■

While we may be tempted to postulate that the naturals satisfy the properties of natural numbers we know from early childhood, let's prove that this is indeed the case, i.e., a direct consequence of our definitions.

Proposition 2.22 $\mathbb{N}$ contains a minimal element, 1.

Proof. Define
$$I = \{n \in \mathbb{N} \,:\, 1 \le n\}.$$
I is an inductive set because it contains 1, and if $x \in I$, i.e.,
$$x \in \mathbb{N} \quad \text{and} \quad x \ge 1,$$
then
$$x+1 \in \mathbb{N} \quad \text{and} \quad x+1 \ge 1,$$
i.e., $x+1 \in I$. Hence $I = \mathbb{N}$. ∎

Proposition 2.23 For every $m,n \in \mathbb{N}$, $m+n \in \mathbb{N}$.

Proof. We will show that
$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(n+m \in \mathbb{N}).$$
Let $n \in \mathbb{N}$ be arbitrary and consider the set
$$I = \{m \in \mathbb{N} \,:\, n+m \in \mathbb{N}\}.$$
That is,
$$m \in I \quad \Longleftrightarrow \quad m \in \mathbb{N} \quad \text{and} \quad n+m \in \mathbb{N}.$$
Since $n \in \mathbb{N}$, and by the properties of $\mathbb{N}$,
$$1 \in \mathbb{N} \qquad \text{and} \qquad n+1 \in \mathbb{N},$$
which implies that $1 \in I$.
If $m \in I$, i.e.,
$$m \in I \quad \Longleftrightarrow \quad m \in \mathbb{N} \quad \text{and} \quad n+m \in \mathbb{N}.$$
then,
$$m+1 \in \mathbb{N} \qquad \text{and} \qquad n+(m+1) = (n+m)+1 \in \mathbb{N},$$
hence $m+1 \in I$. It follows that I is an inductive subset of $\mathbb{N}$, hence it coincides with $\mathbb{N}$. ∎

Proposition 2.24 For every $m,n \in \mathbb{N}$, $mn \in \mathbb{N}$.

Proof. Take it as an exercise. ∎

Proposition 2.25 Let $n \in \mathbb{N}$ satisfy $n > 1$. Then there exist a $k \in \mathbb{N}$, such that
$$n = k+1,$$

i.e.,

$$(\forall n \in \mathbb{N}, n > 0)(n - 1 \in \mathbb{N}).$$

Proof. Suppose, by contradiction, that there exists an $n \in \mathbb{N}$ greater than 1 for which there does not exist an $k \in \mathbb{N}$ such that $n = k + 1$. Consider the set

$$I = \mathbb{N} \setminus \{n\}.$$

Then I is inductive because it contains 1 and there is no element in I whose successor is not it I. Thus, $I = \mathbb{N}$ contradicting the fact that it excludes n. ■

Proposition 2.26 Let $n, m \in \mathbb{N}$ satisfy $n > m$. Then there exist a $k \in \mathbb{N}$, such that

$$n = m + k,$$

which amount to say that $n - m \in \mathbb{N}$.

Proof. We need to show that

$$(\forall m \in \mathbb{N})(\forall n \in \mathbb{N}, n > m)(n - m \in \mathbb{N}).$$

Set

$$I = \{m \in \mathbb{N} \; : \; (\forall n \in \mathbb{N}, n > m)(n - m \in \mathbb{N})\}.$$

That is,

$$m \in I \quad \Longleftrightarrow \quad m \in \mathbb{N} \quad \text{and} \quad (\forall n \in \mathbb{N}, n > m)(n - m \in \mathbb{N}).$$

By the previous proposition,

$$(\forall n \in \mathbb{N}, n > 1)(n - 1 \in \mathbb{N}),$$

which implies that $1 \in I$.
Suppose next that $m \in I$, that is,

$$m \in \mathbb{N} \quad \text{and} \quad (\forall n \in \mathbb{N}, n > m)(n - m \in \mathbb{N})$$

Then,

$$m + 1 \in \mathbb{N} \quad \text{and} \quad (\forall n \in \mathbb{N}, n > m + 1)(n - (m + 1) = (n - 1) - m \in \mathbb{N}),$$

where we used the fact that $n - 1$ is a natural greater than m. Hence, $m + 1 \in I$, which implies that I is an inductive set, hence coincides with $\mathbb{N}$. ■

Proposition 2.27 Let $m \in \mathbb{N}$. Then, there is no $n \in \mathbb{N}$ satisfying

$$m < n < m + 1.$$

Proof. Suppose this were the case. By the previous proposition, $n-m\in\mathbb{N}$, however,

$$n-m<(m+1)-m=1,$$

which contradicts the fact that 1 is the smallest natural. ■

Proposition 2.28 — Well-ordering principle. Every non-empty subset of $\mathbb{N}$ has a minimal element (an element smaller than all the other).

Proof. Let $A\subset\mathbb{N}$ be non-empty. Suppose, by contradiction, that A does not have a minimal element. That is, for every $a\in A$ there exists a $b\in A$ satisfying $b<a$.
Define

$$B=\{n\in\mathbb{N}\ :\ \forall a\in A,\, n<a\}.$$

This is the set of naturals smaller than all the elements in A. Note that A and B are disjoint. If we show that $B=\mathbb{N}$, it will imply that $A=\varnothing$, which is a contradiction.
We will show that B is inductive. Clearly, $1\in B$, otherwise, $1\in A$, and A has a minimal element. Suppose that $m\in B$, i.e., m is smaller than all the elements in A. If $m+1\notin B$, then there is an element in A less or equal $m+1$. This element must be $m+1$, which would be the smallest element in A. ■

2.4.2 Proof by induction

As an example of an inductive definition, we define for all $a\in\mathbb{Q}$ and $n\in\mathbb{N}$, the n-th power, by setting,

$$a^1=a \qquad \text{and} \qquad a^{k+1}=a\cdot a^k.$$

As an example of an inductive proof, consider the following proposition:

Proposition 2.29 Let $a,b>0$ and $n\in\mathbb{N}$. Then

$$a<b \quad \text{if and only if} \quad a^n<b^n.$$

Proof. We proceed by induction. Define

$$P(n)=\{\text{if } a<b \text{ then } a^n<b^n\}.$$

Clearly,

$$P(1)=\text{True}.$$

Suppose that $P(n)=\text{True}$, i.e., if $a<b$ then $a^n<b^n$. Then

$$a^{n+1}=a\cdot a^n<a\cdot b^n<b\cdot b^n=b^{n+1}.$$

It follows that the set natural numbers n for which $P(n)=\text{True}$ is an inductive set, hence it is all of $\mathbb{N}$.
Conversely, if $a^n<b^n$ then $a\ge b$ would imply that $a^n\ge b^n$, hence $a<b$. ■

Another inequality that we will need occasionally is:

Proposition 2.30 — Bernoulli inequality. For all $n \in \mathbb{N}$ and for all $x > (-1)$,

$$(1+x)^n \ge 1 + nx.$$

Proof. The proof is by induction. Let

$$P(n) = \{(1+x)^n \ge 1 + nx \text{ for all } x > -1\}.$$

Clearly, $P(1) = \text{True}$. Suppose that $P(n) = \text{True}$. Then,

$$(1+x)^{n+1} = (1+x)(1+x)^n \ge (1+x)(1+nx) = 1 + (n+1)x + nx^2 \ge 1 + (n+1)x,$$

where in the left-most inequality we have used explicitly the fact that $1+x > 0$. Thus, $P(n+1) = \text{True}$, which means that the set of naturals n for which $P(n) = \text{True}$ is an inductive set. ∎

2.4.3 The integers

We augment the set of naturals by adding the elements

$$0, \quad -1, \quad -(1+1), \quad -(1+1+1), \quad \text{etc.}$$

This forms the set of ***integers*** (השלמים), which we denote by $\mathbb{Z}$ (for the German *Zahlen*—numbers). Formally,

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \,:\, n \in \mathbb{N}\}.$$

Also,

$$\mathbb{Z} = \{k \in \mathbb{F} \,:\, \text{exist } m, n \in \mathbb{N} \text{ such that } k = m - n\}.$$

The integers form a commutative group with respect to addition, but are not a field since multiplicative inverses do not exist:

Proposition 2.31 The element $(1+1)^{-1}$ is not an integer.

Proof. Denote $a = (1+1)^{-1}$. We know that a is positive. If it were an integer it would have to be a natural, hence greater then 1. Suppose that it were true that $1 < (1+1)^{-1}$, then

$$1 + 1 = (1+1) \cdot 1 < (1+1) \cdot (1+1)^{-1} = 1,$$

which is a violation of the axioms of order. ∎

2.4.4 The rationals

The set of integers can be further augmented by adding all ***integer quotients***, m/n, $n \neq 0$. This forms the set of ***rationals***, which we denote by

$$\mathbb{Q} = \{r \in \mathbb{F} \,:\, \exists m,n \in \mathbb{Z}, n \neq 0 \text{ such that } r = m/n\}.$$

It should be noted that the rationals are the set of integer quotients modulo an equivalence relation[1]. A rational element has infinitely many representations as the quotient of two integers.

Proposition 2.32 Let $b,d \neq 0$. Then

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

Proof. Multiply/divide both sides by bd (which cannot be zero if $b,d \neq 0$). ∎

Proposition 2.33 Every positive rational $0 < r \in \mathbb{Q}$ has an ***irreducible*** representation (הצגה מצומצמת), $r = m/n$, such that m and n don't have a common factor.

Proof. Define

$$I = \{n \in \mathbb{N} \,:\, \exists m \in \mathbb{N},\ r = m/n\}.$$

I is a subset of $\mathbb{N}$ and by the well-ordering principle contains an element n that is the smallest. Let m be the corresponding natural such that $r = m/n$. This is an irreducible representation, for if $n = kn'$ and $m = km'$ for $1 \neq k \in \mathbb{N}$, then $r = m'/n'$ with $I \ni n' < n$, which is a contradiction to n being minimal. ∎

It can be checked that the rational elements form an ordered field:

Proposition 2.34 Let $\mathbb{F}$ be an ordered field and let $\mathbb{Q}$ be the subset of rational elements. Then $\mathbb{Q}$ is a ***subfield*** of $\mathbb{F}$.

Proof. We need to show that $\mathbb{Q}$ is closed under addition, multiplication, additive inverse, multiplicative inverse. ∎

This means that $\mathbb{Q}$ is the smallest possible ordered field.

[1] A few words about ***equivalence relations*** (יחס שקילות). An equivalence relation on a set S is a property that any two elements either have or don't. If two elements a,b have this property, we say that a is equivalent to b, and denote it by $a \sim b$. An equivalence relation has to be ***symmetric*** ($a \sim b$ implies $b \sim a$), ***reflexive*** ($a \sim a$ for all a) and ***transitive*** ($a \sim b$ and $b \sim c$ implies $a \sim c$). Thus, with every element $a \in S$ we can associate an ***equivalence class*** (מחלקת שקילות), which is $\{b \,:\, b \sim a\}$. Every element belongs to one and only one equivalence class. An equivalence relation partitions S into a collection of disjoint equivalence classes; we call this set S modulo the equivalence relation $\sim$, and denote it $S/\sim$.

2.5 Upper and lower bounds

Definition 2.6 A set of elements $A \subset \mathbb{F}$ is said to be ***bounded from above*** (חסום מלעיל) if there exists an element $M \in \mathbb{F}$, such that $a \le M$ for all $a \in A$, i.e.,

$$A \text{ is bounded from above} \iff (\exists M \in \mathbb{F})(\forall a \in A)(a \le M).$$

Such an element M is called an ***upper bound*** (חסם מלעיל) for A. A is said to be ***bounded from below*** (חסום מלרע) if

$$(\exists m \in \mathbb{F})(\forall a \in A)(a \ge m).$$

Such an element m is called a ***lower bound*** (חסם מלרע) for A. A set is said to be ***bounded*** (חסום) if it is bounded both from above and below.

Note that if a set is upper bounded, then the upper bound is not unique, for if M is an upper bound, then so are $M+1$, $M+2$, and so on.

Proposition 2.35 A set A is bounded if and only if

$$(\exists M \in \mathbb{F})(\forall a \in A)(|a| \le M).$$

Proof. Suppose A is bounded, then

$$(\exists M_1 \in \mathbb{F})(\forall a \in A)(a \le M_1)$$
$$(\exists M_2 \in \mathbb{F})(\forall a \in A)(a \ge M_2).$$

That is,

$$(\exists M_1 \in \mathbb{F})(\forall a \in A)(a \le M_1)$$
$$(\exists M_2 \in \mathbb{F})(\forall a \in A)((-a) \le (-M_2)).$$

By taking $M = \max(M_1, -M_2)$ we obtain

$$(\forall a \in A)(a \le M \quad \text{and} \quad (-a) \le M),$$

i.e.,

$$(\forall a \in A)(|a| \le M).$$

The other direction isn't much different. ∎

A few notational conventions: for $a < b$ we use the following notations for open (קטע פתוח), closed (קטע סגור), and semi-open (קטע חצי סגור) segments,

$$(a,b) = \{x \in \mathbb{F} : a < x < b\}$$
$$[a,b] = \{x \in \mathbb{F} : a \le x \le b\}$$
$$(a,b] = \{x \in \mathbb{F} : a < x \le b\}$$
$$[a,b) = \{x \in \mathbb{F} : a \le x < b\}.$$

Each of these sets is bounded (above and below).

We also use the following notations for sets that are bounded on one side,

$$
\begin{aligned}
(a,\infty) &= \{x \in \mathbb{F} \,:\, x > a\} \\
[a,\infty) &= \{x \in \mathbb{F} \,:\, x \geq a\} \\
(-\infty,a) &= \{x \in \mathbb{F} \,:\, x < a\} \\
(-\infty,a] &= \{x \in \mathbb{F} \,:\, x \leq a\}.
\end{aligned}
$$

The first two sets are bounded from below, whereas the last two are bounded from above. It should be emphasized that $\pm\infty$ are not members of $\mathbb{F}$! The above is purely a notation. “Being less than infinity" does not mean being less than a field element called infinity.

2.6 The Archimedean property

We are aiming at constructing an ordered field that matches our notions and experiences associated with numbers. Thus far, we showed that an ordered field contains a set, which we called the integers, along with all elements expressible at ratios of integers—the rationals $\mathbb{Q}$. Whether it contains additional elements (non-rational or ***irrational*** elements) is left for the moment unanswered. The point is that $\mathbb{Q}$ already has the property of being an ordered field.

Consider the set of naturals, $\mathbb{N}$. Does it have an upper bound? Our “geometrical picture" of the number axis clearly indicates that this set is unbounded, but *can we prove it*? We can easily prove the following:

Proposition 2.36 The naturals are not bounded from above by any element in $\mathbb{N}$.

Proof. Suppose, by contradiction, that $n \in \mathbb{N}$ were an upper bound for $\mathbb{N}$, i.e.,

$$(\forall k \in \mathbb{N})(k \leq n).$$

However, $m = n+1 \in \mathbb{N}$ and $m > n$, i.e.,

$$(\exists m \in \mathbb{N})(n < m),$$

which is a contradiction. ■

Similarly,

Corollary 2.37 The naturals are not bounded from above by any rational element.

Proof. Suppose that $p/q \in \mathbb{Q}$ was an upper bound for $\mathbb{N}$. Since, by the axioms of order, $p/q \leq p$ (convince yourself this is true), it would imply the existence of a natural element p that is upper bound for $\mathbb{N}$. ■

It may however be possible that an irrational element in $\mathbb{F}$ be an upper bound for $\mathbb{N}$. Can we rule out this possibility? It turns out that this cannot be proved from the axioms of an ordered field. Here is the reason: there exist ordered fields in which the naturals are bounded.

■ **Example 2.3** Consider the set $\mathbb{F}$ of ***rational functions*** (functions expressible as the ratio of polynomials), i.e., the function

$$f(x) = \frac{x^9 - 3x^7}{2x^4 + 4}.$$

It is easy to see that this set forms a field with respect to function addition and multiplication, with $f(x) \equiv 0$ for additive neutral element and $f(x) \equiv 1$ for multiplicative neutral element. Ignore the fact that a rational function may be undefined at a finite collection of points. The "natural elements" are the constant functions, $f(x) \equiv 1$, $f(x) \equiv 2$, etc, i.e.,

$$\mathbb{N}_{\mathbb{F}} = \{f(x) = n \,:\, n \in \mathbb{F}\},$$

and the “rational elements" are ratios of natural elements and their negatives. We endow this set with an order relation by defining $f < g$ if $f(x) < g(x)$ for x sufficiently large. It is easy to see that the set of “natural elements" is bounded, say, by an “irrational" element $f(x) = x$ (which obviously belongs to $\mathbb{F}$). ■

Since, however, the boundedness of the naturals is so basic to our intuition, we may want to impose it as an additional axiom, known as ***the axiom of the Archimedean field***: the set of natural numbers is unbounded from above.

Definition 2.7 An order field $\mathbb{F}$ is called ***Archimedean*** if the set of naturals in not bounded from above. That is,

$$(\forall a \in \mathbb{F})(\exists n \in \mathbb{N})(a < n).$$

Proposition 2.38 In an Archimedean ordered field $\mathbb{F}$:

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(1/n < \varepsilon).$$

Proof. By the previous proposition,

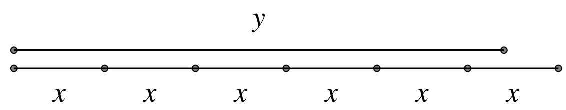
$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})\underbrace{(1/\varepsilon < n)}_{(1/n<\varepsilon)}.$$

■

Corollary 2.39 In an Archimedean ordered field $\mathbb{F}$:

$$(\forall x, y > 0)(\exists n \in \mathbb{N})(y < nx).$$

Comment 2.14 This is really what is meant by the Archimedean property. For every $x, y > 0$, a segment of length y can be covered by a finite number of segments of length x.



Proof. By the previous corollary with $\varepsilon = y/x$,

$$(\forall x, y > 0)(\exists n \in \mathbb{N})\underbrace{(y/x < n)}_{(y<nx)}.$$

■

2.7 Axiom of completeness

We now come to the one additional axiom—the ***axiom of completeness***—with which we can finally characterize the real numbers as a set satisfying the properties of a ***complete ordered field***.

The Greeks knew already that the field of rational numbers is “incomplete", in the sense that there is no rational number whose square equals 2 (whereas they knew by Pythagoras’ theorem that this should be the length of the diagonal of a unit square). In fact, let’s prove it:

Proposition 2.40 There is no $r \in \mathbb{Q}$ such that $r^2 = 2$.

Proof. Suppose, by contradiction, that $r \in \mathbb{Q}$ (we can assume that r is positive) satisfies $r^2 = 2$. We have seen that any (positive) rational can be brought into a form $r = m/n$, $m, n \in \mathbb{N}$ where m and n have no common divisor. By assumption, $m^2/n^2 = 2$, i.e., $m^2 = 2n^2$. This means that m^2 is even, from which follows that m is even, hence $m = 2k$ for some $k \in \mathbb{N}$. Hence, $4k^2 = 2n^2$, or $n^2 = 2k^2$, from which follows that n is even, contradicting the fact that m and n have no common divisor. ■

Proof. Another proof: suppose again that $r = n/m$ is irreducible and $r^2 = 2$. Since, $n^2 = 2m^2$, then

$$m^2 < n^2 < 4m^2,$$

it follows that

$$m < n < 2m < 2n,$$

and

$$0 < n - m < m \qquad \text{and} \qquad 0 < 2m - n < n.$$

Consider now the ratio

$$q = \frac{2m - n}{n - m},$$

whose numerator and denominator are both smaller than the respective numerator and denominator of r. By elementary arithmetic

$$q^2 = \frac{4m^2 - 4mn + n^2}{n^2 - 2nm + m^2} = \frac{4 - 4n/m + n^2/m^2}{n^2/m^2 - 2n/m + 1} = \frac{6 - 4n/m}{3 - 2n/m} = 2,$$

which is a contradiction. ∎

A way to cope with this “missing number" would be to add $\sqrt{2}$ “by hand" to the set of rational numbers, along with all the numbers obtained by field operations involving $\sqrt{2}$ and rational numbers (this is called in algebra a ***field extension***) (**שדה הרחבה**)[2]. But then, what about $\sqrt{3}$? We could add all the square roots of all positive rational numbers. And then, what about $\sqrt[3]{2}$? Shall we add all n-th roots? But then, what about a solution to the equation $x^5 + x + 1 = 0$ (it cannot be expressed in terms of roots as a result of ***Galois' theory***)?

It turns out that a single additional axiom, known as the ***axiom of completeness*** (**אכסיומת השלמות**), completes the set of rational numbers in one fell swoop, such to provide solutions to all those equations. Let us try to look in more detail in what sense is the field of rational numbers “incomplete". Consider the following two sets,

$$A = \{x \in \mathbb{Q} \,:\, 0 < x, x^2 \le 2\}$$
$$B = \{y \in \mathbb{Q} \,:\, 0 < y, 2 \le y^2\}.$$

Every element in B is greater or equal than every element in A (by transitivity and by the fact that $0 < x \le y$ implies $x^2 \le y^2$). Formally,

$$(\forall a \in A \wedge \forall b \in B)(a \le b).$$

Does there exist a rational number that “separates" the two sets, i.e., does there exist an element $c \in \mathbb{Q}$ such that

$$(\forall a \in A \wedge \forall b \in B)(a \le c \le b) \quad ?$$

It can be shown that if there existed such a c it would satisfy $c^2 = 2$, hence such a c does not exist. This observation motivates the following definition:

Definition 2.8 An ordered field $\mathbb{F}$ is said to be ***complete*** if for every two non-empty sets, $A, B \subseteq \mathbb{F}$ satisfying

$$(\forall a \in A \wedge \forall b \in B)(a \le b).$$

[2] It can be shown that this field extension of $\mathbb{Q}$ consists of all elements of the form

$$\{a + b\sqrt{2} \,:\, a, b \in \mathbb{Q}\}.$$

there exists an element $c \in \mathbb{F}$ such that

$$(\forall a \in A \wedge \forall b \in B)(a \le c \le b).$$

We will soon see how this axiom "completes" the field $\mathbb{Q}$.

We next introduce more definitions. Let's start with a motivating example:

Example 2.4 For any set $[a,b) = \{x : a \le x < b\}$, b is an upper bound, but so is any larger element, e.g., $b+1$. In fact, it is clear that b is the least upper bound. ∎

Definition 2.9 Let $A \subseteq \mathbb{F}$ be a non-empty set. An element $M \in \mathbb{F}$ is called a ***least upper bound*** (חסם עליון) for A if (i) it is an upper bound for A, and (ii) if M' is also an upper bound for A then $M \le M'$. That is,

$$\begin{gathered} M \text{ is a least upper bound for } A \\ \Updownarrow \\ (\forall a \in A)(a \le M) \\ (\forall M' \in \mathbb{F})(\text{ if } (\forall a \in A)(a \le M') \text{ then } (M \le M')). \end{gathered}$$

Very important: the least upper bound of a set A may, or may not be an element of A.

We can see right away that a least upper bound, if it exists, is unique:

Proposition 2.41 Let $A \subset \mathbb{F}$ be a non-empty set and let M be a least upper bound for A. If M' is also a least upper bound for A then $M = M'$.

Proof. It follows immediately from the definition of the least upper bound. If M and M' are both least upper bounds, then both are in particular upper bounds, hence $M \le M'$ and $M' \le M$, which implies that $M = M'$. ∎

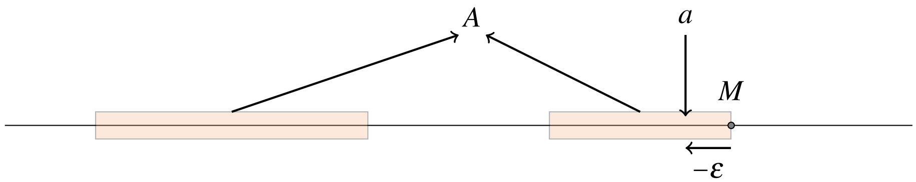
We call the least upper bound of a set (if it exists!) a ***supremum***, and denote it by $\sup A$. Similarly, the ***greatest lower bound*** (חסם תחתון) of a set is called the ***infimum***, and it is denoted by $\inf A$.[3]

We can provide an equivalent definition of the least upper bound:

Proposition 2.42 Let A be a set. A number M is a least upper bound if and only if (i) it is an upper bound, and (ii)

$$(\forall \varepsilon > 0)(\exists a \in A)(a > M - \varepsilon).$$

[3] In some books the notations ***lub*** (least upper bound) and ***glb*** (greatest lower bound) are used instead of sup and inf.



Proof. There are two directions to prove:

1. Suppose first that M were a least upper bound for A, i.e.,
$$(\forall a \in A)(a \le M) \quad \text{and} \quad (\forall M' \in \mathbb{F})(\text{if } (\forall a \in A)(a \le M') \text{ then } (M \le M')).$$
Suppose, by contradiction, that there exists an $\varepsilon > 0$ such that $a \le M - \varepsilon$ for all $a \in A$, i.e., that
$$(\exists \varepsilon > 0)(\forall a \in A)(a < M - \varepsilon),$$
then $M - \varepsilon$ is an upper bound for A, smaller than the least upper bound, which is a contradiction.
2. Conversely, suppose that M is an upper bound and
$$(\forall \varepsilon > 0)(\exists a \in A)(a > M - \varepsilon).$$
Suppose that M was not a least upper bound. Then there exists a smaller upper bound $M' < M$. Take $\varepsilon = M - M'$. Then, $a \le M' = M - \varepsilon$ for all $a \in A$, or in formal notation,
$$(\exists \varepsilon > 0)(\forall a \in A)(a \le M - \varepsilon),$$
which is a contradiction.

■

■ **Examples 2.1** What are the least upper bounds (if they exist) in the following examples:

1. $[-5, 15]$ (answer: 15).
2. $[-5, 15)$ (answer: 15).
3. $[-5, 15] \cup \{20\}$ (answer: 20).
4. $[-5, 15] \cup (17, 18)$ (answer: 18).
5. $[-5, \infty)$ (answer: none).
6. $\{1 - 1/n \ : \ n \in \mathbb{N}\}$ (answer: 1).

■

Definition 2.10 Let $A \subset \mathbb{F}$ be a subset of an ordered field. It is said to have a ***maximum*** if there exists an element $M \in A$ which is an upper bound for A. We denote
$$M = \max A.$$
It is said to have a ***minimum*** if there exists an element $m \in A$ which is a lower

bound for A. We denote

$$m = \min A.$$

Examples 2.2 What are the maxima (if they exist) in the following examples:

1. $[-5, 15]$ (answer: 15).
2. $[-5, 15)$ (answer: none).
3. $[-5, 15] \cup \{20\}$ (answer: 20).
4. $[-5, 15] \cup (17, 18)$ (answer: none).
5. $[-5, \infty)$ (answer: none).
6. $\{1 - 1/n \,:\, n \in \mathbb{N}\}$ (answer: none).

■

Proposition 2.43 If a set A has a maximum, then the maximum is also the least upper bound, i.e.,

$$\sup A = \max A.$$

Similarly, if A has a minimum, then the minimum is also the greatest lower bound,

$$\inf A = \min A.$$

Proof. Let $M = \max A$. Then, by definition, M is an upper bound for A, and for every $\varepsilon > 0$,

$$A \ni M > M - \varepsilon,$$

that is $(\forall \varepsilon > 0)(\exists a \in A)(a > M - \varepsilon)$, which proves that M is the least upper bound. ■

Proposition 2.44 Every (non-empty) finite set in an ordered field has a minimum and a maximum.

Proof. The proof is by induction on the size of the set. ■

Corollary 2.45 Every (non-empty) finite set in an ordered field has a least upper bound and a greatest lower bound.

The question is whether every *infinite* set has a supremum. The answer, in general, is negative, as there are sets that are not even bounded. Then, we may ask whether every upper-bounded set has a supremum.

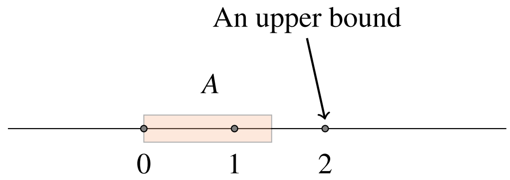
Consider an ordered field $\mathbb{F}$ and the set

$$A = \{x \in \mathbb{F} \,:\, 0 < x,\ x^2 < 2\}.$$

This set is bounded from above, as 2, for example, is an upper bound. Indeed, if $x \in A$, then

$$x^2 < 2 < 2^2,$$

which implies that $x < 2$.



But does A have a least upper bound?

Proposition 2.46 Suppose that A has a least upper bound, then

$$(\sup A)^2 = 2.$$

Proof. We assume that A has a least upper bound, and denote

$$\alpha = \sup A.$$

Clearly, $1 < \alpha < 2$.
Suppose, by contradiction, that $\alpha^2 < 2$. We are going to show that there exists a member of A greater than α, which means that α is not even an upper bound for A—contradiction.
Let

$$0 < \beta < \min\left(\alpha, \frac{2-\alpha^2}{3\alpha}\right).$$

We will prove that $(\alpha+\beta)^2 < 2$, which will imply that $\alpha < \alpha+\beta \in A$.
Then,

$$\begin{aligned}(\alpha+\beta)^2 &= \alpha^2 + 2\alpha\beta + \beta^2 \\ &< \alpha^2 + 3\alpha\beta && \text{(use } \beta < \alpha) \\ &< \alpha^2 + (2-\alpha^2) = 2, && \text{(use } 3\alpha\beta < 2-\alpha^2)\end{aligned}$$

Similarly, suppose by contradiction that $\alpha^2 > 2$. We are going to show that there exists a field element $0 < a < \alpha$, such that $a^2 > 2$, which means that a is an upper bound for A smaller than α—again, a contradiction.
This time we set

$$0 < \beta < \min\left(\alpha, \frac{\alpha^2-2}{2\alpha}\right).$$

Then, $\alpha - \beta > 0$, then

$$\begin{aligned}(\alpha-\beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &> \alpha^2 - 2\alpha\beta && \text{(omit } \beta^2) \\ &> \alpha^2 + (2-\alpha^2) = 2, && \text{(use } -2\alpha\beta > 2-\alpha^2),\end{aligned}$$

which implies that $\alpha - \beta$ is an upper bound for A.

■

Corollary 2.47 For $\mathbb{F} = \mathbb{Q}$, the set

$$A = \{x \in \mathbb{F} \,:\, 0 < x, x^2 < 2\}$$

is non-empty, bounded from above but does not have a supremum.

Proof. This follows from the fact that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$. ■

Proposition 2.48 In a complete ordered field every non-empty set that is upper-bounded has a least upper bound (a supremum).

Proof. Let $A \subset \mathbb{F}$ be a non-empty set that is upper-bounded. Define

$$B = \{b \in \mathbb{F} \,:\, b \text{ is an upper bound for } A\},$$

which by assumption is non-empty. Clearly,

$$(\forall a \in A \wedge \forall b \in B)(a \le b).$$

By the axiom of completeness,

$$(\exists c \in \mathbb{F})(\forall a \in A \wedge \forall b \in B)(a \le c \le b).$$

Clearly c is an upper bound for A smaller or equal than every other upper bound, hence

$$c = \sup A.$$

■

Similarly, we can prove that:

Proposition 2.49 In a complete ordered field every non-empty set that is bounded from below has a greatest lower bound (an infimum).

With that, we define the set of real numbers as a complete ordered field, which we denote by $\mathbb{R}$. It turns out that this defines the set uniquely, up to a relabeling of its elements (i.e., up to an isomorphism). [4]

The axiom of completeness can be used to prove the Archimedean property. In other words, the axiom of completeness solves at the same time the "problems" pointed out in the previous section.

[4] Strictly speaking, we should prove that such an animal exists. Constructing the set of real numbers from the set of rational numbers is beyond the scope of this course. You are strongly recommended, however, to read about it. See for example the construction of ***Dedekind cuts***.

Proposition 2.50 — Archimedean property. In a complete ordered field $\mathbb{N}$ is not bounded from above.

Proof. Suppose $\mathbb{N}$ was bounded. Then it would have a least upper bound M. By the characterization of the least upper bound,

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(n > M - \varepsilon).$$

In particular,

$$(\exists n \in \mathbb{N})(n > M - 0.5),$$

and by the inductive property of $\mathbb{N}$,

$$(\exists n \in \mathbb{N})(n + 1 \in \mathbb{N},\ n + 1 > M + 0.5),$$

contradicting the fact that M is an upper bound for $\mathbb{N}$. ■

We next prove an important fact about the ***density*** (צפיפות) of rational numbers within the reals.

Proposition 2.51 — The rational numbers are dense in the reals. Let $x, y \in \mathbb{R}$, such that $x < y$. There exists a rational number $q \in \mathbb{Q}$ such that $x < q < y$. In formal notation,

$$(\forall x, y \in \mathbb{R} \, : \, x < y)(\exists q \in \mathbb{Q})(x < q < y).$$

Proof. Since the natural numbers are not bounded, there exists an $m \in \mathbb{Z} \subset \mathbb{Q}$ such that $m < x$. Also there exists a natural number $n \in \mathbb{N}$ such that $n > 1/(y - x)$, which in turn implies that $1/n < y - x$. Consider now the set of rational numbers,

$$\left\{ m + \frac{k}{n} \, : \, k = 0, 1, 2, \dots \right\}$$

From the Archimedean property follows that this set is not bounded from above. Hence, the set of natural numbers,

$$I = \left\{ k \in \mathbb{N} \, : \, m + \frac{k}{n} > x \right\}$$

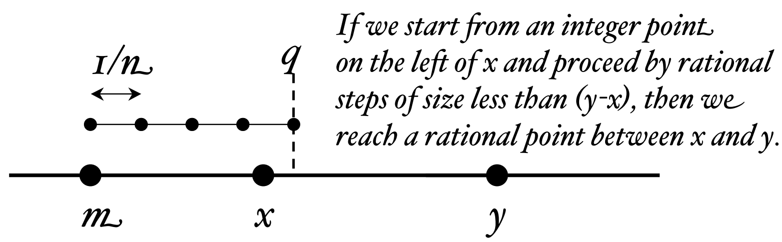
is not empty. By the well-ordering principle, I has a minimal element. That is, there exists a $k^* \in \mathbb{N}$ such that

$$m + \frac{k^*}{n} > x \qquad \text{and} \qquad m + \frac{k^* - 1}{n} \le x.$$

Then,

$$x < m + \frac{k^*}{n} = m + \frac{k^* - 1}{n} + \frac{1}{n} \le x + \frac{1}{n} < y,$$

which completes the proof.



■

The following proposition provides a useful fact about complete ordered fields. We will use it later in the course.

Proposition 2.52 — (למת החתכים). Let $A,B \subset \mathbb{F}$ be two non-empty sets in a complete ordered field, such that

$$(\forall a \in A \wedge \forall b \in B)(a \le b).$$

Then the following three statements are equivalent:

1. $(\exists! M \in \mathbb{F})(\forall a \in A \wedge \forall b \in B)(a \le M \le b)$.
2. $\sup A = \inf B$.
3. $(\forall \varepsilon > 0)(\exists a \in A, b \in B)(b - a < \varepsilon)$.

Comment 2.15 The claim is not that the three items follow from the given data. The claim is that each statement implies the two other.

Proof. Suppose that the first statement holds. This means that M is both an upper bound for A and a lower bound for B. Thus,

$$\sup A \le M \le \inf B.$$

This implies at once that $\sup A \le \inf B$. If $\sup A < \inf B$, then any number m in between would satisfy $a < m < b$ for all $a \in A, b \in B$, contradicting the assumption, hence ① implies ②.
By the properties of the infimum and the supremum,

$$(\forall \varepsilon > 0)(\exists a \in A)(a > \sup A - \varepsilon/2)$$
$$(\forall \varepsilon > 0)(\exists b \in B)(b < \inf B + \varepsilon/2).$$

That is,

$$(\forall \varepsilon > 0)(\exists a \in A)(a + \varepsilon/2 > \sup A)$$
$$(\forall \varepsilon > 0)(\exists b \in B)(b - \varepsilon/2 < \inf B),$$

i.e., $(\forall \varepsilon > 0)(\exists a \in A, b \in B)$ such that

$$b - a < (\inf B + \varepsilon/2) - (\sup A - \varepsilon/2) = \varepsilon + (\inf B - \sup A).$$

Thus, ② implies ③.

It remains to show that ③ implies ①. Suppose M was not unique, i.e., there were $M_1 < M_2$ such that

$$(\forall a \in A \wedge \forall b \in B)(a \le M_1 < M_2 \le b).$$

Set $m = (M_1 + M_2)/2$ and $\varepsilon = (M_2 - M_1)$. Then,

$$(\forall a \in A, b \in B)\underbrace{(a \le m - \varepsilon/2, b \ge m + \varepsilon/2)}_{b-a \ge (m+\varepsilon/2)-(m-\varepsilon/2)=\varepsilon},$$

i.e.,

$$(\exists \varepsilon > 0)(\forall a \in A \wedge \forall b \in B)(b - a > \varepsilon),$$

i.e., if ① does not hold then ③ does not hold. This concludes the proof. ■

2.8 Rational powers

We defined the ***integer powers*** recursively,

$$x^1 = x \qquad x^{k+1} = x \cdot x^k.$$

We also define for $x \ne 0$, $x^0 = 1$ and $x^{-n} = 1/x^n$.

Proposition 2.53 For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, x^n is well-defined, namely,

$$(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(\exists! y \in \mathbb{R})(y = x^n).$$

Proof. Given $x \in R$, define

$$I = \{n \in \mathbb{N} \,:\, (\exists! y \in \mathbb{R})(y = x^n)\}.$$

Then, I is an inductive set. ■

Proposition 2.54 — Properties of integer powers. Let $x, y \ne 0$ and $m, n \in \mathbb{Z}$, then

1. $x^m x^n = x^{m+n}$.
2. $(x^m)^n = x^{mn}$.
3. $(xy)^n = x^n y^n$.
4. $0 < \alpha < \beta$ and $n > 0$ implies $\alpha^n < \beta^n$.
5. $0 < \alpha < \beta$ and $n < 0$ implies $\alpha^n > \beta^n$.
6. $\alpha > 1$ and $n > m$ implies $\alpha^n > \alpha^m$.
7. $0 < \alpha < 1$ and $n > m$ implies $\alpha^n < \alpha^m$.

Proof. The proof is by induction (since the power are defined inductively). We start by proving it for natural powers and then extend it to integer powers.

Take, for example, the first statement. Let $x \in \mathbb{R}$ be given. Fix $m \in \mathbb{N}$ and let

$$I = \{n \in \mathbb{N} \,:\, x^m x^n = x^{m+n}\}.$$

First, $1 \in N$ since $x^m x^1 = x^m x = x^{m+1}$. Suppose that $n \in I$, then

$$x^m x^{n+1} = x^m x^n x = x^{m+n} x = x^{m+n+1},$$

which implies that $n+1 \in I$, hence $I = \mathbb{N}$. ∎

Definition 2.11 Let $x > 0$. An ***n*-th root** of x is a positive number y, such that $y^n = x$.

It is easy to see that if x has an n-th root then it is unique (since $y^n = z^n$, would imply that $y = z$). Thus, we denote it by either $\sqrt[n]{x}$, or by $x^{1/n}$.

Theorem 2.55 — Existence and uniqueness of roots.

$$(\forall x > 0 \wedge \forall n \in \mathbb{N})(\exists! y > 0)(y^n = x).$$

Proof. Consider the set

$$S = \{z \geq 0 \,:\, z^n < x\}.$$

This is a non-empty set (it contains zero) and bounded from above, since

$$\begin{aligned} &\text{if } x \leq 1 \text{ and } z \in S \text{ then } z^n < x \leq 1, \text{ which implies that } z \leq 1 \\ &\text{if } x > 1 \text{ and } z \in S \text{ then } z^n < x < x^n, \text{ which implies that } z < x, \end{aligned}$$

it follows that regardless of the value of x, $z \in S$ implies that $z \leq \max(1, x)$, and the latter is hence an upper bound for S. As a consequence of the axiom of completeness, there exists a unique $y = \sup S$, which is the "natural suspect" for being an n-th root of x. Indeed, we will show that $y^n = x$.

Claim: y is positive

Indeed,

$$0 < \frac{x}{1+x} < 1,$$

hence

$$\left(\frac{x}{1+x}\right)^n < \frac{x}{1+x} < x.$$

Thus, $x/(1+x) \in S$, which implies that

$$0 < \frac{x}{1+x} \leq y.$$

Claim: $y^n \geq x$

Suppose, by contradiction that $y^n < x$. We will show by contradiction that y is not an upper bound for S by showing that there exists an element of S that is greater than y. We will do it by showing the existence of an $\varepsilon > 0$ such that

$$\frac{y}{1-\varepsilon} \in S \qquad \text{i.e.} \qquad \left(\frac{y}{1-\varepsilon}\right)^n < x.$$

This means that we look for an $\varepsilon > 0$ satisfying

$$\frac{y^n}{x} < (1-\varepsilon)^n.$$

Since we can choose $\varepsilon > 0$ at will, we can take it smaller than 1, in which case

$$(1-\varepsilon)^n \geq (1-\varepsilon n)$$

If we choose ε sufficiently small such that

$$1-\varepsilon n > \frac{y^n}{x},$$

then a forteriori $(1-\varepsilon)^n > y^n/x$, and this will be satisfied if we choose

$$\varepsilon < \frac{1-y^n/x}{n},$$

which is possible because the right hand side is positive. Thus, we found a number greater than y, whose n-th power is less than x, i.e., in S. This contradicts the assumption that y is an upper bound for S.

Claim: $y^n \leq x$

Suppose, by contradiction that $y^n > x$. This time we will show that there exists a positive number less than y whose n-th power is greater then x, i.e., it is an upper bound for S, contradicting the assumption that y is the least upper bound for S.

To find such a number we will show that there exists an $\varepsilon > 0$, such that

$$[(1-\varepsilon)y]^n > x,$$

i.e.,

$$(1-\varepsilon)^n > \frac{x}{y^n}.$$

Once again, we can assume that $\varepsilon < 1$, in which case by the Bernoulli inequality $(1-\varepsilon)^n \geq (1-\varepsilon n)$, so that if

$$(1-\varepsilon n) > \frac{x}{y^n},$$

then a forteriori $(1-\varepsilon)^n > x/y^n$. Thus, we need $0 < \varepsilon < 1$ to satisfiy

$$\varepsilon < \frac{1-x/y^n}{n},$$

which is possible because the right hand side is positive.

From the two inequalities follows (by trichotomy) that $y^n = x$. ■

Having shown that every positive number has a unique n-th root, we may proceed to define rational powers.

Definition 2.12 Let $r \in \mathbb{Q}$, then for all $x > 0$

$$x^r = (x^m)^{1/n},$$

where $m/n = r$.

There is one little delicacy with this definition. Rational numbers do not have a unique representation as the ratio of two integers. We thus need to show that the above definition is independent of the representation. In other words, if $ad = bc$, with $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$, then for all $x > 0$

$$x^{a/b} = x^{c/d}.$$

This is a non-trivial fact. We need to prove that $ad = bc$ implies that

$$\sup\{z > 0 \,:\, z^b < x^a\} = \sup\{z > 0 \,:\, z^d < x^c\}.$$

The arguments goes as follows:

$$(x^{a/b})^{bd} = (((x^a)^{1/b})^b)^d = (x^a)^d = x^{ad} = x^{bc} = (x^c)^b = (((x^c)^{1/d})^d)^b = (x^{c/d})^{bd},$$

hence $x^{a/b} = x^{c/d}$.

Proposition 2.56 — Properties of rational powers. Let $x, y > 0$ and $r, s \in \mathbb{Q}$, then

1. $x^r x^s = x^{r+s}$.
2. $(x^r)^s = x^{rs}$.
3. $(xy)^r = x^r y^r$.
4. $0 < \alpha < \beta$ and $r > 0$ implies $\alpha^r < \beta^r$.
5. $0 < \alpha < \beta$ and $r < 0$ implies $\alpha^r > \beta^r$.
6. $\alpha > 1$ and $r > s$ implies $\alpha^r > \alpha^s$.
7. $0 < \alpha < 1$ and $r > s$ implies $\alpha^r < \alpha^s$.

Proof. We are going to prove only two items. Start with the first. Let $r = a/b$ and $s = c/d$. Using the (proved!) laws for integer powers,

$$(x^{a/b} x^{c/d})^{bd} = (x^{a/b})^{bd} (x^{c/d})^{bd} = \cdots = x^{ad} x^{bc} = x^{ad+bc} = (x^{a/b+c/d})^{bd},$$

from which follows that

$$x^{a/b} x^{c/d} = x^{a/b+c/d}.$$

Take then the fourth item. Let $r = a/b$. We already know that $\alpha^a < \beta^a$. Thus it remains to show $\alpha^{1/b} < \beta^{1/b}$. This has to be because $\alpha^{1/b} \geq \beta^{1/b}$ would have implied (from the rules for integer powers!) that $\alpha \geq \beta$. ∎

2.9 Real powers

Having defined rational powers, we proceed to define real powers. Think of it: how would you define, say, $3^{\sqrt{2}}$? One way of doing so is to use the fact that we have a definition for rational powers, along with, once again, the notion of the least-upper bound:

Definition 2.13 For $x > 0$ and $a \in \mathbb{R}$ we define

$$x^a = \begin{cases} \sup\{x^r \,:\, \mathbb{Q} \ni r < a\} & x > 1 \\ 1 & x = 1 \\ (1/x)^{-a} & x < 1. \end{cases}$$

We first need to prove that this definition is valid. The case of $x < 1$ relies on the case $x > 1$, which is well-define only if the set

$$\{x^r \,:\, \mathbb{Q} \ni r < a\}$$

if non-empty and bounded from above. That it is non-empty follows from the fact that for every $a \in \mathbb{R}$ there exists an $r \in \mathbb{Q}$ satisfying $r < a$. That it bounded follows from the fact that there exists a $q \in \mathbb{Q}$ satisfying $q > a$, and that x^q is an upper bound for this set.

And now, it remains to prove that real-valued powers satisfy the same properties as rational powers, namely:

Proposition 2.57 — Properties of real powers. Let $x, y > 0$ and $a, b \in \mathbb{R}$, then

1. $x^a x^b = x^{a+b}$.
2. $(x^a)^b = x^{ab}$.
3. $(xy)^a = x^a y^a$.
4. $0 < \alpha < \beta$ and $a > 0$ implies $\alpha^a < \beta^a$.
5. $0 < \alpha < \beta$ and $a < 0$ implies $\alpha^a > \beta^a$.
6. $\alpha > 1$ and $a > b$ implies $\alpha^a > \alpha^b$.
7. $0 < \alpha < 1$ and $a > b$ implies $\alpha^a < \alpha^b$.

Once again, we will only prove one property. To complicate things, we need to distinguish between the cases $x > 1$ and $x < 1$. We will prove the first property for $x > 1$. But first, we will introduce new definitions, and prove a couple of lemmas:

Definition 2.14 Let $A, B \subset \mathbb{R}$ be non-empty sets and let $\alpha \in \mathbb{R}$. We denote

$$\begin{aligned} A + B &= \{a + b \,:\, a \in A,\ b \in B\} \\ AB &= \{ab \,:\, a \in A,\ b \in B\} \\ \alpha A &= \{\alpha a \,:\, a \in A\}. \end{aligned}$$

Comment 2.16 Note that in general $A + A \neq 2A$.

Proposition 2.58 Let $A, B \subset \mathbb{R}$ be non-empty sets bounded from above. Then,

$$\sup(A+B) = \sup A \ \sup B.$$

Proof. For the ease of notation, write $\sup A = s_A$ and $\sup B = s_B$. For every $a \in A$ and $b \in B$,

$$a + b \le s_A + s_B,$$

which implies that $s_A + s_B$ is an upper-bound for $A + B$.
To prove that it is the least upper bound we need to show that for every $\varepsilon > 0$ there exist $a \in A$ and $b \in B$, satisfying

$$a + b > s_A + s_B - \varepsilon.$$

By the definition of the least upper bounds, there exist $a \in A$ satisfying $a > s_A - \varepsilon/2$ and $b \in B$ satisfying $b > s_B - \varepsilon/2$. Their sum satisfies the required property. ∎

Proposition 2.59 Let $A \subset \mathbb{R}$ be a non-empty set bounded from above and let $\alpha > 0$. Then,

$$\sup(\alpha A) = \alpha \sup A.$$

Proof. Once again, write $\sup A = s_A$. For every $a \in A$,

$$\alpha a \le \alpha s_A,$$

which implies that αs_A is an upper-bound for αA.
To prove that it is the least upper bound we need to show that for every $\varepsilon > 0$ there exist $a \in A$ satisfying

$$\alpha a > \alpha s_A - \varepsilon.$$

By the definition of the least upper bounds, there exist $a \in A$ satisfying $a > s_A - \varepsilon/\alpha$. Then,

$$\alpha a > \alpha\,(s_A - \varepsilon/\alpha)\,,$$

which completes the proof. ∎

Proposition 2.60 Let $A, B \subset \mathbb{R}$ be non-empty sets of positive numbers bounded from above. Then,

$$\sup(AB) = \sup A \ \sup B.$$

Proof. The proof is similar to the two previous ones, but requires some more delicacy. Let again $\sup A = s_A$ and $\sup B = s_B$. For every $a \in A$ and $b \in B$,

$$ab \le a s_B \le s_A s_B,$$

which implies that $s_A s_B$ is an upper-bound for AB.

To prove that it is the least upper bound we need to show that for every $\varepsilon > 0$ there exist $a \in A$ and $b \in B$, satisfying

$$ab > s_A s_B - \varepsilon.$$

By the definition of the least upper bounds, there exist $a \in A$ and $b \in B$ satisfying

$$a > s_A - \frac{\varepsilon}{s_A + s_B} > 0 \qquad \text{and} \qquad b > s_B - \frac{\varepsilon}{s_A + s_B} > 0.$$

Then,

$$ab > \left(s_A - \frac{\varepsilon}{s_A + s_B}\right)\left(s_B - \frac{\varepsilon}{s_A + s_B}\right) = s_A s_B - \varepsilon + \frac{e^2}{(s_A + s_B)^2} > s_A s_B - \varepsilon.$$

■

Proposition 2.61 For every $x > 1$ and $a, b \in \mathbb{R}$,

$$x^a x^b = x^{a+b}.$$

Proof. Denote

$$A = \{x^r \ : \ \mathbb{Q} \ni r < a\} \qquad \text{and} \qquad B = \sup\{x^q \ : \ \mathbb{Q} \ni q < b\}.$$

We need to prove that

$$\sup\{x^p \ : \ \mathbb{Q} \ni p < a + b\} = \sup A \ \sup B.$$

By Proposition 2.60, $\sup A \ \sup B = \sup(AB)$, which means that we have to prove that

$$\sup\{x^{r+q} \ : \ \mathbb{Q} \ni r < a, \ \mathbb{Q} \ni q < b\} = \sup\{x^p \ : \ \mathbb{Q} \ni p < a + b\},$$

where we used the fact that for rational powers $x^r x^q = x^{r+q}$. The proof follows from the fact for every $p < a + b$ there exist $\mathbb{Q} \ni r < a$ and $\mathbb{Q} \ni q < b$ satisfying $r + q > p$, and conversely, for every $r < a$ and $q < b$ there exists a $\mathbb{Q} \ni p < a + b$ satisfying $r + q < p$. ■

2.10 Addendum

2.10.1 Connected sets

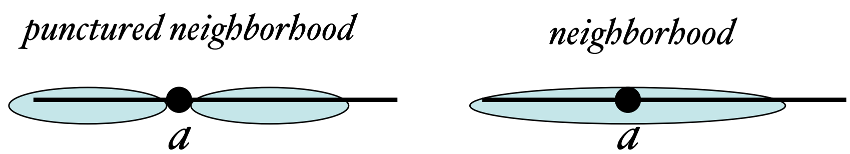
A set $A \subseteq \mathbb{R}$ will be said to be ***connected*** (קשיר) if $x, y \in A$ and $x < z < y$ implies $z \in A$ (that is, every point between two elements in the set is also n the set). We state that a (non-empty) connected set can only be of one of the following forms:

$$\{a\}, \quad (a,b), \quad [a,b), \quad (a,b], \quad [a,b]$$
$$(a,\infty), \quad [a,\infty), \quad (-\infty,b), \quad (-\infty,b], \quad (-\infty,\infty).$$

That is, it can only be a single point, an open, closed or semi-open segments, an open or closed ray, or the whole line.

2.10.2 Neighborhoods

Definition 2.15 Let $x \in \mathbb{R}$. A ***neighborhood*** of x (סביבה) is an open segment (a,b) that contains the point x (note that since the segment is open, x cannot be a boundary point). A ***punctured neighborhood*** of x (סביבה מנוקבת) is a set $(a,b) \setminus \{x\}$ where $a < x < b$.



Definition 2.16 Let $A \subset \mathbb{R}$. A point $a \in A$ is called an ***interior point*** of A (נקודה פנימית) if it has a neighborhood contained in A.

Notation 2.1 We will mostly deal with ***symmetric*** neighborhoods (whether punctured or not), i.e., neighborhoods of a of the form

$$\{x : |x-a| < \delta\}$$

for some $\delta > 0$. We will introduce the following notations for neighborhoods,

$$B(a,\delta) = (a-\delta, a+\delta)$$
$$B^\circ(a,\delta) = \{x : 0 < |x-a| < \delta\}.$$

We will also define ***one-sided neighborhoods***,

$$B_+(a,\delta) = [a, a+\delta)$$
$$B_+^\circ(a,\delta) = (a, a+\delta)$$
$$B_-(a,\delta) = (a-\delta, a]$$
$$B_-^\circ(a,\delta) = (a-\delta, a).$$

■ **Example 2.5** For $A = [0,1]$, the point $1/2$ is an interior point, but not the point 0.
■

■ **Example 2.6** The set $\mathbb{Q} \subset \mathbb{R}$ has no interior points, and neither does its complement, $\mathbb{R} \setminus \mathbb{Q}$. ■