Chapter 2

Measure spaces

2.1 Sets and σ-algebras

2.1.1 Definitions

Let $\mathbb{X}$ be a (non-empty) set; at this stage it is only a set, i.e., not yet endowed with any additional structure. Our eventual goal is to define a measure on subsets of $\mathbb{X}$. As we have seen, we may have to restrict the subsets to which a measure is assigned. There are, however, certain requirements that the collection of sets to which a measure is assigned—**measurable sets** (קבוצות מדידות)—should satisfy. If two sets are measurable, we would like their union, intersection and complements to be measurable as well. This leads to the following definition:

Definition 2.1 Let $\mathbb{X}$ be a non-empty set. A (boolean) ***algebra of subsets*** *(אלגברה של קבוצות) of $\mathbb{X}$, is a non-empty collection of subsets of $\mathbb{X}$, closed under finite unions (איחוד סופי) and complementation. A (boolean)* ***σ-algebra*** *(סיגמה אלגברה) Σ of subsets of $\mathbb{X}$ is a non-empty collection of subsets of $\mathbb{X}$, closed under countable unions (איחוד בן מנייה) and complementation.*

Comment: In the context of probability theory, the elements of a σ-algebra are called **events** (מאורעות).

Proposition 2.2 Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$. Then, $\mathcal{A}$ contains $\mathbb{X}$, the empty set, and it is closed under finite intersections.

Proof: Since $\mathcal{A}$ is not empty, it contains at least one set $A \in \mathcal{A}$. Then,

$$\mathbb{X} = A \cup A^c \in \mathcal{A} \qquad \text{and} \qquad \emptyset = \mathbb{X}^c \in \mathcal{A}.$$

Let $A_1, \dots, A_n \subset \mathcal{A}$. By de Morgan's laws,

$$\bigcap_{k=1}^{n} A_k = \left(\bigcup_{k=1}^{n} A_k^c \right)^c \in \mathcal{A}.$$

■

Given in 2018

✎ *Exercise 2.1* Let $\mathbb{X}$ be a set and let $\mathcal{A}$ be a collection of subsets containing $\mathbb{X}$ and closed under set subtraction (i.e, if $A, B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$. Show that $\mathcal{A}$ is an algebra.

The following useful proposition states that for an algebra to be a σ-algebra, it suffices to require closure under countable *disjoint* union:

Proposition 2.3 Let $\mathbb{X}$ be a non-empty set, and let $\mathcal{A} \subset \mathscr{P}(\mathbb{X})$ be an algebra of subsets of $\mathbb{X}$, closed under countable disjoint *unions. Then, $\mathcal{A}$ is a σ-algebra.*

Proof: We need to show that $\mathcal{A}$ is closed under countable unions. Let $(A_n) \subset \mathcal{A}$, and define recursively the sequence of disjoint sets (B_n),

$$B_1 = A_1 \in \mathcal{A} \qquad \text{and} \qquad B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right) \in \mathcal{A}.$$

Then,

$$\bigcup_{n=1}^{\infty} A_n = \coprod_{n=1}^{\infty} B_n \in \mathcal{A}.$$

(The inclusion $\coprod_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$ is obvious. Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then, there exists a minimal n such that $x \in A_n$. By definition of the B_n's, $x \in B_n$, hence $x \in \coprod_{n=1}^{\infty} B_n$, which proves the reverse inclusion.) ■

Trick #1: Turn a sequence of sets into a sequence of disjoint sets having the same
—1h(2018)— union.

*Definition 2.4 A non-empty set endowed with a σ-algebra of subsets is called a **measurable space** (מרחב מדיד).*

To conclude, a σ-algebra is a collection of sets closed under countably-many set-theoretic operations.

Examples:

1. Let $\mathbb{X}$ be a non-empty set. Then,
$$\{\varnothing, \mathbb{X}\}$$
is its smallest σ-algebra (the trivial σ-algebra).

2. Let $\mathbb{X}$ be a non-empty set. Then, its **power set** (קבוצת החזקה) $\mathscr{P}(\mathbb{X})$ is its maximal σ-algebra.

3. Let $\mathbb{X}$ be a non-empty set. Then,
$$\Sigma = \{A \subset \mathbb{X} \; : \; A \text{ is countable or } A^c \text{ is countable}\}$$
is a σ-algebra (the **countable or co-countable sets**). Clearly, Σ is non-empty and is closed under complementation. Let $(A_n) \subset \Sigma$. If all A_n are countable, then their union is also countable, hence in Σ. Conversely, if one of the A_n is co-countable, then their union is also co-countable, hence in Σ.

The following proposition states that every intersection of σ-algebras is a σ-algebra. This property will be used right after to prove that every collection of subsets defines a unique σ-algebra, which is the smallest σ-algebra containing that collection.

Proposition 2.5 Let $\{\Sigma_\alpha \subset \mathscr{P}(\mathbb{X}) \; : \; \alpha \in J\}$ be a (not necessarily countable) collection of σ-algebras. Then, their intersection is a σ-algebra.

Proof: Set
$$\Sigma = \bigcap_{\alpha \in J} \Sigma_\alpha.$$
Since for every $\alpha \in J$, $\varnothing, \mathbb{X} \in \Sigma_\alpha$, it follows that Σ contains $\varnothing$ and $\mathbb{X}$. Let $A \in \Sigma$; by definition,
$$A \in \Sigma_\alpha \qquad \forall \alpha \in J.$$

Since each of the Σ_α is a σ-algebra,

$$A^c \in \Sigma_\alpha \qquad \forall \alpha \in J,$$

hence $A^c \in \Sigma$, proving that Σ is closed under complementation. Likewise, let $(A_n) \subset \Sigma$. Then,

$$A_n \in \Sigma_\alpha \qquad \forall n \in \mathbb{N} \quad \text{and} \quad \forall \alpha \in J.$$

It follows that

$$\bigcup_{n=1}^\infty A_n \in \Sigma_\alpha \qquad \forall \alpha \in J,$$

hence

$$\bigcup_{n=1}^\infty A_n \in \Sigma,$$

proving that Σ is closed under countable unions. ■

✎ *Exercise 2.2* Let $\mathbb{X}$ be a set. (a) Show that the collection of subsets that are either finite or co-finite is an algebra. (b) Show that it is a σ-algebra if and only if $\mathbb{X}$ is a finite set.

Given in 2018

✎ *Exercise 2.3* (a) Let $\mathcal{A}_n$ be an increasing sequence of algebras of subsets of $\mathbb{X}$, i.e., $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. Show that

$$\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{A}_n$$

is an algebra. (b) Give an example in which $\mathcal{A}_n$ is an increasing sequence of σ-algebras, and its union is not a σ-algebra.

Let $\mathscr{E} \subset \mathscr{P}(\mathbb{X})$ be a collection of sets and consider the family of all σ-algebras of subsets of $\mathbb{X}$ containing $\mathscr{E}$. This family is not empty since is includes at least the power set $\mathscr{P}(\mathbb{X})$. By Proposition 2.5,

$$\sigma(\mathcal{E}) = \bigcap\{\Sigma \,:\, \Sigma \text{ is a } \sigma\text{-algebra containing } \mathcal{E}\}$$

is a σ-algebra; it is the smallest σ-algebra containing $\mathcal{E}$; it is called the σ-algebra **generated by** (נוצרת על ידי) $\mathcal{E}$.

A comment about nomenclature: we will repeatedly deal with sets of sets and sets of sets of sets; to avoid confusion, we will use the terms **collection** (אוסף), **class** (מחלקה) and **family** (משפחה) for sets whose elements are sets.

The following proposition comes up useful:

Proposition 2.6 *Let $\mathcal{E} \subset \mathscr{P}(\mathbb{X})$ and let $\Sigma \subset \mathscr{P}(\mathbb{X})$ be a σ-algebra. If*

$$\mathcal{E} \subset \Sigma,$$

then

$$\sigma(\mathcal{E}) \subset \Sigma.$$

Proof: Since Σ is a σ-algebra containing $\mathcal{E}$, then, by definition, it contains $\sigma(\mathcal{E})$. ■

Corollary 2.7 *Let $\mathcal{E}, \mathcal{F} \subset \mathscr{P}(\mathbb{X})$. If*

$$\mathcal{E} \subset \sigma(\mathcal{F}) \qquad \text{and} \qquad \mathcal{F} \subset \sigma(\mathcal{E}),$$

then

$$\sigma(\mathcal{E}) = \sigma(\mathcal{F}).$$

✎ ***Exercise 2.4*** Let $\mathbb{X}$ be a non-countable set. Let

$$\mathcal{E} = \{\{x\} \ : \ x \in \mathbb{X}\}.$$

What is the σ-algebra generated by $\mathcal{E}$?

✎ ***Exercise 2.5*** Let $A_1, \dots, A_n$ be a finite number of subsets of $\mathbb{X}$.

(a) Prove that if the $A_1, \dots, A_n$ are disjoint and their union is $\mathbb{X}$, then

$$|\sigma(\{A_1, \dots, A_n\})| = 2^n.$$

(b) Prove that for arbitrary $A_1, \dots, A_n$,

$$\sigma(\{A_1, \dots, A_n\})$$

contains a finite number of elements.

✎ ***Exercise 2.6*** Let $(\mathbb{X}, \Sigma)$ be a measurable space. Let $\mathcal{C}$ be a collection of sets. Let P be a property of sets in $\mathbb{X}$ (i.e., a function $P : \mathscr{P}(\mathbb{X}) \to \{\text{true}, \text{false}\}$), such that

(a) $P(\varnothing) = \text{true}$.

(b) $P(C)$ = true for all $C \in \mathcal{C}$.

(c) $P(A)$ = true implies that $P(A^c)$ = true.

(d) if $P(A_n)$ = true for all n, then $P(\bigcup_{n=1}^{\infty} A_n)$ = true.

Show that $P(A)$ = true for all $A \in \sigma(\mathcal{C})$.

Given in 2018

✎ *Exercise 2.7* Prove that every member of a σ-algebra is **countably-generated**: i.e., let $\mathcal{C}$ be a collection of subsets of $\mathbb{X}$ and let $A \in \sigma(\mathcal{C})$. Prove that there exists a countable sub-collection $\mathcal{C}_A \subset \mathcal{C}$, such that $A \in \sigma(\mathcal{C}_A)$.

Given in 2018

✎ *Exercise 2.8* Let $f : \mathbb{X}_1 \to \mathbb{X}_2$ be a function between two sets. Prove that if Σ_2 is a σ-algebra on $\mathbb{X}_2$, then

$$\Sigma_1 = \{f^{-1}(A) \,:\, A \in \Sigma_2\}$$

is a σ-algebra on $\mathbb{X}_1$.

Given in 2018

✎ *Exercise 2.9* Let $(\mathbb{X}, \Sigma)$ be a measurable space and let $A_n \in \Sigma$ be a sequence of measurable set. The **superior limit** of this sequence is the set of points x which belong to A_n for infinitely-many n's, i.e.,

$$\limsup_{n\to\infty} A_n = \{x \in \mathbb{X} \,:\, |\{n \,:\, x \in A_n\}| = \infty\}.$$

The **inferior limit** of this sequence is the set of points x which belong to A_n for all but finitely-many n's, i.e.,

$$\liminf_{n\to\infty} A_n = \{x \in \mathbb{X} \,:\, |\{n \,:\, x \notin A_n\}| < \infty\}.$$

(a) Prove that $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

(b) Prove that $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

(c) Prove that $(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c$.

(d) Prove that $(\liminf_{n\to\infty} A_n)^c = \limsup_{n\to\infty} A_n^c$.

(e) Denote by χ_A the **indicator function** of a measurable set A. Prove that

$$\chi_{\limsup_{n\to\infty} A_n} = \limsup_{n\to\infty} \chi_{A_n} \qquad \text{and} \qquad \chi_{\liminf_{n\to\infty} A_n} = \liminf_{n\to\infty} \chi_{A_n}.$$

2.1.2 The Borel σ-algebra on $\mathbb{R}$

A σ-algebra is a structure on a set. Another set structure you are familiar with is a topology. Open sets and measurable sets are generally distinct entities, however, we often want to assign a measurable structure to a topological space. For that, there is a natural construction:

Definition 2.8 *Let* $(\mathbb{X}, \tau_{\mathbb{X}})$ *be a* ***topological space*** *(מרחב טופולוגי). The* σ*-algebra generated by the open sets is called the* ***Borel*** σ***-algebra*** *and is denoted*

$$\mathscr{B}(\mathbb{X}) = \sigma(\tau_{\mathbb{X}}).$$

Its elements are called ***Borel sets*** *(קבוצות בורל) (roughly speaking, it is the collection of all sets obtained by countably-many set-theoretic operations on the open sets).*

Since we have special interest in the topological spaces $\mathbb{R}^n$ (endowed with the topology induced by the Euclidean metric), we want to get acquainted with its Borel sets. We first investigate the Borel σ-algebra $\mathscr{B}(\mathbb{R})$, and then construct σ-algebras for products of measurable spaces.

Proposition 2.9 *The Borel* σ*-algebra* $\mathscr{B}(\mathbb{R})$ *is generated by each of the following collections of subsets,*

$$\begin{gathered}\{(a,b) \ : \ a<b\}\\ \{[a,b] \ : \ a<b\}\\ \{[a,b) \ : \ a<b\}\\ \{(a,b] \ : \ a<b\}\\ \{(a,\infty) \ : \ a\in\mathbb{R}\}\\ \{[a,\infty) \ : \ a\in\mathbb{R}\}.\end{gathered}$$

Proof: Since

$$\begin{gathered}(a,b) \subset \mathscr{B}(\mathbb{R})\\ [a,b] = \cap_{n=1}^{\infty}(a-1/n, b+1/n) \subset \mathscr{B}(\mathbb{R})\\ [a,b) = \cap_{n=1}^{\infty}(a-1/n, b) \subset \mathscr{B}(\mathbb{R})\\ (a,\infty) = \cup_{n>a}^{\infty}(a,n) \subset \mathscr{B}(\mathbb{R})\\ [a,\infty) = \cap_{m=1}^{\infty}\cup_{n>a}^{\infty}(a-1/m, n) \subset \mathscr{B}(\mathbb{R}),\end{gathered}$$

and since $\mathscr{B}(\mathbb{R})$ is a σ-algebra, it follows from Proposition 2.6 that the σ-algebra generated by each of the collections on the right-hand side is a subset of $\mathscr{B}(\mathbb{R})$.

It remains to prove the reverse inclusion. Since every open set in $\mathbb{R}$ is a countable union of open intervals ($\mathbb{R}$ is **first countable** (אקסיומת מנייה ראשונה)), then

$$\tau_{\mathbb{R}} \subset \sigma(\{(a,b) \ : \ a<b\}),$$

and by Proposition 2.6,

$$\mathscr{B}(\mathbb{R}) = \sigma(\tau_{\mathbb{R}}) \subset \sigma(\{(a,b) \,:\, a < b\}).$$

Proving the other cases is not much harder. ■

Given in 2018

✎ *Exercise 2.10* Show that every open set in $\mathbb{R}$ is a countable disjoint union of open segments. (You may use the fact that the rational numbers form a countable set.)

✎ *Exercise 2.11* Prove that

$$\mathscr{B}(\mathbb{R}) = \sigma(\{[a,\infty) \,:\, a \in \mathbb{R}\}).$$

Comment: Let $\mathcal{C}$ be a collection of subsets of $\mathbb{X}$ containing the empty set. We define

$$\mathcal{C}_{\sigma c} = \left\{ \bigcup_{n=1}^{\infty} C_n, \; \left(\bigcup_{n=1}^{\infty} C_n\right)^c \,:\, C_n \in \mathcal{C} \right\},$$

which is this collection, augmented by all possible countable unions of elements and their complements. Define now recursively,

$$\mathcal{C}^{(0)} = \mathcal{C} \qquad \text{and} \qquad \mathcal{C}^{(n+1)} = \mathcal{C}^{(n)}_{\sigma c}.$$

Finally, let

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{C}^{(n)}.$$

Intuitively, we might expect that $\mathcal{D} = \sigma(\mathcal{C})$. This turns out not to be true, showing how large a σ-algebra might be.

Theorem 2.10 *The Borel σ-algebra on $\mathbb{R}$ has the cardinality of the continuum,*

$$\operatorname{card}(\mathscr{B}(\mathbb{R})) = \operatorname{card}(\mathbb{R}).$$

In particular, the cardinality of $\mathscr{B}(\mathbb{R})$ is strictly smaller than the cardinality of $\mathscr{P}(\mathbb{R})$.

Proof: This is a set-theoretic statement, whose proof we leave for the interested reader to complete. ■

Another set-theoretic statement is:

Theorem 2.11 *A σ-algebra is either finite or not countable.*

Proof:[Sketch] One can show that an infinite σ-algebra contains at least an $\aleph_0$ number of disjoint set, A_n. Then, all the sets of the form

$$\coprod_{n=1}^{\infty} A_n^{s_n},$$

where $s_n = \pm 1$ with $A_n^1 = A_n$ and $A_n^{-1} = A_n^c$ are distinct, and there are $2^{\aleph_0}$ many of those. ■

2.1.3 Products of measurable spaces

Let $(\mathbb{X}_n)$ be a sequence of non-empty sets. The product set

$$\mathbb{X} = \bigtimes_{n=1}^{\infty} \mathbb{X}_n$$

consists of sequences in which the n-th element belongs to $\mathbb{X}_n$. For every

$$x = (x_1, x_2, \dots) \in \mathbb{X},$$

we define the projections

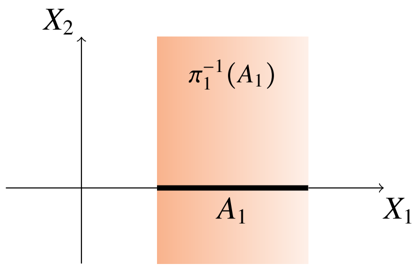
$$\pi_n : \mathbb{X} \to \mathbb{X}_n, \qquad \pi_n : x \mapsto x_n.$$

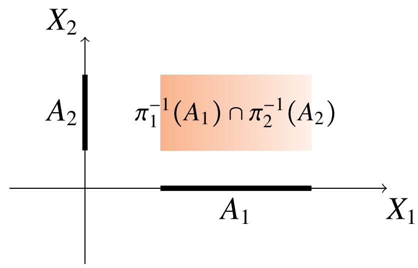
Definition 2.12 *Let $(\mathbb{X}_n, \Sigma_n)$ be a sequence of measurable spaces and let $\mathbb{X} = \bigtimes_{n=1}^{\infty} \mathbb{X}_n$. The **product σ-algebra** on $\mathbb{X}$ is*

$$\bigotimes_{n=1}^{\infty} \Sigma_n \overset{def}{=} \sigma\left(\{\pi_n^{-1}(A_n) \ : \ A_n \in \Sigma_n, \ n \in \mathbb{N}\}\right).$$

Proposition 2.13 *Let $(\mathbb{X}_n, \Sigma_n)$ be sequence of measurable spaces. Then,*

$$\bigotimes_{n=1}^{\infty} \Sigma_n = \sigma\left(\left\{\bigtimes_{n=1}^{\infty} A_n \ : \ A_n \in \Sigma_n\right\}\right).$$





Proof: Denote

$$\mathcal{E} = \left\{ \bigtimes_{n=1}^{\infty} A_n \;:\; A_n \in \Sigma_n \right\} \qquad \text{and} \qquad \mathcal{F} = \left\{ \pi_n^{-1}(A_n) \;:\; A_n \in \Sigma_n,\;\; n \in \mathbb{N} \right\}.$$

We need to prove that $\sigma(\mathcal{E}) = \sigma(\mathcal{F})$. By Corollary 2.7, it suffices to show that

$$\mathcal{E} \subset \sigma(\mathcal{F}) \qquad \text{and} \qquad \mathcal{F} \subset \sigma(\mathcal{E}).$$

Indeed, for every element of $\mathcal{F}$,

$$\pi_n^{-1}(A_n) = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{n-1} \times A_n \times \mathbb{X}_{n+1} \times \cdots \in \mathcal{E} \subset \sigma(\mathcal{E}),$$

and for every element of $\mathcal{E}$,

$$\bigtimes_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \pi_n^{-1}(A_n) \in \sigma(\mathcal{F}).$$

■

Proposition 2.14 Suppose that for every $n \in \mathbb{N}$, Σ_n is generated by $\mathcal{E}_n$ and $\mathbb{X} \in \mathcal{E}_n$. Then, $\bigotimes_{n=1}^{\infty} \Sigma_n$ is generated by either of the collections of sets

$$\mathcal{F}' = \left\{ \pi_n^{-1}(E_n) \;:\; E_n \in \mathcal{E}_n,\;\; n \in \mathbb{N} \right\}.$$

and

$$\mathcal{E}' = \left\{ \bigtimes_{n=1}^{\infty} E_n \;:\; E_n \in \mathcal{E}_n \right\}.$$

Given in 2018

✎ *Exercise 2.12* Prove Proposition 2.14.

Proposition 2.15 Let $\mathbb{X}_1, \dots, \mathbb{X}_n$ be metric spaces and let $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$ be the product space equipped with the product metric (the ℓ_2-norm of the metrics). Then

$$\bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j) \subset \mathscr{B}(\mathbb{X}).$$

If the spaces are separable, then this is an equality.

Proof: First. since for $x, y \in \mathbb{X}$,

$$d_{\mathbb{X}}^2(x, y) = \sum_{j=1}^{n} d_{\mathbb{X}_j}^2(x_j, y_j),$$

it follows that convergence in $\mathbb{X}$ occurs if and only if each component converges, and in particular, the projections $\pi_j : \mathbb{X} \to \mathbb{X}_j$ are continuous.

By Proposition 2.14,

$$\bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j) = \sigma\left(\{\pi_j^{-1}(U_j) \;:\; U_j \text{ open in } \mathbb{X}_j,\ j = 1, \dots, n\}\right).$$

Since π_n is continuous, every set of the form $\pi_j^{-1}(U_j)$, where U_j open in $\mathbb{X}_j$, is open in $\mathbb{X}$, namely,

$$\{\pi_j^{-1}(U_j) \;:\; U_j \text{ open in } \mathbb{X}_j\} \subset \tau_{\mathbb{X}},$$

from which follows that

$$\bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j) \subset \sigma(\tau_{\mathbb{X}}) = \mathscr{B}(\mathbb{X}).$$

Next, suppose that the $\mathbb{X}_j$ are separable; let $C_j \subset \mathbb{X}_j$ be countable dense sets, so that

$$\bigtimes_{j=1}^{n} C_j \subset \mathbb{X}$$

is a countable dense set in $\mathbb{X}$. It follows that the collection of open sets,

$$\mathcal{F} = \left\{ \bigtimes_{j=1}^{n} B(p_j, r) \;:\; p_j \in C_j,\ \ r \in \mathbb{Q}^+ \right\}$$

is a countable basis for $\tau_{\mathbb{X}}$. Since $B(p_j, r) \in \mathscr{B}(\mathbb{X}_j)$, it follows that

$$\mathcal{F} \subset \bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j) \qquad \text{hence} \qquad \sigma(\mathcal{F}) \subset \bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j).$$

On the other hand, since every element in $\tau_{\mathbb{X}}$ is a (necessarily countable) union of elements of $\mathcal{F}$,

$$\tau_{\mathbb{X}} \subset \sigma(\mathcal{F}) \qquad \text{hence} \qquad \mathscr{B}(\mathbb{X}) \subset \sigma(\mathcal{F}),$$

hence

$$\mathscr{B}(\mathbb{X}) \subset \bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{X}_j),$$

which completes the proof. ■

Corollary 2.16 For every $n \in \mathbb{N}$,

$$\mathscr{B}(\mathbb{R}^n) = \bigotimes_{j=1}^{n} \mathscr{B}(\mathbb{R}).$$

Proof: $\mathbb{R}^n = \times_{k=1}^{n} \mathbb{R}$ and $\mathbb{R}$ is separable. ■

Given in 2018

✎ *Exercise 2.13* Show that every open set in $\mathbb{R}^2$ is a countable union of open rectangles of the form $(a,b) \times (c,d)$. Is it also true if we require the rectangles to be disjoint (Hint: a contradiction is obtained by a topological argument)?

—3h(2017)—

2.1.4 Monotone classes

The notion of a monotone class will be used later in this course. The motivation for it is as follows: given an algebra $\mathcal{A}$ of sets, the σ-algebra generated by $\mathcal{A}$ is hard to characterize. As we will see $\sigma(\mathcal{A})$ can be characterized as being the monotone class generated by $\mathcal{A}$, which is easier to perceive.

*Definition 2.17 Let $\mathbb{X}$ be a set. A **monotone class** on $\mathbb{X}$ (מחלקה מונוטונית) is a collection of subsets of $\mathbb{X}$ closed under countable increasing unions and countable* *decreasing intersections. That is, if $\mathbb{M} \subset \mathscr{P}(\mathbb{X})$ is a monotone class, and $A_n \in \mathbb{M}$ is monotonically increasing, then*

$$\bigcup_{n=1}^{\infty} A_n \in \mathbb{M}.$$

Likewise, if $B_n \in \mathbb{M}$ is monotonically decreasing, then

$$\bigcap_{n=1}^{\infty} B_n \in \mathbb{M}.$$

Comment: Every σ-algebra is a monotone class. Also, the intersection of every collection of monotone classes is a monotone class. It follows that every non-empty collection $\mathcal{E}$ of sets generates a monotone class—the minimal monotone class containing it—which we denote by $\mathbb{M}(\mathcal{E})$. —3h(2018)—

Lemma 2.18 If an algebra $\mathcal{A}$ is a monotone class, then it is a σ-algebra.

Trick #2: Turn a sequence of sets into an increasing sequence of sets having the same union.

Proof: Let $A_j \in \mathcal{A}$, then define

$$B_n = \bigcup_{j=1}^{n} A_j.$$

Clearly, $B_n \in \mathcal{A}$ and they are monotonically increasing. Since $\mathcal{A}$ is also a monotone class,

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A},$$

proving that $\mathcal{A}$ is a σ-algebra. ■

The following proposition establishes a connection between monotone classes and σ-algebras:

Proposition 2.19 Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$. Then, $\mathbb{M}(\mathcal{A})$ is a σ-algebra.

Proof: By the previous lemma, it suffices to prove that $\mathbb{M}(\mathcal{A})$ is an algebra, i.e., that it is closed under pairwise union and complementation. Specifically, we will proceed as follows:

1. $B \in \mathbb{M}(\mathcal{A})$ implies $B^c \in \mathbb{M}(\mathcal{A})$.
2. $B \in \mathbb{M}(\mathcal{A})$ and $C \in \mathcal{A}$ implies $B \cup C \in \mathbb{M}(\mathcal{A})$.
3. $B, D \in \mathbb{M}(\mathcal{A})$ implies $B \cup D \in \mathbb{M}(\mathcal{A})$.

1. Let
$$\mathcal{U} = \{B \in \mathbb{M}(\mathcal{A}) \;:\; B^c \in \mathbb{M}(\mathcal{A})\} \subset \mathbb{M}(\mathcal{A}),$$
be the subset of $\mathbb{M}(\mathcal{A})$ closed under complementation. On the one hand, since $\mathcal{A}$ is an algebra, $\mathcal{A} \subset \mathcal{U}$. On the other hand, $\mathcal{U}$ is a monotone class, for if A_n is an increasing sequence in $\mathcal{U}$, then
$$\bigcup_{n=1}^{\infty} A_n \in \mathbb{M}(\mathcal{A}) \qquad \text{and} \qquad (\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c \in \mathbb{M}(\mathcal{A}),$$
which implies that
$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{U}.$$
Similarly, since $\mathcal{U}$ is closed under complementation, it is also closed under the intersection of decreasing sequences. It follows that $\mathcal{U} = \mathbb{M}(\mathcal{A})$, i.e., $\mathbb{M}(\mathcal{A})$ is closed under complementation.
2. Let $C \in \mathcal{A}$, and define
$$\Gamma_C = \{B \in \mathbb{M}(\mathcal{A}) \;:\; B \cup C \in \mathbb{M}(\mathcal{A})\} \subset \mathbb{M}(\mathcal{A}).$$
Clearly, $\mathcal{A} \in \Gamma$. Moreover, Γ_C is a monotone class, since if A_n is an increasing sequence in Γ_C (likewise, if A_n is a decreasing sequence and considering intersections), then,
$$\bigcup_{n=1}^{\infty} A_n \in \mathbb{M}(\mathcal{A}) \qquad \text{and} \qquad \forall B \in \mathbb{M}(\mathcal{A}),\; B \cup \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (B \cup A_n) \in \mathbb{M}(\mathcal{A}),$$
which implies that
$$\Gamma_C = \mathbb{M}(\mathcal{A}) \qquad \text{for all } C \in \mathcal{A}.$$

3. Let $D \in \mathbb{M}(\mathcal{A})$ and consider Γ_D. By the previous item, it follows that Γ_D contains $\mathcal{A}$ and we can show in the same way that it is a monotone class. It follows that
$$\Gamma_D = \mathbb{M}(\mathcal{A}) \qquad \text{for all } D \in \mathbb{M}(\mathcal{A}).$$

This concludes the proof. ∎

Note that we used the following recurring approach:

Trick #3: In order to prove that all the elements of a σ-algebra Σ (or a monotone class $\mathbb{M}$) satisfy a Property P, consider the set of all elements of Σ (or $\mathbb{M}$) satisfying Property P, and show that it is a σ-algebra (or a monotone class) containing a generating collection of sets.

Theorem 2.20 (Monotone Class Theorem) Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$. Then,
$$\mathbb{M}(\mathcal{A}) = \sigma(\mathcal{A}).$$

Proof: Since $\sigma(\mathcal{A})$ is a monotone class containing $\mathcal{A}$, it follows that
$$\mathbb{M}(\mathcal{A}) \subset \sigma(\mathcal{A}).$$
Conversely, since $\mathbb{M}(\mathcal{A})$ is a σ-algebra containing $\mathcal{A}$, then
$$\sigma(\mathcal{A}) \subset \mathbb{M}(\mathcal{A}),$$
which completes the proof. ∎

✎ *Exercise 2.14* In each of the following cases, find the σ-algebra and the monotone class generated by a collection $\mathcal{E}$ of sets:

(a) Let $\mathbb{X}$ be a set and $P : \mathbb{X} \to \mathbb{X}$ a permutation (i.e., a one-to-one and only map). Then, $\mathcal{E}$ is the collection of all sets that are invariant under P.
(b) $\mathbb{X}$ is the Euclidean plane and $\mathcal{E}$ is the collection of all sets that may be covered by countably-many horizontal lines.

2.2 Measures

2.2.1 Definitions

Definition 2.21 *Let $\mathbb{X}$ be a non-empty set. A **set function** on $\mathbb{X}$ (פונקציה על קבוצות) is a function whose domain is a collection of subsets of $\mathbb{X}$.*

Definition 2.22 *Let $\mathbb{X}$ be a non-empty set and let μ be a set function whose domain is a collection $\mathcal{E}$ of subsets of $\mathbb{X}$ and whose range is the extended real line. The function μ is called **finitely-additive** (אדיטיבית סופית) if whenever $A, B \in \mathcal{E}$ are disjoint set whose union belongs to $\mathcal{E}$,*

$$\mu(A \amalg B) = \mu(A) + \mu(B).$$

*It is called **countably-additive** or **σ-additive** (סיגמה אדיטיבית) if whenever $A_n \in \mathcal{E}$ are disjoint sets whose union belongs to $\mathcal{E}$,*

$$\mu\left(\coprod_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 2.23 *Let $(\mathbb{X}, \Sigma)$ be a measurable space. A **measure** (מידה) on Σ is an extended real-valued σ-additive set function $\mu : \Sigma \to [0, \infty]$, satisfying $\mu(\varnothing) = 0$. The triple $(\mathbb{X}, \Sigma, \mu)$ is called a **measure space** (מרחב מידה).*

We next classify families of measure spaces:

Definition 2.24 *A measure space $(\mathbb{X}, \Sigma, \mu)$ is called*

*(a) **Finite** (סופי) if $\mu(\mathbb{X}) < \infty$.*
*(b) **σ-Finite** (סיגמה סופי) if $\mathbb{X} = \cup_{n=1}^{\infty} \mathbb{X}_n$ with $\mu(\mathbb{X}_n) < \infty$ for all n.*
*(c) **Semi-finite** (סופי למחצה) if for every $A \in \Sigma$ for which $\mu(A) > 0$, there exists a $B \subset A$ such that $0 < \mu(B) < \infty$.*

Examples:

1. A **probability space** (מרחב הסתברות) is a finite measure space satisfying $\mu(\mathbb{X}) = 1$.

2. For every measurable space, the set function, defined by

$$\mu(A) = \begin{cases} 0 & A = \emptyset \\ \infty & \text{otherwise,} \end{cases}$$

is a measure (the **infinite measure**).

3. Let $\mathbb{X}$ be a non-empty set and let $\Sigma = \mathscr{P}(\mathbb{X})$. Every function $f : \mathbb{X} \to [0, \infty]$ defines a measure μ on Σ via,

$$\mu(A) = \sum_{x \in A} f(x).$$

Note that we may have here an uncountable sum, which is defined as the supremum of all finite partial sums. If $f(x) = 1$ for all x, then μ is called the **counting measure** of $\mathbb{X}$ (מידת המנייה). If $f(x_0) = 1$ and $f(x) = 0$ for all $x \neq x_0$, then μ is called the **Dirac measure (מידת דיראק) at** x_0.

—4h(2018)—

✎ *Exercise 2.15* Let $\mathbb{X}$ be an infinite set with $\Sigma = \mathscr{P}(\mathbb{X})$. Define

$$\mu(A) = \begin{cases} 0 & A \text{ is finite (or empty)} \\ \infty & A \text{ is infinite.} \end{cases}$$

Show that μ is finitely-additive, but not σ-additive (hence not a measure).

2.2.2 General properties of measures

Proposition 2.25 Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Then,

(a) Monotonicity (מונוטוניות): if $A \subset B$, then $\mu(A) \leq \mu(B)$.
(b) Sub-additivity (תת אדיטיביות): for every sequence (A_n) of measurable sets,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

(both sides may be infinite).

(c) *Lower-semicontinuity (רציפות למחצה מלרע): If (A_n) is an increasing sequence of measurable sets, namely, $A_1 \subset A_2 \subset \dots$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(Why does the limit on the right-hand side exist?)

(d) *Upper-semicontinuity (רציפות למחצה מלעיל): If (A_n) is a decreasing sequence of measurable sets, namely, $A_1 \supset A_2 \supset \dots$, and there exists some $k \in \mathbb{N}$ for which $\mu(A_k) < \infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof:

(a) Monotonicity is immediate, as $A \subset B$ implies that

$$\mu(B) = \mu(A \amalg (B \setminus A)) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

(b) To prove sub-additivity, we replace (A_n) by a sequence of disjoint sets (B_n) (Trick #1),

$$B_1 = A_1 \qquad \text{and} \qquad B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right).$$

Then, $B_n \subset A_n$ and

$$\coprod_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

By σ-additivity and monotonicity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\coprod_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

where those equalities and inequalities hold also if the terms are infinite.

(c) To prove lower-semicontinuity we use the same trick, except that the monotonicity of (A_n) implies that for every n,

$$A_n = \coprod_{k=1}^{n} B_k.$$

Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\coprod_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k)$$
$$= \lim_{n\to\infty} \mu\left(\coprod_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \mu(A_n).$$

(d) Without loss of generality, we may assume that $\mu(A_1) < \infty$. To prove upper-semicontinuity, we would like to use the fact that if (A_n) is decreasing, then (A_n^c) is increasing. It follows from the previous item that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \mu\left(\bigcup_{n=1}^{\infty} A_n^c\right) = \lim_{n\to\infty} \mu(A_n^c),$$

however, this is not obviously helpful; unlike in probability theory, it is not generally true that $\mu(A_n^c) = 1 - \mu(A_n)$. However, if $\mu(A_1) < \infty$, then we can complement with respect to A_1. That is, $(A_1 \setminus A_n)$ is increasing, hence from the previous item,

$$\mu\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\right) = \lim_{n\to\infty} \mu(A_1 \setminus A_n).$$

Now,

$$\bigcup_{n=1}^{\infty} (A_1 \setminus A_n) = \bigcup_{n=1}^{\infty} (A_1 \cap A_n^c) = A_1 \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right) = A_1 \cap \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right),$$

hence

$$\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right),$$

which completes the proof.

■

✎ *Exercise 2.16* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. For $A, B \in \Sigma$, their **symmetric difference** is

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that for every $A, B, C \in \Sigma$,

$$\mu(A \Delta B) \le \mu(A \Delta C) + \mu(C \Delta B).$$

✎ *Exercise 2.17* Let μ be a set function on a measurable space $(\mathbb{X}, \Sigma)$. Show that if $\mu(\varnothing) = 0$, μ is finitely-additive and countably-subadditive, then μ is a meaure.

Given in 2018

✎ *Exercise 2.18* Construct an example showing that upper-semicontinuity requires some finiteness assumption, as it may well be that $\mu(A_n) = \infty$ for all n, however, $\mu\left(\bigcap_{n=1}^\infty A_n\right) < \infty$.

✎ *Exercise 2.19* Let $(\mathbb{X}, \Sigma)$ be a measurable space. Let $\mu : \Sigma \to [0, \infty]$ satisfy $\mu(\varnothing) = 0$ and be *finitely* additive. Prove that μ is a measure if and only if it is lower-semicontinuous.

✎ *Exercise 2.20* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Let $E \in \Sigma$. Prove that the set function

$$\mu_E(A) = \mu(A \cap E)$$

is a measure on $(\mathbb{X}, \Sigma)$.

Given in 2018

✎ *Exercise 2.21* Every probability measure on a metric space is regular: Let $(\mathbb{X}, d)$ be a metric space and let $\Sigma = \mathscr{B}(\mathbb{X})$ be the σ-algebra of Borel sets. Let μ be a probability measure on $(\mathbb{X}, \Sigma)$ (the important fact is that it is finite). Prove that μ is **regular** in the following sense: for every $A \in \Sigma$ and for every $\varepsilon > 0$, there exist a closed set C and an open set U, such that

$$C \subset A \subset U \qquad \text{and} \qquad \mu(U \setminus C) < \varepsilon.$$

Hint: (a) Show that this holds for every closed set. (b) Show that the collection of measurable sets for which this assertion holds is a σ-algebra.

Given in 2018

✎ *Exercise 2.22* Fatou's lemma for measures: Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Let $(A_n) \subset \Sigma$. Show that

$$\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n),$$

where $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty A_k$. Further, show that if

$$\mu\left(\bigcup_{n=1}^\infty A_n\right) < \infty,$$

then

$$\mu(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} \mu(A_n),$$

where $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k$.

Given in 2018

✎ *Exercise 2.23* The following statement is known as the **Borel-Cantelli lemma**: Let A_n be a sequence of measurable sets in a measure space $(\mathbb{X}, \Sigma, \mu)$. Suppose that

$$\sum_{n=1}^\infty \mu(A_n) < \infty.$$

Prove that almost every $x \in \mathbb{X}$ is an element of only finitely many A_n's, i.e.,

$$\mu\left(\{x : |\{n : x \in A_n\}| = \infty\}\right) = 0.$$

✎ *Exercise 2.24* Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let $\mathcal{C} \subset \Sigma$ be a sub-σ-algebra. Denote by $\nu = \mu|_{\mathcal{C}}$ the restriction of μ to $\mathcal{C}$.

(a) Show that ν is a measure.
(b) Assume that μ is a finite measure. Is ν necessarily finite?
(c) Does ν inherit σ-finiteness from μ?

✎ *Exercise 2.25* Let $\mathbb{X}$ be a countable set endowed with the σ-algebra $\mathscr{P}(\mathbb{X})$. Show that every measure on this space can be uniquely represented in the form

$$\mu = \sum_{x \in \mathbb{X}} c_x \, \delta_x,$$

for some $c_x \in [0, \infty]$.

2.2.3 Complete measures

Definition 2.26 Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. A measurable set A is called a ***null set*** *(קבוצה זניחה) if $\mu(A) = 0$. A property of points in $\mathbb{X}$ is said to occur* ***almost everywhere*** *(a.e.) (מתקיימת כמעט בכל מקום) if the set of points in which it does not occur is measurable and has measure zero (compare this to the notion of a property occurring almost always in probability).*

Generally, the definition of a measure space does not require subsets of null sets to be measurable. Thus, even though one would clearly like to assign such sets a zero measure, this is not possible. Yet, measure spaces in which subsets of null sets are measurable turn out to have nice properties.

As an example of how completeness may be relevant, suppose that we already have the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, which assigns a measure zero to all singletons. We want to use this measure to define the Lebesgue measure on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}))$. Such a measure assigns measure zero to every line. Let $A \subset \mathbb{R}$ be a non-measurable set, and consider the set

$$\{0\} \times A \subset \mathbb{R}^2.$$

It is not $\mathscr{B}(\mathbb{R}^2)$-measurable even though we would want it to have measure zero.

This leads us to the following definition:

Definition 2.27 A measure space is called ***complete*** *(שלם), if every subset of a null set is measurable (and by monotonicity, it is also a null set).*

The following theorem asserts that every measure space has a canonical extension into a complete measure space.

Theorem 2.28 (Completion) Let $(\mathbb{X},\Sigma,\mu)$ *be a measure space. Let*

$$\mathcal{N} = \{A \in \Sigma \,:\, \mu(A) = 0\}$$

be the collection of its null sets. Then,

$$\Sigma' = \{A \cup B \,:\, A \in \Sigma,\;\; B \subset N \text{ for some } N \in \mathcal{N}\}$$

is a σ*-algebra. Moreover,* μ *has a unique extension* μ' *on* Σ'*, which is complete, and is called the* ***completion*** *(השלמה) of* μ*.*

Comment: This is probably not your first encounter with a completion theorem, e.g., the completion of a metric space.

Proof: First, note that both Σ and $\mathcal{N}$ are closed under countable unions (a countable union of null sets is a null set). Hence, if

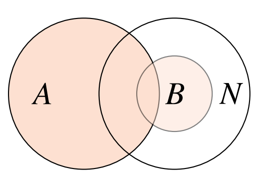
$$C_n = A_n \cup B_n \in \Sigma', \qquad A_n \in \Sigma,\;\; B_n \subset N_n \in \mathcal{N},$$

then, since $\bigcup_{n=1}^\infty B_n \subset \bigcup_{n=1}^\infty N_n \in \mathcal{N}$,

$$\bigcup_{n=1}^\infty C_n = \left(\bigcup_{n=1}^\infty A_n\right) \cup \left(\bigcup_{n=1}^\infty B_n\right) \in \Sigma'.$$

Next let,

$$C = A \cup B \in \Sigma', \qquad A \in \Sigma,\;\; B \subset N \in \mathcal{N}.$$



Then (see figure),

$$C^c = (A \cup N)^c \cup (N \smallsetminus A \smallsetminus B) \in \Sigma',$$

—4h(2017)— proving that Σ' is indeed a σ-algebra.

It is quite clear now to define the extended measure $\mu'(A \cup B)$, as we must have

$$\mu(A) = \mu'(A) \le \mu'(A \cup B) \le \mu'(A \cup N) = \mu(A \cup N) \le \mu(A) + \mu(N) = \mu(A),$$

i.e.,

$$\mu'(A \cup B) = \mu(A).$$

We first need to show that this extension does not depend on the representation: let

$$A_1 \cup B_1 = A_2 \cup B_2,$$

where $A_1, A_2 \in \Sigma$, $B_1 \subset N_1 \in \mathcal{N}$ and $B_2 \subset N_2 \in \mathcal{N}$. Then,

$$A_1 \subset A_1 \cup B_1 = A_2 \cup B_2 \subset A_2 \cup N_2,$$

and by monotonicity,

$$\mu(A_1) \le \mu(A_2) + \mu(N_2) = \mu(A_2).$$

By symmetry, $\mu(A_1) = \mu(A_2)$.

Now that we have shown that μ' is well-defined, it is easy to see that it extends μ. For $A \in \Sigma$,

$$A = A \cup \varnothing,$$

where $\varnothing \subset \varnothing \in \mathcal{N}$, hence

$$\mu'(A) = \mu(A),$$

namely, $\mu'|_\Sigma = \mu$.

We next prove that μ' is a measure, i.e., that it is σ-additive. This follows from the fact that null sets are closed under countable unions. That is, if A_n is a sequence of measurable sets, $B_n \subset N_n \in \mathcal{N}$, and $A_n \cup B_n$ are disjoint, then

$$\mu'\left(\coprod_{n=1}^\infty (A_n \cup B_n)\right) = \mu'\left(\coprod_{n=1}^\infty A_n \cup \coprod_{n=1}^\infty B_n\right) = \mu\left(\coprod_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \mu'(A_n \cup B_n).$$

We next prove that μ' is complete. Let $A \in \Sigma$ and $B \subset N \in \mathcal{N}$ satisfy

$$\mu'(A \cup B) = \mu(A) = 0.$$

Then, $A \in \mathcal{N}$ and so is $A \cup N$. If $C \subset A \cup B$, then

$$C = \varnothing \cup C \in \Sigma',$$

proving that μ' is complete.

It remains to prove that the extension is unique. Let ν be an extension of μ defined on Σ'. For $A \in \Sigma$ and $B \subset N \in \mathcal{N}$,

$$\mu(A) = \nu(A) \le \nu(A \cup B) \le \nu(A \cup N) \le \nu(A) + \nu(N) = \mu(A) + \mu(N) = \mu(A),$$

proving that $\nu(A \cup B) = \mu(A) = \mu'(A \cup B)$. ■

2.3 Borel measures on $\mathbb{R}$: a first attempt

Our ultimate goal is to define a measure μ on $\mathbb{R}$ satisfying

$$\mu((a,b)) = b - a.$$

The measure μ has to be defined for all open segments, and as a result, it must be defined at least for all sets in the σ-algebra generated by the collection of open segments. i.e., for all sets in the Borel σ-algebra $\mathscr{B}(\mathbb{R})$.

In this section we shall start developing such a measure. In fact, we shall start constructing a more general family of Borel measures on the real line. The generalization is based on the following observation:

Proposition 2.29 Let $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ be a measure space, such that μ is finite for every finite interval. Consider the ***cumulative distribution function*** *(פונקצית ההתפלגות המצטברת)*

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases} \tag{2.1}$$

Then,

(a) For every $a < b$,

$$\mu((a,b]) = F(b) - F(a).$$

(b) F is monotonically-increasing.

(c) F is right-continuous (including at $-\infty$).

Comment: You may be familiar with the cumulative distribution function from a Probability course. The standard length measure on the real line corresponds to the case $F(x) = x$.

Proof:

(a) For every $0 \le a < b$, $(a,b] = (0,b] \smallsetminus (0,a]$, hence

$$\mu((a,b]) = \mu((0,b]) - \mu((0,a]) = F(b) - F(a).$$

Likewise, for every $a < 0 < b$, $(a,b] = (a,0] \amalg (0,b]$, hence

$$\mu((a,b]) = \mu((a,0]) + \mu((0,b]) = F(b) - F(a),$$

and similarly for the case $a < b < 0$.

(b) The monotonicity of F follows from the monotonicity of μ.

(c) Without loss of generality, assume that $x \ge 0$. By the upper-semicontinuity of μ (for sets of finite measure!), for every sequence $x_n \searrow x$,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((0,x_n]) = \mu\left(\bigcap_{n=1}^{\infty}(0,x_n]\right) = \mu((0,x]) = F(x).$$

The same argument works for $x < 0$ and $x = -\infty$.

■

We will proceed in the reverse direction, and construct for every monotonically-increasing right-continuous function F, a Borel measure on the real line; this construction does not cover all possible Borel measures on the real line, but as we shall see, it covers all possible Borel measures that are finite on finite intervals. Thus far, given such a function F, we have a set-function for semi-open sets of the form $(a,b]$. Note that we may well let a and b assume the values $\pm\infty$. The measure, of the semi-open intervals $(-\infty,b]$ and $(a,\infty]$ will be finite or not depending on whether F has a finite limit at $\pm\infty$.

Thus, we have an early notion of a set-function for all sets in the family

$$\mathcal{E} = \{\varnothing\} \cup \{(a,b] : -\infty \le a < b\} \cup \{(a,\infty) : a \in \mathbb{R}\}.$$

The reason for working with semi-open intervals is essentially technical, and related to the following definition:

Definition 2.30 *Let $\mathbb{X}$ be an non-empty set. An* ***elementary family*** *(משפחה יסודית) of subsets of $\mathbb{X}$, is a collection $\mathcal{E}$ of subsets of $\mathbb{X}$ (the* ***elementary sets****), satisfying*

(a) $\varnothing \in \mathcal{E}$.
(b) *$\mathcal{E}$ is closed under pairwise intersection (hence under finite intersection).*
(c) *For every $E \in \mathcal{E}$, E^c is a finite disjoint union of elementary sets; that is*

$$\forall E \in \mathcal{E}, \qquad \exists n \in \mathbb{N}, \ (F_k)_{k=1}^n \subset \mathcal{E}, \quad \text{such that} \quad E^c = \coprod_{k=1}^n F_k.$$

Example: The semi-open intervals are an elementary family of subsets of $\mathbb{R}$. ▲▲▲

An elementary family of sets is somewhat reminiscent of a subbase (בסיס למחצה) in a topological space, in that it is a basis for constructing an algebra, as shown by the following proposition:

Proposition 2.31 *Let $\mathcal{E}$ be an elementary family of subsets of $\mathbb{X}$. Then, the collection of finite disjoint unions of elementary sets,*

$$\mathcal{A} = \left\{ \coprod_{k=1}^n E_k \ : \ n \in \mathbb{N} \cup \{0\}, \ \ E_1, \dots, E_n \in \mathcal{E} \right\},$$

is an algebra.

Proof: Clearly, $\varnothing \in \mathcal{A}$. Next, $\mathcal{A}$ is closed under pairwise intersection: Let

$$A = \coprod_{k=1}^n E_k \in \mathcal{A} \qquad \text{and} \qquad B = \coprod_{j=1}^m F_j \in \mathcal{A},$$

then

$$A \cap B = \coprod_{k=1}^n \coprod_{j=1}^m (E_k \cap F_j) \in \mathcal{A},$$

where we used the fact that $\mathcal{E}$ is closed under pairwise intersection. Finally, $\mathcal{A}$ is closed under complementation: Let

$$A = \coprod_{k=1}^n E_k, \qquad \text{where} \qquad E_k^c = \coprod_{j=1}^{n_k} F_{k,j} \in \mathcal{A}.$$

Then by Item (b),

$$A^c = \bigcap_{k=1}^{n} E_k^c \in \mathcal{A}.$$

■

The next step is to use monotonically-increasing, right-continuous functions $F : \mathbb{R} \to \mathbb{R}$ to construct a set-function on the algebra $\mathcal{A}$. Measures, however, are defined on σ-algebras, which brings us to the following definition:

Definition 2.32 Let $\mathbb{X}$ be an non-empty set and let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a ***pre-measure*** *(קדם מידה) if $\mu_0(\varnothing) = 0$ and μ_0 is σ-additive. Note that the disjoint union of $(E_n) \subset \mathcal{A}$ is not necessarily in $\mathcal{A}$; however, if it is, then*

$$\mu_0\left(\coprod_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

Comment: While a pre-measure looks "almost" like a measure, its restriction to an algebra, rather than a σ-algebra, makes it easier to construct, hence a convenient starting point for constructing measures.

Proposition 2.33 Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically-increasing and right-continuous (including at $-\infty$). Let $\mathcal{A}$ be the algebra of disjoint unions of semi-open intervals. Then, the set-function $\mu_0 : \mathcal{A} \to [0, \infty]$ defined by $\mu_0(\varnothing) = 0$ and

$$\mu_0(A) = \sum_{j=1}^{n} (F(b_k) - F(a_k)) \qquad \text{for} \qquad A = \coprod_{k=1}^{n} (a_k, b_k] \tag{2.2}$$

is a pre-measure for $\mathcal{A}$. (Note that one of the a_k may be $-\infty$.)

Proof: We will break the proof into a few lemmas. ■

Lemma 2.34 The function μ_0 defined by (2.2) *is well-defined; it does not depend on the representation of elements in $\mathcal{A}$.*

Proof: For example, suppose that

$$I \equiv (a,b] = \coprod_{k=1}^{n} (a_k, b_k] \equiv \coprod_{k=1}^{n} I_k.$$

The $(a_k), (b_k)$ can be re-ordered, such that

$$a = a_1 < b_1 = a_2 < b_2 = \cdots = a_n < b_n = b.$$

By the definition of μ_0,

$$\mu_0\left(\coprod_{k=1}^{n} I_k\right) = \sum_{j=1}^{n} (F(b_k) - F(a_k)) = F(b_n) - F(a_1) = \mu_0(I).$$

With some technical work, we may show that μ_0 is uniquely-defined for any element of $\mathcal{A}$. ■

—6h(2017)—

Lemma 2.35 The function μ_0 defined by (2.2) *is finitely-additive.*

Proof: This is immediate. Let

$$A = \coprod_{i=1}^{n} (a_i, b_i] \qquad \text{and} \qquad B = \coprod_{j=1}^{m} (c_j, d_j]$$

be disjoint finite disjoint unions of semi-open intervals. Then,

$$\mu_0(A \amalg B) = \sum_{i=1}^{n} (F(b_i) - F(a_i)) + \sum_{j=1}^{m} (F(d_j) - F(c_j)) = \mu_0(A) + \mu_0(B).$$

■

Lemma 2.36 Let

$$I = (a,b] = \coprod_{n=1}^{\infty} (a_n, b_n] = \coprod_{n=1}^{\infty} I_n,$$

then

$$\mu_0(I) = \sum_{n=1}^{\infty} \mu_0(I_n).$$

Proof: On the one hand, since for every n,

$$\coprod_{k=1}^{n} I_k \subset I,$$

and since μ_0 is finitely-additive,

$$\mu_0\left(\coprod_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} \mu_0(I_k) \le \mu_0(I).$$

Letting $n \to \infty$,

$$\sum_{n=1}^{\infty} \mu_0(I_n) \le \mu_0(I).$$

—6h(2018)—

In the other direction, let $\varepsilon > 0$ be given. Since F is right-continuous, there exists a $\delta > 0$ such that

$$F(a+\delta) - F(a) < \varepsilon,$$

i.e.,

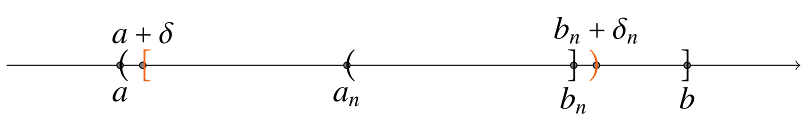
$$\mu_0((a, a+\delta]) < \varepsilon.$$

Likewise, for every $n \in \mathbb{N}$, there exists a $\delta_n > 0$, such that

$$F(b_n + \delta_n) - F(b_n) < \frac{\varepsilon}{2^n},$$

namely.

$$\mu_0((b_n, b_n + \delta_n]) < \frac{\varepsilon}{2^n}.$$



The union,

$$\bigcup_{n=1}^{\infty} (a_n, b_n + \delta_n)$$

is an open cover of the compact set $[a+\delta, b]$. Thus, there exists a finite sub-cover,

$$\bigcup_{k=1}^{m} (a_{n_k}, b_{n_k} + \delta_{n_k}) \supset [a+\delta, b].$$

By monotonicity,

$$\mu_0((a+\delta,b]) \leq \sum_{k=1}^{m} \mu_0((a_{n_k}, b_{n_k}+\delta_{n_k}]).$$

By the definition of δ and δ_k,

$$\begin{aligned}
\mu_0(I) &= \mu_0((a,a+\delta]) + \mu_0((a+\delta,b]) \\
&\leq \varepsilon + \sum_{k=1}^{m} \mu_0((a_{n_k}, b_{n_k}+\delta_{n_k}]) \\
&\leq \varepsilon + \sum_{k=1}^{m} \left(\mu_0((a_{n_k}, b_{n_k}]) + \frac{\varepsilon}{2^{n_k}} \right) \\
&\leq \varepsilon + \sum_{n=1}^{\infty} \left(\mu_0((a_n, b_n]) + \frac{\varepsilon}{2^{n}} \right) \\
&= 2\varepsilon + \sum_{n=1}^{\infty} \mu_0(I_n).
\end{aligned}$$

Since ε is arbitrary,

$$\mu_0(I) \leq \sum_{n=1}^{\infty} \mu_0(I_n),$$

which completes the proof. ■

Lemma 2.37 *The function μ_0 defined by* (2.2) *is σ-additive.*

Proof: Suppose that $A_n \in \mathcal{A}$ are disjoint, with

$$A_n = \coprod_{k=1}^{p_n} I_{n,k},$$

such that

$$\coprod_{n=1}^{\infty} A_n = \coprod_{m=1}^{q} J_m \in \mathcal{A}.$$

We need to show that

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \coprod_{m=1}^{q} \mu_0(J_m),$$

i.e., that

$$\sum_{n=1}^{\infty}\coprod_{k=1}^{p_n}\mu_0(I_{n,k}) = \coprod_{m=1}^{q}\mu_0(J_m).$$

Now for every $m = 1,\dots,q$,

$$\coprod_{n=1}^{\infty}\coprod_{k=1}^{p_n}(J_m \cap I_{n,k}) = J_m,$$

hence by the previous lemma,

$$\sum_{n=1}^{\infty}\coprod_{k=1}^{p_n}\mu_0(J_m \cap I_{n,k}) = \mu_0(J_m).$$

Summing over m we recover the desired result. ■

In conclusion, given any monotonically-increasing right-continuous function F, we have a pre-measure μ_0, defined on the algebra $\mathcal{A}$ of finite disjoint unions of semi-open intervals. Moreover, $\sigma(\mathcal{A}) = \mathscr{B}(\mathbb{R})$. What is now missing is a method for extending pre-measures into measures. We undertake this task in the next section. This is not a trivial task, as σ-algebras may be very large collections, for which we do not have an explicit representation.

2.4 Outer-measures

2.4.1 Definition

Definition 2.38 (outer-measure) Let $\mathbb{X}$ be a non-empty set. An ***outer-measure*** *(מידה חיצונית) on $\mathbb{X}$ is a function $\mu^* : \mathscr{P}(\mathbb{X}) \to [0,\infty]$ satisfying*

(a) $\mu^*(\varnothing) = 0$.
(b) Monotonicity: if $Y \subset Z$ then $\mu^*(Y) \le \mu^*(Z)$.
(c) Countable sub-additivity (תת אדיטיביות מנייתית): for every sequence $(Y_n) \subset \mathscr{P}(\mathbb{X})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} Y_n\right) \le \sum_{n=1}^{\infty}\mu^*(Y_n).$$

Unlike measures, outer-measures, which are defined for *all* subsets of $\mathbb{X}$, are easy to construct. The following proposition provides a canonical construction:

Proposition 2.39 Let $\mathcal{E} \subset \mathscr{P}(\mathbb{X})$ be a collection of sets including $\varnothing$ and $\mathbb{X}$, and let $\rho : \mathcal{E} \to [0, \infty]$ be a set function satisfying $\rho(\varnothing) = 0$. Then,

$$\mu^*(Y) = \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) \ : \ E_n \in \mathcal{E}, \ \ Y \subset \bigcup_{n=1}^{\infty} E_n \right\} \tag{2.3}$$

is an outer-measure.

Proof: First, note that μ^* is well-defined and non-negative, as the set

$$\left\{ \sum_{n=1}^{\infty} \rho(E_n) \ : \ E_n \in \mathcal{E}, \ \ Y \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

is non-empty (choose $E_1 = \mathbb{X}$, and $E_i = \varnothing$ for all $n \geq 2$). Also, $\mu^*(\varnothing) = 0$ since we can take $E_n = \varnothing$ for all n. If $Y \subset Z$, then a covering of Z is also a covering of Y, i.e.,

$$\left\{ (E_n) \subset \mathcal{E} \ : \ Y \subset \bigcup_{n=1}^{\infty} E_n \right\} \supset \left\{ (E_n) \subset \mathcal{E} \ : \ Z \subset \bigcup_{n=1}^{\infty} E_n \right\},$$

which is an inclusion between sets of sequences. Hence,

$$\left\{ \sum_{n=1}^{\infty} \rho(E_n) \ : \ E_n \in \mathcal{E}, \ \ Y \subset \bigcup_{n=1}^{\infty} E_n \right\} \supset \left\{ \sum_{n=1}^{\infty} \rho(E_n) \ : \ E_n \in \mathcal{E}, \ \ Z \subset \bigcup_{n=1}^{\infty} E_n \right\},$$

which is an inclusion between sets of numbers. The infimum of the left-hand side is less or equal than the infimum of the right-hand side, i.e.,

$$\mu^*(Y) \leq \mu^*(Z).$$

It remains to show that μ^* is countably sub-additive. Let $(Y_n) \subset \mathscr{P}(\mathbb{X})$. If $\mu^*(Y_n) = \infty$ for some n, then there is nothing to prove. Otherwise, by the definition of $\mu^*(Y_n)$, there exist for every $\varepsilon > 0$, $E_{n,k} \in \mathcal{E}$, such that

$$Y_n \subset \bigcup_{k=1}^{\infty} E_{n,k} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \rho(E_{n,k}) < \mu^*(Y_n) + \frac{\varepsilon}{2^n}.$$

Then,

$$\bigcup_{n=1}^{\infty} Y_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k},$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_{n,k}) < \sum_{n=1}^{\infty}\mu^*(Y_n) + \varepsilon,$$

proving that

$$\mu^*\left(\bigcup_{n=1}^{\infty} Y_n\right) \le \sum_{n=1}^{\infty}\mu^*(Y_n) + \varepsilon.$$

Since ε is arbitrary, μ^* is countably sub-additive. ■

Example: We can define an outer-measure on $\mathbb{R}$ by taking

$$\mathcal{E} = \{(a,b) \,:\, a < b\} \cup \varnothing \cup \mathbb{R},$$

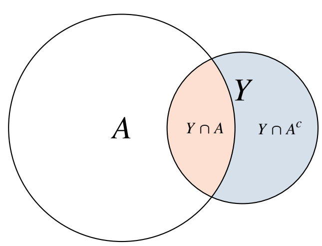
and setting $\rho((a,b)) = b - a$, $\rho(\varnothing) = 0$ and $\rho(\mathbb{R}) = \infty$. We will do something in that spirit shortly. ▲▲▲ —7h(2018)—

2.4.2 Carathéodory's theorem

We next follow the path of Carathéodory and show how an outer-measure may be used to define a measure on a collection of measurable sets. (Constantin Carathéodory (1873–1950) was a Greek mathematician who spent most of his professional career in Germany.)

*Definition 2.40 Let μ^* be an outer-measure on $\mathbb{X}$. A set $A \subset \mathbb{X}$ is called **μ^*-measurable** if*

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \cap A^c) \qquad \forall Y \subset \mathbb{X}.$$



Comment: By the sub-additivity of the outer-measure, it is always the case that

$$\mu^*(Y) \le \mu^*(Y \cap A) + \mu^*(Y \cap A^c),$$

hence A is μ^*-measurable if and only if

$$\mu^*(Y) \geq \mu^*(Y \cap A) + \mu^*(Y \cap A^c) \qquad \forall Y \subset \mathbb{X}.$$

Since this property holds trivially if $\mu^*(Y) = \infty$, it suffices to verify it for μ^*-finite sets Y.

✎ *Exercise 2.26* Let $\mathbb{X}$ be a non-empty set, and let

$$\mu^*(Y) = \begin{cases} 0 & Y = \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

(a) Show that μ^* is an outer-measure. (b) Find all the μ^*-measurable sets.

Proposition 2.41 Let μ^* be an outer-measure on $\mathbb{X}$. Then, the collection

$$\Sigma = \{A \subset \mathbb{X} \,:\, A \text{ is } \mu^*\text{-measurable}\}$$

of all μ^*-measurable sets is a σ-algebra, which we denote by $\sigma(\mu^*)$.

Proof: We will proceed as follows:

(i) Show that Σ is non-empty, by showing that $\varnothing \in \Sigma$.
(ii) Show that $A \in \Sigma$ implies that $A^c \in \Sigma$,
(iii) Show that Σ is closed under *finite* union.
(iv) Show that Σ is closed under *countable* union.

(i) For every $Y \subset \mathbb{X}$,

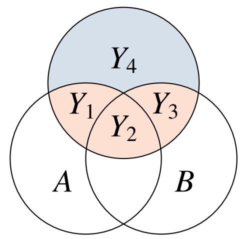
$$\mu^*(Y) = \underbrace{\mu^*(Y \cap \varnothing)}_{\mu^*(\varnothing)=0} + \underbrace{\mu^*(Y \cap \varnothing^c)}_{\mu^*(Y)},$$

i.e., $\varnothing \in \Sigma$.

(ii) Σ is closed under complementation, since A and A^c play a symmetric role in the definition of μ^*-measurability.

(iii) Let $A, B \in \Sigma$. We need to show that for every $Y \subset \mathbb{X}$,

$$\mu^*(Y) \geq \mu^*(\underbrace{Y \cap (A \cup B)}_{Y_1 \sqcup Y_2 \sqcup Y_3}) + \mu^*(\underbrace{Y \cap (A \cup B)^c}_{Y_4}).$$



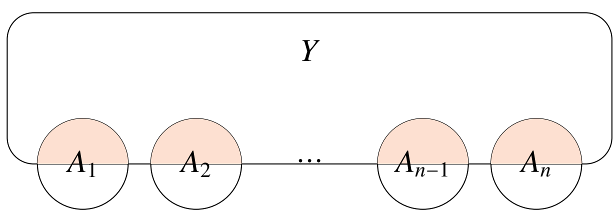
Since A, B are μ^*-measurable,

$$\begin{aligned}\mu^*(Y) &= \mu^*(Y_1 \amalg Y_2) + \mu^*(Y_3 \amalg Y_4)\\ &= \mu^*(Y_1) + \mu^*(Y_2) + \mu^*(Y_3) + \mu^*(Y_4)\\ &\geq \mu^*(Y_1 \amalg Y_2 \amalg Y_3) + \mu^*(Y_4),\end{aligned}$$

where in the last passage we used the sub-additivity of μ^*.

(iv) By Proposition 2.3, it suffices to show that Σ is closed under countable *disjoint* unions. Let $(A_n) \subset \Sigma$ be a sequence of disjoint μ^*-measurable sets, and define

$$B_n = \coprod_{k=1}^{n} A_k \qquad \text{and} \qquad B = \coprod_{n=1}^{\infty} A_n.$$



Let $Y \subset \mathbb{X}$. Since $A_n \in \Sigma$ for every n,

$$\begin{aligned}\mu^*(Y \cap B_n) &= \mu^*(Y \cap B_n \cap A_n) + \mu^*(Y \cap B_n \cap A_n^c)\\ &= \mu^*(Y \cap A_n) + \mu^*(Y \cap B_{n-1}),\end{aligned}$$

from which follows inductively that

$$\mu^*(Y \cap B_n) = \sum_{k=1}^{n} \mu^*(Y \cap A_k). \tag{2.4}$$

By the closure of Σ under finite unions, $B_n \in \Sigma$; it follows that for every n,

$$\begin{aligned}\mu^*(Y) &= \mu^*(Y \cap B_n) + \mu^*(Y \cap B_n^c)\\ &= \sum_{k=1}^{n} \mu^*(Y \cap A_k) + \mu^*(Y \cap B_n^c)\\ &\geq \sum_{k=1}^{n} \mu^*(Y \cap A_k) + \mu^*(Y \cap B^c),\end{aligned}$$

where we used (2.4), replaced B_n^c by the smaller B^c and used the monotonicity of μ^*. Letting $n \to \infty$,

$$\mu^*(Y) \geq \sum_{k=1}^{\infty} \mu^*(Y \cap A_k) + \mu^*(Y \cap B^c). \tag{2.5}$$

Finally, using again the countable sub-additivity of μ^*,

$$\mu^*(Y \cap B) = \mu^*\left(Y \cap \left(\coprod_{n=1}^{\infty} A_n\right)\right) = \mu^*\left(\coprod_{n=1}^{\infty}(Y \cap A_n)\right) \leq \sum_{k=1}^{\infty} \mu^*(Y \cap A_k), \tag{2.6}$$

showing that

$$\mu^*(Y) \geq \mu^*(Y \cap B) + \mu^*(Y \cap B^c),$$

—8h(2017)— i.e., $B \in \Sigma$. ■

✎ *Exercise 2.27* Let μ^* be an outer-measure on a set $\mathbb{X}$. Let $A, B \subset \mathbb{X}$, such that at least one of them is μ^*-measurable. Prove that

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

✎ *Exercise 2.28* Let μ^* be an outer-measure on $\mathbb{X}$. Suppose that μ^* is finitely-additive on $\mathscr{P}(\mathbb{X})$, i.e., for all disjoint $A, B \subset \mathbb{X}$, $\mu^*(A \amalg B) = \mu^*(A) + \mu^*(B)$. Show that $\sigma(\mu^*) = \mathscr{P}(\mathbb{X})$.

Theorem 2.42 (Carathéodory) Let μ^* be an outer-measure on $\mathbb{X}$. Then, the restriction $\mu = \mu^*|_{\sigma(\mu^*)}$ is a measure on $(\mathbb{X}, \sigma(\mu^*))$. Moreover, this measure is complete.

Proof: The property $\mu(\emptyset) = 0$ follows from the defining properties of outer-measures and the fact that $\emptyset \in \sigma(\mu^*)$. To show that μ is σ-additive, let $(A_n) \subset \sigma(\mu^*)$ be disjoint and let $B = \coprod_{n=1}^{\infty} A_n$. We have just shown (see (2.5) and (2.6)) that for every $Y \subset \mathbb{X}$,

$$\mu^*(Y) = \sum_{k=1}^{\infty} \mu^*(Y \cap A_k) + \mu^*(Y \cap B^c) = \mu^*(Y \cap B) + \mu^*(Y \cap B^c).$$

Substituting $Y = B$ and using the fact that $\mu^*|_{\sigma(\mu^*)} = \mu$,

$$\mu(B) = \mu^*(B) = \sum_{k=1}^{\infty} \mu^*(B \cap A_k) + \mu^*(B \cap B^c) = \sum_{k=1}^{\infty} \mu(A_k).$$

It remains to show that μ is complete, i.e., that $\sigma(\mu^*)$ contains every subset of every μ-null set. Let $N \in \sigma(\mu^*)$ satisfy $\mu(N) = 0$ and let $A \subset N$. By the sub-additivity and the monotonicity of the outer-measure, for every $Y \subset \mathbb{X}$,

$$\mu^*(Y) \le \mu^*(Y \cap A) + \mu^*(Y \cap A^c) \le \mu^*(N) + \mu^*(Y) = 0 + \mu^*(Y),$$

i.e., A is μ^*-measurable. ■

To summarize, we may construct a measure on $\mathbb{X}$ as follows: take any collection $\mathcal{E}$ of sets including $\varnothing$ and $\mathbb{X}$, along with a function $\rho : \mathcal{E} \to [0,\infty]$, which only needs to vanish on the empty set. Then, construct an outer-measure μ^* via formula (2.3). The outer-measure defines a σ-algebra of μ^*-measurable sets, $\sigma(\mu^*)$. The restriction of μ^* to this σ-algebra is a complete measure. —8h(2018)—

✎ *Exercise 2.29* Let μ^* be an outer-measure on $\mathbb{X}$, and let $\mu = \mu^*|_{\sigma(\mu^*)}$ be the induced complete measure on the σ-algebra of μ^*-measurable sets. Let ν^* be the outer-measure defined by (2.3), with $\mathcal{E} = \sigma(\mu^*)$ and $\rho = \mu$, namely,

$$\nu^*(Y) = \inf\left\{ \sum_{n=1}^\infty \mu(E_n) \;:\; E_n \in \sigma(\mu^*),\;\; Y \subset \bigcup_{n=1}^\infty E_n \right\}.$$

(a) Prove that

$$\mu^*(Y) \le \nu^*(Y) \qquad \forall Y \subset \mathbb{X}.$$

(b) Prove that $\mu^*(Y) = \nu^*(Y)$ if and only if there exists an $A \in \sigma(\mu^*)$, such that $Y \subset A$ and $\mu^*(Y) = \mu(A)$.

The following proposition asserts that if one starts with a σ-finite measure μ, constructs from it an outer-measure μ^*, and proceeds to derive a measure $\bar{\mu}$ using Carathéodory's theorem, then one obtains the closure of the original measure.

Proposition 2.43 *Let $(\mathbb{X},\Sigma,\mu)$ be a σ-finite measure space. Let μ^* be the outer-measure induced by μ. We denote by $\bar{\mu}$ the restriction of μ^* to $\sigma(\mu^*)$. Prove that $(\mathbb{X},\sigma(\mu^*),\bar{\mu})$ is the completion of $(\mathbb{X},\Sigma,\mu)$. That is, all the sets $A \in \sigma(\mu^*)$ are of the form*

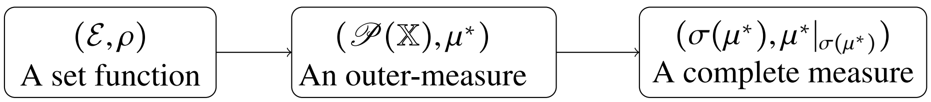
$$A = B \cup Y,$$

where $B \in \Sigma$ and $Y \subset N$ with $\mu(N) = 0$.

✎ *Exercise 2.30* Prove Proposition 2.43.

2.4.3 Extension of pre-measures

Thus far, we have the following construction:



Note that it is not clear a priori which measure μ results from a given set function ρ. A useful construction would start from a clever choice of set function ρ, by taking $\mathcal{E}$ sufficiently large and ρ satisfying properties expected for μ.

In this section, we show how to use Caratheódory's theorem to extend a pre-measure defined on an algebra $\mathcal{A}$ into a measure on the σ-algebra $\sigma(\mathcal{A})$. In particular, this will complete the construction of Borel measures on $\mathbb{R}$ with given cumulative distribution function F.

Proposition 2.44 Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$ and let μ_0 be a pre-measure on $\mathcal{A}$. Let μ^* be the outer-measure defined by

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty}\mu_0(E_n) \;:\; E_n \in \mathcal{A},\;\; A \subset \bigcup_{n=1}^{\infty} E_n\right\}.$$

Then,

(a) $\mu^*|_{\mathcal{A}} = \mu_0$.

(b) Every set in $\mathcal{A}$ is μ^*-measurable, i.e., $\mathcal{A} \subset \sigma(\mu^*)$.

Comment: Since $\mathcal{A}$ is an algebra, we can replace any sequence E_n from a disjoint sequence F_n, such that

$$\coprod_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n.$$

Since by monotonicity,

$$\sum_{n=1}^{\infty}\mu_0(F_n) \le \sum_{n=1}^{\infty}\mu_0(E_n),$$

such a substitution does not change the infimum.

Proof: (a) Let $A \in \mathcal{A}$. Setting $E_1 = A$ and $E_n = \varnothing$ for $n > 1$,

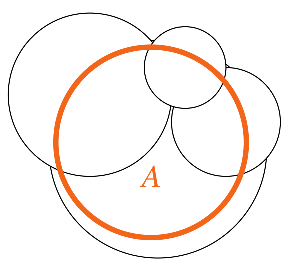
$$A \subset \coprod_{n=1}^{\infty} E_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu_0(E_n) = \mu_0(A),$$

i.e.,

$$\mu^*(A) \le \mu_0(A).$$

To show the reverse inequality, suppose that $A \subset \coprod_{n=1}^{\infty} E_n$, with $E_n \in \mathcal{A}$. Then,

$$A = A \cap \left(\coprod_{n=1}^{\infty} E_n \right) = \coprod_{n=1}^{\infty} (A \cap E_n).$$



We have a situation where a countable disjoint union in $\mathcal{A}$ is an element of $\mathcal{A}$. By the σ-additivity of the pre-measure,

$$\mu_0(A) = \sum_{n=1}^{\infty} \mu_0(A \cap E_n) \le \sum_{n=1}^{\infty} \mu_0(E_n).$$

Since this holds for any sequence (E_n) covering A, it follows that

$$\mu_0(A) \le \mu^*(A).$$

—9h(2017)—

(b) Let $A \in \mathcal{A}$ and let $Y \subset \mathbb{X}$. By the definition of the outer-measure, there exists for every $\varepsilon > 0$ a sequence $(E_n) \subset \mathcal{A}$, such that $Y \subset \coprod_{n=1}^{\infty} E_n$ and

$$\sum_{n=1}^{\infty} \mu_0(E_n) < \mu^*(Y) + \varepsilon.$$

Since μ_0 is σ-additive on $\mathcal{A}$,

$$\begin{aligned}\mu^*(Y)+\varepsilon &> \sum_{n=1}^{\infty}\mu_0(E_n)\\ &= \sum_{n=1}^{\infty}\left(\mu_0(E_n\cap A)+\mu_0(E_n\cap A^c)\right)\\ &= \sum_{n=1}^{\infty}\mu^*(E_n\cap A)+\sum_{n=1}^{\infty}\mu^*(E_n\cap A^c)\\ &\geq \mu^*\left(\coprod_{n=1}^{\infty}E_n\cap A\right)+\mu^*\left(\coprod_{n=1}^{\infty}E_n\cap A^c\right)\\ &\geq \mu^*(Y\cap A)+\mu^*(Y\cap A^c),\end{aligned}$$

where the passage to the second line follows form the fact that $A, E_n \in \mathcal{A}$, the passage to the third line follows from $\mu^*|_{\mathcal{A}} = \mu_0$ and the passage to the fourth line follows from the sub-additivity and monotonicity of μ^*. Since ε is arbitrary, it follows that A is μ^*-measurable. ■

Theorem 2.45 (Extension of a pre-measure) Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{X}$ and let μ_0 be a pre-measure on $\mathcal{A}$; let μ^* be the outer-measure induced by μ_0. Then,

(a) $\mu = \mu^*|_{\sigma(\mathcal{A})}$ is a measure on $\sigma(\mathcal{A})$ extending μ_0.

(b) If ν is some other measure on $\sigma(\mathcal{A})$ extending μ_0, then

$$\nu(A) \leq \mu(A),$$

(c) $\nu(A) = \mu(A)$ for all μ-finite sets.

(d) If μ_0 is σ-finite, then the extension is unique.

Proof: (a) By Proposition 2.44, $\mathcal{A} \subset \sigma(\mu^*)$, hence

$$\sigma(\mathcal{A}) \subset \sigma(\mu^*).$$

Since

$$\mu^*|_{\sigma(\mu^*)} \text{ is a (complete) measure,}$$

and since the restriction of a measure on a sub-σ-algebra is a measure, it follows that

$$\mu = \mu^*|_{\sigma(\mathcal{A})} \text{ is a measure.}$$

Since μ^* extends μ_0, so does μ.

(b-c) Next, let ν be a measure on $\sigma(\mathcal{A})$ extending μ_0. If $\mu(A) = \infty$ then the assertion $\nu(A) \le \mu(A)$ is trivial. Otherwise, let $A \in \sigma(\mathcal{A})$ be μ-finite. Since $\mu(A) = \mu^*(A)$, there exists for every $\varepsilon > 0$ a sequence $E_n \in \mathcal{A}$ such that $A \subset \coprod_{n=1}^\infty E_n$, and

$$\sum_{n=1}^\infty \mu_0(E_n) < \mu^*(A) + \varepsilon = \mu(A) + \varepsilon.$$

(This is where we used the finiteness of $\mu(A)$.) On the other hand, by the sub-additivity of ν,

$$\nu(A) \le \sum_{n=1}^\infty \nu(E_n) = \sum_{n=1}^\infty \mu_0(E_n),$$

proving that $\nu(A) \le \mu(A)$.

For the reverse inequality, set

$$E = \coprod_{n=1}^\infty E_n.$$

By the countable additivity of the measure,

$$\mu(E) = \sum_{n=1}^\infty \mu_0(E_n) \le \mu(A) + \varepsilon,$$

hence

$$\mu(E) - \mu(A) \le \varepsilon.$$

By the σ-additivity of the measures μ and ν, and the fact that μ and ν coincide on $\mathcal{A}$,

$$\nu(E) = \sum_{n=1}^\infty \nu(E_n) = \sum_{n=1}^\infty \mu(E_n) = \mu(E).$$

It follows that

$$\mu(A) \le \mu(E) = \nu(E) = \nu(A) + \nu(E \setminus A) \le \nu(A) + \mu(E \setminus A) \le \nu(A) + \varepsilon,$$

proving that $\mu(A) = \nu(A)$.

(d) Finally, suppose that μ_0 is σ-finite, i.e.,

$$\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n,$$

with $\mathbb{X}_n \in \mathcal{A}$ and $\mu_0(\mathbb{X}_n) < \infty$; without loss of generality we may assume that the $\mathbb{X}_n$ are disjoint. Then, for every $A \in \sigma(\mathcal{A})$,

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap \mathbb{X}_n) = \sum_{n=1}^{\infty} \nu(A \cap \mathbb{X}_n) = \nu(A).$$

■

✎ *Exercise 2.31* Let μ^* be an outer-measure on $\mathbb{X}$ induced by a pre-measure μ_0 on an algebra $\mathcal{A}$. Suppose that E is μ^*-measurable, and there exist $A, B \subset \mathbb{X}$, such that $E = A \amalg B$ and

$$\mu^*(E) = \mu^*(A) + \mu^*(B).$$

Prove that A, B are both μ^*-measurable.

✎ *Exercise 2.32* Let $\mathcal{A}$ be an algebra of sets of $\mathbb{X}$. Let $\mathcal{A}_\sigma$ be the collection of countable unions of sets in $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a pre-measure on $\mathcal{A}$ and let μ^* be the induced outer-measure.

(a) Show that for every $Y \subset \mathbb{X}$ and $\varepsilon > 0$, there exists an $A \in \mathcal{A}_\sigma$, such that $Y \subset A$ and

$$\mu^*(A) \le \mu^*(Y) + \varepsilon.$$

(b) Show that if $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists for every $\varepsilon > 0$ a $B \in \mathcal{A}_{\sigma\delta}$, such that $E \subset B$ and $\mu^*(B \smallsetminus E) < \varepsilon$.

(c) Show that the restriction $\mu^*(E) < \infty$ in (b) is not needed if μ_0 is σ-finite.

✎ *Exercise 2.33* Let μ^* be an outer-measure on $\mathbb{X}$ induced from a pre-measure μ_0 on an algebra $\mathcal{A}$, and let $\mu = \mu^*|_{\sigma(\mu^*)}$ be the induced complete measure on the σ-algebra of μ^*-measurable sets. Let μ^{**} be the outer-measure defined by (2.3), with $\mathcal{E} = \sigma(\mu^*)$ and $\rho = \mu$, namely,

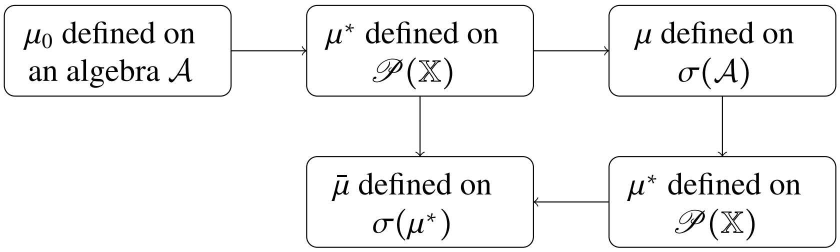
$$\mu^{**}(Y) = \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n) \;:\; E_n \in \sigma(\mu^*),\;\; Y \subset \bigcup_{n=1}^{\infty} E_n \right\}.$$

Prove that

$$\mu^*(Y) = \mu^{**}(Y) \qquad \forall Y \in \mathbb{X},$$

In other words, μ_0 and μ induce the same outer-measure, and as a result, if one starts with a pre-measure the process of extracting a measure cannot be further iterated to yield another measure. (Note that this exercise differs from Ex. 2.29 in that μ^* is induced by a pre-measure.)

To conclude, we have the following picture:



The fact that μ induces the same outer-measure μ^* as the one that generated it was proved in Ex. 2.33. Finally, by Proposition 2.43, the complete measure $\bar{\mu}$ is the completion of μ.

✎ *Exercise 2.34* Let μ_0 be a finite pre-measure on $(\mathbb{X}, \Sigma)$, and let μ^* be the induced outer-measure. For $Y \subset \mathbb{X}$, we define its **inner-measure** (**מידה פנימית**),

$$\mu_*(Y) = \mu_0(\mathbb{X}) - \mu^*(Y^c).$$

Prove that $A \in \sigma(\mu^*)$ if and only if its outer-measure coincides with its inner-measure.

2.5 Measures on $\mathbb{R}$

2.5.1 Borel measures on the real line

We may now combine together the construction of the pre-measures induced by a monotonically-increasing, right-continuous function F and Carathéodory's theorem applied by pre-measures (Theorem 2.45). Then, we will focus our attention on the standard Lebesgue measure, which corresponds to the choice of $F(x) = x$. —10h(2017)—

Proposition 2.46 (a) Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically-increasing and right-continuous. Then, F defines a unique Borel measure, denoted μ_F, on $\mathbb{R}$, such that

$$\mu_F((a,b]) = F(b) - F(a).$$

(b) Conversely, every Borel measure μ on $\mathbb{R}$, which is finite on all bounded Borel sets, is of the form $\mu = \mu_F$, where

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0. \end{cases} \tag{2.7}$$

Proof: (a) By Proposition 2.33, F defines a pre-measure μ_0 on $\mathcal{A}$. By the extension Theorem 2.45, μ_0 extends to a measure on $\sigma(\mathcal{A}) = \mathscr{B}(\mathbb{R})$, i.e., into a Borel measure μ_F. Moreover, this measure is σ-finite, as

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} (n, n+1] \qquad \text{and} \qquad \mu_0((n,n+1]) = F(n+1) - F(n) < \infty.$$

(b) Given a Borel measure μ and defining F as in (2.7), we obtain that μ and μ_F agree on semi-open intervals, hence on $\mathcal{A}$. By the uniqueness of the extension, they are equal ■

The measure μ_F is not complete. Its completion, λ_F, which equals $\mu_F^*|_{\Sigma_F}$, where $\Sigma_F = \sigma(\mu_F^*)$, and μ_F^* is induced by μ_F, is called the **Lebesgue-Stieltjes measure** associated with F.

We proceed to study regularity properties of the family of Borel measures μ_F on $\mathbb{R}$;

Lemma 2.47 Let $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu_F)$ be a Borel measure. For every $A \in \mathscr{B}(\mathbb{R})$,

$$\mu_F(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu_F((a_n, b_n]) \ : \ A \subset \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Proof: This is an immediate consequence of the the definition of the outer-measure μ_F^*, which is generated by the pre-measure μ_0 on the algebra of finite unions of semi-open intervals, and the fact that μ_F and μ_0 coincide for semi-open intervals.
■

Lemma 2.48 *Every open interval on $\mathbb{R}$ is a countable disjoint union of semi-open intervals. In particular,*

$$\left\{\sum_{n=1}^{\infty}\mu_F((a_n,b_n]) \,:\, A\subset\bigcup_{n=1}^{\infty}(a_n,b_n]\right\}\supset\left\{\sum_{n=1}^{\infty}\mu_F((a_n,b_n)) \,:\, A\subset\bigcup_{n=1}^{\infty}(a_n,b_n)\right\}$$

Proof: Clearly,

$$(a,b)=\coprod_{k=1}^{\infty}(a^k,b^k],$$

where $a^1=a$ and $b^k\nearrow b$. Using the same notation, if

$$A\subset\bigcup_{n=1}^{\infty}(a_n,b_n),$$

then

$$A\subset\bigcup_{n=1}^{\infty}\coprod_{k=1}^{\infty}(a_n^k,b_n^k],$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_F((a_n^k,b_n^k])=\mu_F((a_n,b_n)),$$

proving the second part. ■

Lemma 2.49 *Let $(\mathbb{R},\mathscr{B}(\mathbb{R}),\mu_F)$ be a Borel measure. For every $A\in\mathscr{B}(\mathbb{R})$,*

$$\mu_F(A)=\inf\left\{\sum_{n=1}^{\infty}\mu_F((a_n,b_n)) \,:\, A\subset\bigcup_{n=1}^{\infty}(a_n,b_n)\right\},$$

i.e., the semi-open intervals have been replaced by open intervals.

Proof: For $A\in\mathscr{B}(\mathbb{R})$, denote

$$\nu(A)=\inf\left\{\sum_{n=1}^{\infty}\mu_F((a_n,b_n)) \,:\, A\subset\bigcup_{n=1}^{\infty}(a_n,b_n)\right\}.$$

It follows from Lemma 2.48 that

$$\mu_F(A) \le \nu(A).$$

Conversely, let $A \in \mathscr{B}(\mathbb{R})$ and let $\varepsilon > 0$. By definition, there exist $(a_n, b_n]$, such that

$$A \subset \bigcup_{n=1}^{\infty} (a_n, b_n] \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu_F((a_n, b_n]) \le \mu_F(A) + \varepsilon.$$

Since F is right-continuous, there exists for every n a $\delta_n > 0$, such that

$$F(b_n + \delta_n) - F(b_n) < \frac{\varepsilon}{2^n},$$

i.e.,

$$\mu_F((a_n, b_n + \delta_n)) \le \mu_F((a_n, b_n + \delta_n]) \le \mu_F((a_n, b_n]) + \frac{\varepsilon}{2^n}.$$

Then,

$$A \subset \bigcup_{n=1}^{\infty} (a_n, b_n + \delta_n),$$

and by the definition of ν as an infimum,

$$\nu(A) \le \sum_{n=1}^{\infty} \mu_F((a_n, b_n + \delta_n)) \le \sum_{n=1}^{\infty} \mu_F((a_n, b_n]) + \varepsilon \le \mu_F(A) + 2\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we obtain

$$\nu(A) \le \mu_F(A),$$

—10h(2018)— which completes the proof. ■

Proposition 2.50 Let μ_F be a Borel measure on $\mathbb{R}$ (as usual, finite on bounded intervals). Then,

*(a) μ is **outer-regular**,*

$$\mu_F(A) = \inf\{\mu_F(U) \,:\, U \supset A \text{ is open}\}.$$

*(b) μ is **inner-regular**,*

$$\mu_F(A) = \sup\{\mu_F(K) \,:\, K \subset A \text{ is compact}\}.$$

Comment: In the general context of Borel measures, a measure that is locally-finite (every point has a neighborhood having finite measure), inner- and outer-regular is called a **Radon measure**. What we are thus proving is that every locally-finite Borel measure on $\mathbb{R}$ is a Radon measure. Radon measures play an important role in functional analysis.

Proof:

(a) By Lemma 2.49, there exist for every $\varepsilon > 0$ open intervals (a_n, b_n), such that

$$A \subset \bigcup_{n=1}^{\infty}(a_n, b_n) \qquad \text{and} \qquad \sum_{n=1}^{\infty}\mu_F((a_n, b_n)) \le \mu_F(A) + \varepsilon.$$

Setting $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$, and using the countable sub-additivity of the measure μ_F,

$$\mu_F(U) \le \sum_{n=1}^{\infty}\mu_F((a_n, b_n)) \le \mu_F(A) + \varepsilon,$$

from which we deduce that

$$\inf\{\mu_F(U) \ : \ U \supset A \text{ is open}\} \le \mu_F(A).$$

The other direction is trivial, as $A \subset U$ implies $\mu_F(A) \le \mu_F(U) = \mu_F(U)$.

(b) Suppose first that A is bounded. If it is closed, then it is compact, and the statement is trivial. Otherwise, for $\varepsilon > 0$, there exists by the first part an open set $U \supset \bar{A} \smallsetminus A$, such that

$$\mu_F(U) \le \mu_F(\bar{A} \smallsetminus A) + \varepsilon.$$

Let $K = \bar{A} \smallsetminus U$. It is compact, contained in A, and

$$\mu_F(K) = \mu_F(\bar{A}) - \mu_F(U) \ge \mu_F(\bar{A}) - \mu_F(\bar{A} \smallsetminus A) - \varepsilon = \mu_F(A) - \varepsilon,$$

and this since holds for all $\varepsilon > 0$,

$$\sup\{\mu_F(K) \ : \ K \subset A \text{ is compact}\} \ge \mu_F(A).$$

The reverse inequality is once again trivial.

Remains the case where A is not bounded. Let

$$A_j = A \cap (j, j+1], \qquad j = \dots, -2, -1, 0, 1, 2, \dots.$$

By the result for bounded sets, there exists for every $\varepsilon > 0$ a compact set $K_j \subset A_j$, such that

$$\mu_F(K_j) > \mu_F(A_j) - \frac{2^{-|j|}\varepsilon}{3}.$$

Set

$$H_n = \coprod_{j=-n}^{n} K_j.$$

The H_n are compact and

$$H_n \subset \coprod_{j=-\infty}^{\infty} K_j \subset \coprod_{j=-\infty}^{\infty} A_j = A.$$

On the other hand,

$$\mu_F(H_n) = \sum_{j=-n}^{n} \mu_F(K_j) > \mu_F\left(\coprod_{j=-n}^{n} A_k\right) - \varepsilon.$$

Letting n to infinity, we obtain that

$$\lim_{n\to\infty} \mu_F(H_n) \geq \mu_F(A) - \varepsilon,$$

and since ε is arbitrary, we obtain the desired result.

■

Finally, we use the above regularity result to show that modulo sets of measure zero, all the Borel sets on $\mathbb{R}$ are of relatively simple form:

Theorem 2.51 *Let $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu_F)$ by a Borel measure. Then every Borel set $A \in \mathscr{B}(\mathbb{R})$ can be represented in either way:*

(a) $A = B \setminus N$, *where B is a G_δ set and $\mu(N) = 0$.*
(b) $A = B \cup N$, *where B is an F_σ set and $\mu(N) = 0$.*

✎ *Exercise 2.35* Prove Theorem 2.51.

2.5.2 The Lebesgue measure on $\mathbb{R}$

The **Lebesgue measure** (מידת לְבֶּג) on $\mathbb{R}$ is the complete measure generated by the function $F(x) = x$. We denote it by m; the σ-algebra is denoted by $\mathcal{L}$ and its elements are called **Lebesgue-measurable** (מדידות לְבֶּג). That is,

$$m((a,b]) = b - a.$$

With a certain abuse of notation, we also denote by m the Borel measure induced by $F(x) = x$ and call it Lebesgue measure as well. Generally, we will keep working with the incomplete Borel measure, unless stated otherwise.

By the definition of the outer-measure μ^*, for every $A \in \mathscr{B}(\mathbb{R})$ (or every $A \in \mathcal{L}$),

$$m(A) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) \ : \ A \subset \bigcup_{n=1}^{\infty}(a_n, b_n]\right\}.$$

Proposition 2.52 *Every singleton has Lebesgue measure zero.*

Proof: For every $x \in \mathbb{R}$,

$$m(\{x\}) = m\left(\bigcap_{n=1}^{\infty}(x - 1/n, x]\right) = \lim_{n\to\infty} m\left((x - 1/n, x]\right) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

where in the second inequality we used the upper-semicontinuity of the measure.
■

Corollary 2.53 *Every countable set has Lebesgue measure zero.*

✎ *Exercise 2.36* Let N be a null set of $(\mathbb{R}, \mathscr{B}(\mathbb{R}), m)$. Prove that N^c is dense in $\mathbb{R}$.

The following theorem shows that the Lebesgue measure satisfies natural invariance properties:

Theorem 2.54 *For $\alpha \in \mathbb{R}$ and $\beta > 0$, denote the affine transformation*

$$T_{\alpha,\beta}(x) = \alpha x + \beta.$$

Then,

$$T_{\alpha,\beta}(\mathscr{B}(\mathbb{R})) = \mathscr{B}(\mathbb{R}),$$

and for evert $A \in \mathscr{B}(\mathbb{R})$,

$$m(T_{\alpha,\beta}(A) = \alpha\, m(A).$$

Proof: If is easy to see that $T_{\alpha,\beta}(\mathscr{B}(\mathbb{R}))$ is a σ-algebra containing all the open segments, hence

$$T_{\alpha,\beta}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R}).$$

Since this holds for every α, β, It follows that

$$\mathscr{B}(\mathbb{R}) = T_{1/\alpha,-\beta/\alpha}\left(T_{\alpha,\beta}(\mathscr{B}(\mathbb{R}))\right) \subset T_{\alpha,\beta}(\mathscr{B}(\mathbb{R})).$$

Next, define

$$\nu(A) = \alpha\, m(T^{-1}_{\alpha,\beta}(A)).$$

It is easy to see that ν is a measure on $\mathscr{B}(\mathbb{R})$: indeed,

$$\nu(\varnothing) = \alpha\, m(\varnothing) = 0,$$

and for disjoint $A_n \in \mathscr{B}(\mathbb{R})$,

$$\nu\left(\coprod_{n=1}^{\infty} A_n\right) = \alpha\, m\left(T^{-1}_{\alpha,\beta}\left(\coprod_{n=1}^{\infty} A_n\right)\right) = \alpha\, m\left(\coprod_{n=1}^{\infty} T^{-1}_{\alpha,\beta}(A_n)\right) = \alpha \sum_{n=1}^{\infty} m(T^{-1}_{\alpha,\beta}(A_n)) = \sum_{n=1}^{\infty} \nu(A_n).$$

Furthermore, every every semi-open segment,

$$\nu((a,b]) = \alpha\, m\left((a-\beta)/\alpha, (b-\beta)/\alpha]\right) = m((a,b]).$$

If follows that ν and m are equal on the algebra of finite unions of semi-open segments, and by the uniqueness of the extension $\nu = m$, or

$$m(A) = \alpha, m(T^{-1}_{\alpha,\beta}(A)).$$

This completes the proof. ■

✎ *Exercise 2.37* Prove that Theorem 2.54 remains valid if $(\mathbb{R}, \mathscr{B}(\mathbb{R}), m)$ is replaced by $(\mathbb{R}, \mathcal{L}, m)$.

✎ *Exercise 2.38* Let $A \in \mathcal{L}$ satisfy $0 < m(A) < \infty$. Prove that there exists for every $\alpha \in (0,1)$ an open interval I, such that

$$\alpha\, m(I) \le m(A \cap I) \le m(I)$$

✎ *Exercise 2.39* Let $A \subset E \subset B \subset \mathbb{R}$, such that A, B are Lebesgue-measurable and $m(A) = m(B)$. Prove that E is Lebesgue-measurable.

Proof: Since

$$E \smallsetminus A \subset B \smallsetminus A$$

and $m(B \smallsetminus A) = 0$, it follows that $E \smallsetminus A$ is measurable, hence so is

$$E = A \cap (E \smallsetminus A).$$

✎ *Exercise 2.40* Let $A \subset \mathbb{R}$ be Lebesgue-measurable, with $m(A) > 0$. Prove that

$$A - A = \{x - y \; : \; x \in A, y \in A\}$$

contains a non-empty open segment.

✎ *Exercise 2.41* Let $A \subset \mathbb{R}$ be of positive Lebesgue measure. Prove that A contains two points whose difference is rational; same for irrational.

✎ *Exercise 2.42* Let μ be a locally-finite Borel measure on $\mathbb{R}$ (every point is in an open segment of finite measure) which is invariant under translation. Prove that μ is proportional to the standard Borel measure m.

✎ *Exercise 2.43* Let A be a Lebesgue-measurable set in $\mathbb{R}$, such that $m(A) = 1$. Show that A has a subset having measure $1/2$.

2.5.3 Measure and topology

The concept of a measure is intuitively associated with the size of a set. There are also set-theoretical notions of magnitude (cardinality), as well as topological notions of magnitude: for example, sets that are both open and dense are usually considered to be "large". As we will see, the topological notion of magnitude and the measure-theoretic notion of magnitude are not always consistent.

As we have seen (Proposition 2.52), every singleton in $\mathbb{R}$ has Lebesgue measure zero. It follows that $\mathbb{Q}$, which is a countable union of singletons, has Lebesgue measure zero. On the other hand,

$$m([0,1] \smallsetminus \mathbb{Q}) = m([0,1]) - m([0,1] \cap \mathbb{Q}) = 1.$$

Thus, in this example of rationals versus irrationals, the two notions of magnitude coincide.

Let (r_n) be an enumeration of the rationals in $[0,1]$, and let I_n be an open interval centered at r_n of length $\varepsilon/2^n$, where $0 < \varepsilon < 1$. Then,

$$U = \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (0,1)$$

is open and dense in $[0,1]$. On the other hand, by the sub-additivity of the measure,

$$m(U) \le \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus, open and dense sets in $[0,1]$ can have arbitrarily small Lebesgue measure.

In contrast, set

$$K = [0,1] \setminus U.$$

Then, K is closed and nowhere dense, and yet

$$m(K) = m([0,1]) - m(U) \ge 1 - \varepsilon,$$

showing that closed and nowhere dense sets in $[0,1]$ can have Lebesgue measure arbitrarily close to 1.

Note that there are no open intervals having Lebesgue measure zero, however, there are Lebesgue-null sets having the cardinality of the continuum, as the following example shows. Recall that the **Cantor set** C, is the subset of $[0,1]$ consisting of those numbers whose base-3 expansion contains only the digits 0 and 2. It can be constructed inductively, by first removing from $[0,1]$ the segment $[1/3,2/3]$, then removing from each of the two remaining segments their mid-third, i.e., removing 2 segments of length $1/9$, then 4 segments of length $1/27$, and so on.

The Cantor set has the cardinality of the continuum, since we can map bijectively a base-3 expansion using the digits 0 and 2 to a base-2 expansion using the digits 0 and 1, i.e., the Cantor set can be mapped bijectively to the unit segment.

Proposition 2.55 The Cantor set C is measurable and has measure zero.

Proof: The Cantor set can be represented as

$$C = [0,1] \setminus \left(\coprod_{n=1}^{\infty} \coprod_{k=1}^{2^{n-1}} I_{n,k} \right),$$

where $m(I_{n,k}) = 1/3^n$. C is clearly measurable. Moreover,

$$m(C) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 - \frac{1}{3}\frac{1}{1-2/3} = 0.$$

■

—11h(2018)—

✎ *Exercise 2.44* A set $A \subset \mathbb{R}$ is called a G_δ-set if it is a countable intersection of open sets (recall that a dense G_δ set is called **residual** (**קבוצה שמנה**)). Show that there exists a G_δ set containing all the rationals (i.e., a residual set) having Lebesgue measure zero.

✎ *Exercise 2.45* Find two measurable sets $A, B \in \mathscr{B}(\mathbb{R})$, such that $m(A) = m(B) = 0$ and $m(A + B) > 0$, where

$$A + B = \{a + b \ : \ a \in A \ \ b \in B\}.$$

Finally, we have an argument showing that there exist Borel-measurable sets that are not Lebesgue-measurable.

Proposition 2.56 $\mathcal{L}$ is strictly larger than $\mathscr{B}(\mathbb{R})$.

Proof: W asserted (without a proof) that the cardinality of $\mathscr{B}(\mathbb{R})$ is that of the continuum,

$$\text{card}(\mathscr{B}(\mathbb{R})) = 2^{\aleph_0}.$$

The cardinality if the Cantor set is also 2^{α_0}, and since it has measure zero, all its subsets must be in $\mathcal{L}$, i..e, $\mathcal{L}$ has the a cardinality larger than that of the continuum.
■

—12h(2017)—