Chapter 1

Metric Spaces

1.1 Definition and examples

In your first calculus class, you encountered the notion of a **limit of a function at a point**. Recall the definition:

Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0, y_0 \in \mathbb{R}$. We say that

$$\lim_{x \to x_0} f(x) = y_0$$

if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall x : 0 < |x - x_0| < \delta \qquad |f(x) - y_0| < \varepsilon.$$

Let's examine what is needed for this definition to make sense and how much we can generalize it. The domain and the range are both \mathbb{R} . Can we replace them by other sets? Clearly, we could take them to be "nice enough" subsets of \mathbb{R} . In addition, it is assumed that we can subtract elements in the domain $(x - x_0)$ and elements in the range $(f(x) - y_0)$. Vectors, for example, can be subtracted. Can we replace the domain and the range by any vector space? The answer is positive, but we need something to replace the absolute value. In vector spaces, the role of the absolute value (the distance from zero) is played by a **norm**.

Thus, we have the following generalization:

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Let $f: V \to W$ and let $x_0 \in V$ and $y_0 \in W$. We say that

$$\lim_{x \to x_0} f(x) = y_0$$

if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \forall x : 0 < \|x - x_0\|_V < \delta \qquad \|f(x) - y_0\|_W < \varepsilon.$$

Expressions such that $||x - x_0||_V$ and $||f(x) - y_0||_W$ can be interpreted as distances between elements in two respective vector spaces. In fact, we could reformulate the above as follows:

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces. Let $f: V \to W$ and let $x_0 \in V$ and $y_0 \in W$. We say that

$$\lim_{x \to x_0} f(x) = y_0$$

if $\forall \varepsilon > 0$ there exists a $\delta > 0$, such that the distance between f(x) and y_0 is less than ε whenever the distance between x and x_0 is greater than zero and less than δ .

In this reformulation, we observe that we don't really need the domain and the range of f to be vector spaces. All we need is a notion of distance. Sets endowed with a distance are called metric spaces, and they are the subject of this chapter.

1.1.1 Basic definitions

Definition 1.1 (Metric space) A metric space (מרחב מטרי) is a pair (\mathbb{X},d) , where \mathbb{X} is a non-empty set, and $d: \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ is a distance function, or a metric (מטריקה). The metric d assigns to every pair of points $x,y \in \mathbb{X}$ a non-negative number, d(x,y), which we call the distance between x and y. A metric d must satisfy three defining properties:

1. Symmetry: for every $x, y \in \mathbb{X}$,

$$d(x,y) = d(y,x).$$

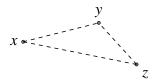
2. **Positivity**: for every $x, y \in \mathbb{X}$,

$$d(x,y) \ge 0$$
,

with equality if and only if x = y.

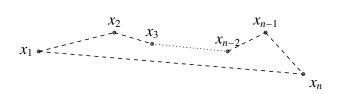
3. The triangle inequality: for every $x, y, z \in \mathbb{X}$,

$$d(x,z) \le d(x,y) + d(y,z).$$



It follows, by induction, from the triangle inequality that for every *finite* set of points $(x_i)_{i=1}^n$ in a metric space,

$$d(x_1,x_n) \leq \sum_{j=1}^{n-1} d(x_j,x_{j+1}).$$



Likewise, it follows from the triangle inequality that

$$d(x,y) \ge d(x,z) - d(y,z).$$

Since this expression must hold if the switch the roles of x and y, we obtain the reverse triangle inequality,

$$d(x,y) \ge |d(x,z) - d(y,z)|.$$
 (1.1)

Comment: Metric spaces were first introduced by Maurice Frechet in 1906, unifying work on function spaces by Cantor, Volterra, Arzelà, Hadamard, Ascoli, and others. Topological spaces (מרחבים מופולוגיים), of which metric spaces are a special instance, were first introduced by Felix Haussdorf in 1914; their current formulation is a slight generalization from 1922 by Kazimierz Kuratowski.

Examples:

- (a) Both \mathbb{R} and \mathbb{C} are metric spaces when endowed with the distance function d(x,y) = |x-y|. We will often refer to "the metric space \mathbb{R} ", without explicit mention of the metric.
- (b) Let $F : \mathbb{R} \to \mathbb{R}$ be any strictly monotonic function (say increasing). Then, \mathbb{R} endowed with the distance function

$$d(x,y) = |F(x) - F(y)|$$

is a metric space. Symmetry and positivity are immediate. It only remains to verify the triangle inequality. For $x, y, z \in \mathbb{R}$,

$$d(x,z) = |F(x) - F(z)|$$

$$\leq |F(x) - F(y)| + |F(y) - F(z)|$$

$$= d(x,y) + d(y,x).$$

Note that the strict monotonicity of F is only needed to ensure the positivity of d.

(c) \mathbb{R}^n endowed with the **Euclidean distance**,

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

is a metric space. Unless otherwise specified, \mathbb{R}^n will always be assumed to be endowed with the Euclidean metric.

- (d) The surface of earth is a metric space, with d being the (shortest) distance along great circles.
- (e) Let (G, V) be a non-directed graph. For $u, v \in V$, a **path** (מסילה) from u to v is a finite sequence of vertices,

$$\gamma = (x_0, x_1, x_2, \dots, x_{n_1}, x_n)$$

such that $(x_j, x_{j+1}) \in E$ for each j = 0, ..., n-1, $x_0 = u$ and $x_n = v$. The length $\ell(\gamma)$ of that path is the number of edges, n. Define then,

$$d(u, v) = \min\{\ell(\gamma) : \gamma \text{ is a path from } u \text{ to } v\}.$$

We claim that d is a metric on V. Indeed, it is positive because d(u, u) = 0 and $d(u, v) \ge 1$ for $u \ne v$. It is symmetric, because every path from u to v is a path from v to u, namely, we have an equality between sets,

$$\{\ell(\gamma) : \gamma \text{ is a path from } u \text{ to } v\} = \{\ell(\gamma) : \gamma \text{ is a path from } v \text{ to } u\}.$$

It remains to show that d satisfies the triangle inequality. If γ_1 and γ_2 are two paths, such that γ_1 ends at the point where γ_2 starts, then we can concatenate (לשרשר) them; denote the compound path by $\gamma_2 * \gamma_1$. Then,

$$\ell(\gamma_2 * \gamma_1) = \ell(\gamma_1) + \ell(\gamma_2).$$

Let $u, v, w \in V$. Let γ_1 be a shortest path from u to v and let γ_2 be a shortest path from v to w. Then $\gamma_2 * \gamma_1$ is a (not necessarily shortest!) path from u to w, hence

$$d(u,w) \le \ell(\gamma_2 * \gamma_1) = \ell(\gamma_1) + \ell(\gamma_2) = d(u,v) + d(v,w).$$

(f) Any non-empty set $\mathbb X$ can be endowed with the **discrete metric** (הבדידה המטריקה),

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

This is a particularly boring metric, since it provides no information about the structure of the space. We will often use it, however, as a clarifying example (or as a counter-example). Note that the discrete metric coincide with the graph metric, if we endow \mathbb{X} with the structure of a complete graph (every two points are connected by an edge).

TA material 1.1 Let \mathbb{X} be a non-empty set. A function $d: \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is called a **pseudo-metric** if

- 1. For every $x, y \in \mathbb{X}$, d(x, y) = d(y, x).
- 2. For every $x \in \mathbb{X}$, d(x, x) = 0.
- 3. For every $x, y, z \in \mathbb{X}$, $d(x, z) \le d(x, y) + d(y, z)$.
- (a) Show that d induces an equivalence relation on \mathbb{X} ,

$$x \sim y \iff d(x,y) = 0.$$

(b) Show that $\tilde{d}: (\mathbb{X}/\sim) \times (\mathbb{X}/\sim) \to [0,\infty)$ given by

$$\tilde{d}([x],[y]) = d(x,y)$$

is well-defined and is a metric on \mathbb{X}/\sim .

 $\textcircled{\mathbb{Z}}$ \mathcal{E} \mathcal{E}

- (a) $(A \cup B) \cap C = (A \cup C) \cup (B \cup C)$.
- (b) $(A \cap B)^c = A^c \cup B^c$.
- (c) $(A \cup B)^c = A^c \cap B^c$.
- (d) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.
- (e) $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.
- (f) $f^{-1}(E^c) = (f^{-1}(E))^c$.
- (g) $f(A \cup B) = f(A) \cup f(B)$.
- (h) $f(A \cap B) \subset f(A) \cap f(B)$ (note that this is not an equality).
- (i) $A \subset f^{-1}(f(A))$.
- (j) $E \supset f(f^{-1}(E))$.
- (k) Is there any relation between f(A) and $f(A^c)$?

Exercise 1.2 Prove that the set of infinite sequences of real numbers, endowed with the function

$$d((x_n),(y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

is a metric space.

 $\textcircled{\mathbb{Z}}$ Exercise 1.3 For each of the following pairs $(\mathbb{X}.d)$, determine whether it is a metric space:

(a) \mathbb{X} is the set of Riemann-integrable functions on [0,1] and

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx.$$

(b) \mathbb{X} is the set of continuous functions on [0,1] and

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx.$$

(c) \mathbb{X} is the set of continuously-differentiable functions on [0,1] and

$$d(f,g) = \max_{0 < x < 1} |f'(x) - g'(x)|.$$

(d) \mathbb{X} is the set of continuously-differentiable functions on [0,1] and

$$d(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)| + \max_{0 \le x \le 1} |f'(x) - g'(x)|.$$

Given a metric space (\mathbb{X}, d) and a non-empty subset $Y \subset \mathbb{X}$, there is a canonical metric defined on Y:

Proposition 1.2 Let (X,d) *be an arbitrary metric space, and let* $Y \subset X$. *Then the set* Y *with the function d restricted to* $Y \times Y$ *is a metric space.*

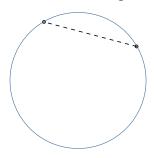
We will call $d|_{Y\times Y}$ the metric on Y **induced** by the metric on \mathbb{X} (מטריקה מושרית). A **subspace** of a metric space always refers to a subset endowed with the induced metric.

Proof: This is really immediate. Since $y_1, y_2 \in Y$ are also points is X,

$$d|_{Y\times Y}(y_1,y_2)=d(y_1,y_2)\geq 0$$

with equality if and only if $y_1 = y_2$. By the same argument, $d|_{Y \times Y}$ is symmetric and satisfies the triangle inequality.

Example: Let $\mathbb{X} = \mathbb{R}^2$ endowed with the Euclidean metric, and let $Y = S^1$ be the unit circle. Then, the induced metric on Y is depicted below:



Note that this metric is different from the "natural" metric on S^1 which determined by the shortest distance along the arc.

Definition 1.3 Let (X,d) be a matric space. A **sphere** (ספירה) of radius r > 0 centered at $a \in X$ is the set of points,

$$S_r(a) = \{x \in \mathbb{X} : d(x,a) = r\}.$$

(Note that this set may be empty.) An **open ball** (כרור פתוח) (or for short, a **ball**) of radius r centered at a is the set

$$\mathcal{B}_r(a) = \{x \in \mathbb{X} : d(x,a) < r\}.$$

A closed ball (כדור סגור) of radius r centered at a is defined by

$$\hat{\mathcal{B}}_r(a) = \{x \in \mathbb{X} : d(x,a) \le r\}.$$

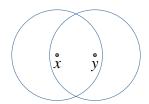
Comment: In \mathbb{R}^n these definitions coincide with our intuitive pictures of spheres and balls. Note, however, that for a space endowed with the discrete metric,

$$\mathcal{B}_r(a) = \begin{cases} \{a\} & r \leq 1 \\ \mathbb{X} & r > 1. \end{cases}$$

We now show number of elementary properties of balls in metric spaces:

Lemma 1.4 For every $x, y \in \mathbb{X}$ and every r > 0,

$$x \in \mathcal{B}_r(y)$$
 if and only if $y \in \mathcal{B}_r(x)$.



Proof: This is immediate as

$$x \in \mathcal{B}_r(y) \iff d(x,y) < r \iff y \in \mathcal{B}_r(x).$$

Lemma 1.5 Let $x \in \mathbb{X}$ and suppose that $y \in \mathcal{B}_r(x)$ for every r > 0. Then, y = x.

Proof: The assumption is that

$$d(y, x) < r$$
 for all $r > 0$.

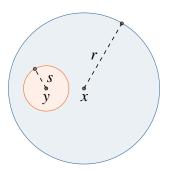
Since the metric is non-negative, it follows that d(y, x) = 0, and by the positivity of the metric, y = x.

Lemma 1.6 Let $x \in \mathbb{X}$ and r > 0. Let $y \in \mathcal{B}_r(x)$ and let s > 0 satisfy

$$d(x, y) + s < r.$$

Then,

$$\mathcal{B}_s(y) \subset \mathcal{B}_r(x)$$
.



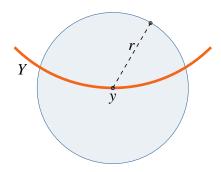
Proof: Let $z \in \mathcal{B}_s(y)$. Then, by the triangle inequality

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + s < r$$

hence $z \in \mathcal{B}_r(x)$. Since this holds for every $z \in \mathcal{B}_s(y)$, we conclude that $\mathcal{B}_s(y) \subset \mathcal{B}_r(x)$.

Proposition 1.7 Let (X, d_X) be a metric space and let $Y \subset \mathbb{X}$ be a non-empty subspace. Denote the induced metric on Y by d_Y . Likewise, denote by $\mathcal{B}_r^{\mathbb{X}}$ and \mathcal{B}_r^Y the open balls with respect to the metrics d_X and d_Y . Then, for every $y \in Y$,

$$\mathcal{B}_r^Y(y) = Y \cap \mathcal{B}_r^{\mathbb{X}}(y).$$



Proof: We need to show that two sets are equal. Just follows the definitions,

$$\begin{split} \mathcal{B}_{r}^{Y}(y) &= \{z \in Y : d_{Y}(z, y) < r\} \\ &= \{z \in Y : d_{X}(z, y) < r\} \\ &= \{z \in X : z \in Y, d_{X}(z, y) < r\} \\ &= \{z \in X : z \in Y\} \cap \{z \in X : d_{X}(z, y) < r\} \\ &= Y \cap \mathcal{B}_{r}^{\mathbb{X}}(y). \end{split}$$

Recall the definition of a bounded set in \mathbb{R} from your first calculus course. We have the following generalization for metric spaces:

Definition 1.8 Let (\mathbb{X},d) be a metric space. A set $A \subset \mathbb{X}$ is called **bounded** $(\square \square \square \square)$ if it is enclosed in some open ball. That is,

$$\exists a \in \mathbb{X} \quad and \quad \exists r > 0 \quad such that \quad A \subset \mathcal{B}_r(a).$$

(Note that a does not need to be an element of A; in particular, the empty set is always bounded.)

The following lemma shows that the center of the bounding ball is immaterial to the definition of boundedness:

Lemma 1.9 If $A \subset \mathbb{X}$ is bounded, then

$$\forall b \in \mathbb{X} \quad \exists \rho > 0 \quad such that \quad A \subset \mathcal{B}_{\rho}(b).$$

Proof: Suppose that *A* is bounded. That is,

$$A \subset \mathcal{B}_r(a)$$

for some $a \in \mathbb{X}$ and r > 0. Let $b \in \mathbb{X}$. Then, for every $x \in A$,

$$d(x,b) \le d(x,a) + d(a,b) < r + d(a,b).$$

Since this holds for every $x \in A$,

$$A \subset \mathcal{B}_{\rho}(b)$$

for
$$\rho = r + d(a, b)$$
.

Lemma 1.10 Every open ball $\mathcal{B}_r(a)$ contains a closed ball centered at a.

Proof: By definition,

$$\hat{\mathcal{B}}_{r/2}(a) \subset \mathcal{B}_r(a).$$

1.1.2 Normed spaces

Recall the following definition from your first-year linear algebra class:

Definition 1.11 Let \mathbb{X} be a vector space over \mathbb{R} (we might consider vector spaces over \mathbb{C} as well). A **norm** (נורמה) on \mathbb{X} is a function $\|\cdot\|: \mathbb{X} \to \mathbb{R}$ satisfying

1. **Positivity**: $||x|| \ge 0$ with equality if and only if x = 0.

- 2. **Homogeneity**: for every $x \in \mathbb{X}$ and $a \in \mathbb{R}$, ||ax|| = |a||x||.
- 3. The triangle inequality: for every $x, y \in X$,

$$||x + y|| \le ||x|| + ||y||$$
.

A vector space X endowed with a norm is called a **normed space** (מרחב נורמי).

Normed space are an important instance of metric spaces, as the following proposition asserts:

Proposition 1.12 Let $(\mathbb{X}, \|\cdot\|)$ *be a normed space. Then,*

$$d(x,y) = \|x - y\|$$

defines a metric on \mathbb{X} . That is, every normed space is automatically a metric space with a canonical metric.

Proof: Immediate.

Note, however, that normed spaces carry properties that are not pertinent to all metric spaces. Metric spaces need not be vector spaces. In addition, normed spaces, as metric spaces, are **translational invariant**,

$$d(x+z,y+z)=d(x,y),$$

and homogeneous,

$$d(\lambda x, \lambda y) = |\lambda| d(x, y),$$

where λ is a scalar.

 $\textcircled{\mathbb{Z}}$ *Exercise 1.4* Let (\mathbb{X},d) be a metric space. Suppose that \mathbb{X} is also a vector space over \mathbb{R} and that

$$d(x+z,y+z) = d(x,y)$$
, and $d(\lambda x, \lambda y) = |\lambda| d(x,y)$.

Show that there exists a norm on \mathbb{X} such that d is the corresponding metric.

Examples:

- (a) Both \mathbb{R} and \mathbb{C} are normed space, with ||x|| = |x|, and the resulting metric is the same as the one above.
- (b) For $\mathbb{X} = \mathbb{R}^n$ and $1 \le p < \infty$, the function $\|\cdot\|_p : \mathbb{X} \to \mathbb{R}$ defined by

$$||x||_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

is a norm, hence defines a metric. We denotes this family of metric spaces by ℓ_p^n . It is easy to see that this function satisfies the positivity and the homogeneity conditions. Let's check that it satisfies also the triangle inequality for the particular cases of p = 1 and p = 2. For p = 1, this follows from the triangle inequality in \mathbb{R} ,

$$||x + y||_1 = \sum_{i=1}^n |x_i + y_i| \le \sum_{i=1}^n (|x_i| + |y_i|) = ||x||_1 + ||y||_1.$$

For p = 2, this follows from the **Cauchy-Schwarz inequality**,

$$||x + y||_{2}^{2} = \sum_{i=1}^{n} |x_{i} + y_{i}|^{2}$$

$$= \sum_{i=1}^{n} (|x_{i}|^{2} + |y_{i}|^{2}) + 2 \sum_{i=1}^{n} |x_{i}y_{i}|$$

$$\leq \sum_{i=1}^{n} (|x_{i}|^{2} + |y_{i}|^{2}) + 2 \left(\sum_{i=1}^{n} |x_{i}|^{2}\right)^{1/2} \left(\sum_{i=1}^{n} |y_{i}|^{2}\right)^{1/2}$$

$$= ||x||_{2}^{2} + ||y||_{2}^{2} + 2||x||_{2}||y||_{2}$$

$$= (||x||_{2} + ||y||_{2})^{2}.$$

(c) The function $\|\cdot\|_{\infty}: \mathbb{R}^n \to \mathbb{R}$ defined by

$$||x||_{\infty} = \max_{1 \le i \le n} |x_i|.$$

is a norm. The corresponding metric space is denoted by ℓ_{∞}^n .

(d) The set of infinite sequences (x_i) for which

$$||x||_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

is finite is an example of an infinite-dimensional normed space. We denote this space by ℓ_p .

(e) Consider C([a,b]), the set of continuous real-valued functions on [a,b]. It is a vector space over \mathbb{R} , where addition and multiplication by a scalar are defined pointwise, namely, for $f,g \in C([a,b])$ and $\alpha,\beta \in \mathbb{R}$,

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x).$$

The function $\|\cdot\|_{\infty}: C([a,b]) \to \mathbb{R}$ defined by

$$||f||_{\infty} = \max_{a < x < b} |f(x)|$$

is a norm, therefore C([a,b]) is a metric space if endowed with the distance function

$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|.$$

Solution Exercise 1.5 Draw the unit balls $\mathcal{B}_1(0)$ in \mathbb{R}^2 for the metrics induced by the 1-, 2-, and ∞-norms.

 $\$ \mathbb{E} $xercise\ 1.6$ Show that the functions $\|\cdot\|_p:\mathbb{R}^n\to\mathbb{R}$, defined by

$$||x||_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$
 1

satisfy the definition of a norm. (Hint: the triangle inequality is proved through Young's inequality, followed by Hölder's inequality, followed by Minkowski's inequality. Prove all three inequalities.)

 \triangle *Exercise 1.7* Let $x = (x_1, x_2, ..., x_n)$. Prove that the function

$$\varphi(p) = \|x\|_p$$

defined for $1 is monotonically decreasing, and that <math>\varphi(p) \to ||x||_{\infty}$ as $p \to \infty$.

---2h(2019)---

1.2 Point set topology in metric spaces

Throughout this section we will assume a metric space (\mathbb{X}, d) . We will denote sequences x_1, x_2, \ldots in \mathbb{X} by $(x_n)_{n=1}^{\infty}$, or in short by x_n , when the identity of the running index is evident.

1.2.1 Limits and convergence

Definition 1.13 (Limit of a sequence) A sequence x_n in a metric space (\mathbb{X}, d) is called **convergent** (מתכנסת) if there exists a point $x \in \mathbb{X}$ such that

$$\lim_{n\to\infty}d(x_n,x)=0.$$

That is,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad such that \quad \forall n > N \qquad d(x_n, x) < \varepsilon.$$

The point x is called a **limit** (גבול) of the sequence x_n , and we write

$$\lim_{n\to\infty} x_n = x \qquad or \qquad x_n \to x.$$

We must first make sure that in a convergent sequence, the notion of *the* limit is well-defined:

Proposition 1.14 If a sequence converges then its limit is unique.

Proof: Suppose that a sequence x_n converges to x and to y. Then, for every $\varepsilon > 0$ there exists a (sufficiently large) n such that

$$d(x_n, x) < \varepsilon$$
 and $d(x_n, y) < \varepsilon$,

from which we conclude that for every $\varepsilon > 0$,

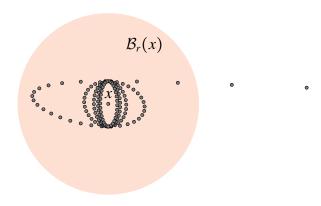
$$d(x,y) \le d(x,x_n) + d(x_n,y) < 2\varepsilon$$
,

i.e.,
$$d(x, y) = 0$$
, namely $x = y$.

The following is an equivalent characterization of a convergent sequence:

Proposition 1.15 A sequence x_n in a metric space (\mathbb{X}, d) converges to $x \in \mathbb{X}$ if and only if for every r > 0 the sequence is eventually in $\mathcal{B}_r(x)$. That is, there exists an N, which depends on r, such that for every n > N,

$$x_n \in \mathcal{B}_r(x)$$
.



Proof: This follows immediately from the definition.

Examples:

(a) In every metric space, a constant sequence $x_n = x_0$ converges to x_0 .

(b) Let $(X, \|\cdot\|)$ be a normed space. Then,

$$\lim_{n\to\infty} x_n = x \qquad \text{if and only if} \qquad \lim_{n\to\infty} \|x_n - x\| = 0.$$

(c) In particular, let $\mathbb{X} = C([a,b])$ endowed with the maximum norm. Let $f_n, f \in C([a,b])$. Then,

$$\lim_{n\to\infty} f_n = f \qquad \text{if and only if} \qquad \lim_{n\to\infty} \max_{a< x < b} |f_n(x) - f(x)| = 0.$$

That is, convergence in C([a,b]) coincides with **uniform convergence** (חות במידה שווה).

Proposition 1.16 A metric d on \mathbb{X} is continuous in the following sense: if $x_n \to x$ and $y_n \to y$, then

$$\lim_{n\to\infty}d(x_n,y_n)=d(x,y).$$

Proof: By the triangle inequality,

$$d(x,y) - d(x,x_n) - d(y_n,y) \le d(x_n,y_n) \le d(x_n,x) + d(x,y) + d(y,y_n).$$



Letting $n \to \infty$ and using the definition of the limit, both left- and right-hand sides tend to d(x, y).

Corollary 1.17 In a normed space, $x_n \rightarrow x$ implies that

$$\lim_{n\to\infty}\|x_n\|=\|x\|.$$

(Note that the converse is not true.)

Proof: Taking y_n , y = 0 and applying the previous proposition,

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} d(x_n, y_n) = d(x, y) = \|x\|.$$

Proposition 1.18 Let x_n be a sequence converging to x. Then, every subsequence of x_n converges to x as well.

Proof: We need to show that if

$$\lim_{n\to\infty}d(x_n,x)=0,$$

then, for every increasing sequence (n_k) of indexes,

$$\lim_{k\to\infty}d(x_{n_k},x)=0.$$

This follows from the convergence of subsequences in \mathbb{R} (which you proved in your first calculus course).

The following is a very useful characterization for when sequences do *not* converge.: a sequence in which the distance between every two elements is bounded form zero uniformly does not converge.

Proposition 1.19 Let x_n *be a sequence in a metric space* \mathbb{X} *and suppose that there exists an* $\varepsilon > 0$ *such that*

$$d(x_i, x_j) \ge \varepsilon$$
 for every $i \ne j$.

Then, the sequence x_n *does not converge.*

Proof: Suppose by contradiction that $x_n \to x$. Then, there exists an $N \in \mathbb{N}$, such that for every n > N,

$$d(x_n, x) < \varepsilon/2$$
.

By the triangle inequality,

$$d(x_{N+1}, x_{N+2}) \le d(x_{N+1}, x) + d(x, x_{N+2}) < \varepsilon,$$

which is a contradiction.

Proposition 1.20 Convergence in \mathbb{R}^k with respect to the Euclidean norm is equivalent to the convergence of each component: for $x_n = (x_n^1, \dots, x_n^k)$ and $x = (x^1, \dots, x^k)$,

$$\lim_{n\to\infty} x_n = x \qquad \text{if and only if} \qquad \lim_{n\to\infty} x_n^j = x^j \quad \text{for } j=1,\ldots,k.$$

Proof: Suppose that $x_n \to x$. Then, for every j = 1, ..., k,

$$|x_n^j - x^j| \le \left(\sum_{i=1}^k |x_n^j - x_j|^2\right)^{1/2},$$

hence the left-hand side tends to zero for every j as $n \to \infty$.

Conversely, suppose that $x_n^j \to x^j$ for j = 1, ..., k. Then,

$$\lim_{n \to \infty} \left(\sum_{i=1}^{k} |x_n^j - x^j|^2 \right)^{1/2} = 0$$

by limit arithmetics.

Exercise 1.8 Let (\mathbb{X}, ρ) be a metric space and define for every $x, y \in \mathbb{X}$,

$$d(x,y) = \frac{\rho(x,y)}{1 + \rho(x,y)}.$$

- 1. Show that d is a metric on \mathbb{X} .
- 2. Show that convergence with respect to this new metric d is equivalent to convergence with respect to the original metric ρ . That is, prove that a sequence x_n is convergent in (\mathbb{X}, d) if and only if it is convergent in (\mathbb{X}, ρ) .

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$$d_1((x_n), (y_n)) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

$$d_2((x_n), (y_n)) = \begin{cases} 2^{-\min(i|x_i \neq y_i)} & (x_n) \neq (y_n) \\ 0 & \text{otherwise} \end{cases}.$$

- 1. Show that these are indeed metrics.
- 2. Show that these two metrics are equivalent.

 $\textcircled{\mathbb{Z}}$ *Exercise* 1.10 Let (\mathbb{X}_n, d_n) be an infinite sequence of metric spaces. We define the product space

$$\mathbb{X} = \prod_{n=1}^{\infty} \mathbb{X}_n = \{(x_1, x_2, \dots,) : x_1 \in \mathbb{X}_1, x_2 \in \mathbb{X}_2, \dots\}$$

of infinite sequence for which the *i*-th term is in \mathbb{X}_n .

(a) Prove that

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a metric on X.

- (b) Prove that a sequence $x^k \in \mathbb{X}$ converges to $x \in \mathbb{X}$ if and only if $\lim_{k \to \infty} x_n^k = x_n$ for every k.
- (c) Conclude that on the space of infinite real numbers (i.e., $\mathbb{X}_n = \mathbb{R}$ for all n) there exists a metric for which convergence coincides with element-by-element convergence. Show, however, that there is no norm satisfying this convergence property (hint: consider the sequence $e_i = (0, \dots, 0, 1, 0, \dots)$).

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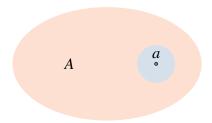
1.2.2 Open and closed sets

A topological space (מרחב מופולוגי) is a non-empty set endowed with a structure in which some if its subsets are called open (and the notion of openness must satisfy some axiomatic properties). Metric spaces are an important sub-class of topological spaces, with the notion of open sets defined below:

Definition 1.21 In a metric space (\mathbb{X},d) , a set $A \subset \mathbb{X}$ is called **open** (פתוח) if every point $a \in A$ is the center of an open ball contained in A. That is,

$$\forall a \in A \quad \exists r > 0 \quad such that \quad \mathcal{B}_r(a) \subset A.$$

In other words, a set is open if every point in it has a "neighborhood" contained in that set.



Comments:

- (a) We can replace "open ball" by "closed ball", since $\mathcal{B}_r(a) \subset A$ implies $\hat{\mathcal{B}}_{r/2}(a) \subset A$.
- (b) By definition, X itself is always an open set.
- (c) By convention, the empty set \emptyset is considered to be open as well (it satisfies the conditions in a null sense).

One can define open sets in the following equivalent way:

Definition 1.22 Let (X,d) be a metric space and let $A \subset X$. A point $a \in A$ is called an **interior point** (נקודה פנימית) of A if there exists an r > 0 such that $\mathcal{B}_r(a) \subset A$. A set is called open if all its points are interior points.

Proposition 1.23 In every metric space (\mathbb{X}, d) ,

- 1. The intersection of any finite number of open sets is open.
- 2. The union of any collection (not necessarily countable) of open sets is open.
- 3. An open ball is an open set.
- 4. A set is open if and only if it is the union of open balls.

Proof:

1. It suffices to show that the intersection of two open sets is open, since the extension to finitely-many will follow by induction (but note that induction does not hold for infinitely-many sets, even if there are countably-many). Let $A, B \subset X$ be open sets of \mathbb{X} and set $C = A \cap B$. If $C = \emptyset$ then we are done, for the empty set is open. Otherwise, let $c \in C$. Since $c \in A$ and $c \in A$ is open, there exist an $c \in A$ such that

$$\mathcal{B}_{r_1}(c) \subset A$$
.

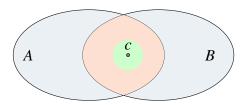
Likewise, since $c \in B$ and B is open, there exist an $r_2 > 0$, such that

$$\mathcal{B}_{r_2}(c) \subset B$$
.

Set $r = \min(r_1, r_2)$. Then,

$$\mathcal{B}_r(c) \subset A$$
 and $\mathcal{B}_r(c) \subset B$ hence $\mathcal{B}_r(c) \subset C$,

proving that *C* is open.



2. Let $\{A_{\alpha} : \alpha \in I\}$ be a collection of open sets and let

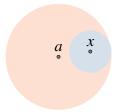
$$A=\bigcup_{\alpha\in I}A_{\alpha}.$$

Every point $a \in A$ belongs to at least one open set A_{β} , therefore there exists an r > 0 such that $\mathcal{B}_r(a) \subset A_{\beta} \subset A$.

3. Let $\mathcal{B}_r(a)$ be an open ball. By Lemma 1.5, for every $x \in \mathcal{B}_r(a)$,

$$\mathcal{B}_{\rho}(x) \subset \mathcal{B}_{r}(a)$$
,

for $\rho = r - d(a, x)$, proving that $\mathcal{B}_r(a)$ is an open set.



4. We have already seen that every open ball is open and that any union of open sets is open. Thus, every union of open balls in open. It remains to show that every open set can be represented as a union of open balls. By definition, to every point x in a open set A corresponds an open ball $\mathcal{B}_{r(x)}(x) \subset A$. Then,

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} \mathcal{B}_{r(x)}(x) \subset A,$$

proving that A is a union of open balls.

An infinite intersection of open sets is not necessarily open. Take for example $\mathbb{X} = \mathbb{R}$ with the standard metric. For every integer n, the open ball $\mathcal{B}_{1/n}(0) = (-1/n, 1/n)$ is open, however

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$$

is not open. There is no r > 0 for which $\mathcal{B}_r(0) \subset \{0\}$.

Proposition 1.24 In a discrete space every set is open.

Proof: Let A be a non-empty set in a discrete metric space (X, d). Then,

$$A=\bigcup_{a\in A}\mathcal{B}_{1/2}(a).$$

(Alternatively, every singleton is open and every set is a union of singletons.)

Definition 1.25 A set $A \subset \mathbb{X}$ is called **closed** (סגור) if its complement A^c is open.

The following proposition is a direct consequence of the Proposition 1.23, using de Morgan's laws,

Proposition 1.26 In a metric space (X, d),

- 1. Every finite union of closed sets is closed.
- 2. Every arbitrary intersection of closed sets is closed.

Exercise 1.11 Show, by example, that it is not generally true that an infinite union of closed sets is closed.

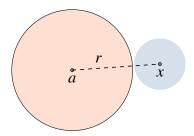
Comments:

- (a) Being closed does *not* mean not being open. The set [0,1) in \mathbb{R} is neither open nor closed.
- (b) By duality, a set is open if and only if its complement is closed.
- (c) In every metric space (\mathbb{X}, d) , the sets \mathbb{X} and \emptyset are both open and closed (one sometimes uses the term **clopen** to designate the property of being both open and closed).
- (d) A metric space is called **connected** (קשיר) if \mathbb{X} and \emptyset are the only sets that are both open and closed. The reason for this terminology is the following: suppose that a non-trivial set A is both open and closed, then \mathbb{X} is the disjoint union of two open sets, A and A^c , i.e., you can disconnect it into two separated open sets. Discrete spaces are the extreme example of a non-connected set.
- (e) For every $x \in \mathbb{X}$ the **singleton** (יחידון) $\{x\}$ is a closed set, since for every $y \in \{x\}^c$,

$$\mathcal{B}_{d(x,y)}(y) \subset \{x\}^c$$
.

(f) By Proposition 1.26, the union of every finite collection of points is closed as well; it is not generally true, however, that a countable union of points is closed.

(g) Every closed ball $\hat{\mathcal{B}}_r(a)$ is closed. Indeed, if $x \in (\hat{\mathcal{B}}_r(a))^c$ then $\mathcal{B}_{d(x,a)-r}(x) \subset (\hat{\mathcal{B}}_r(a))^c$.



(h) In a set endowed with the discrete metric, every subset is both open and closed. This follows from the fact that every set is open.

Examples:

- (a) Consider $\mathbb{X} = \mathbb{R}^2$ with the Euclidean metric and let $A = \mathcal{S}^1$, the unit circle. Then, A is closed because every point not in \mathcal{S}^1 is the center of a ball not in \mathcal{S}^1 .
- (b) Let $\mathbb{X} = C([a,b])$, let $h_1, h_2 \in \mathbb{X}$ with $h_1 < h_2$ and let

$$A = \{ f \in C([a,b]) : \forall x \in [a,b] \ h_1(x) < f(x) < h_2(x) \}.$$

Let $f \in A$ and set

$$\varepsilon = \min_{x \in [a,b]} \min\{|f(x) - h_1(x)|, |f(x) - h_2(x)|\}.$$

Then, every function in

$$\mathcal{B}_{\varepsilon}(f) = \left\{ g \in C(\left[a,b\right]) \ : \ \forall x \in \left[a,b\right] \ f(x) - \varepsilon < g(x) < f(x) + \varepsilon \right\}$$

is an element of A, hence A is an open set.

S *Exercise* 1.12 Let \mathbb{X} be a normed space. For $A, B \subset \mathbb{X}$ we define

$$A + B = \{a + b : a \in A, b \in B\}.$$

- (a) Prove that if either A or B is open then A + B is open.
- (b) Give an example where both A and B are closed however A + B is not closed.

1.2.3 Limits and topology

The following proposition elucidates what is "closed" about a closed set:

Proposition 1.27 A subset $C \subset \mathbb{X}$ is closed if and only if

$$x_n \in C$$
 and $\lim_{n \to \infty} x_n = x$ implies $x \in C$.

That is, a closed set is a set that contains all its limit points.

Proof: There are two directions to prove:

1. A closed set contains the limits of all convergent sequences: Let C be closed (i.e., its complement is open), and let

$$x_n \in C$$
 and $\lim_{n \to \infty} x_n = x$.

Suppose by contradiction that $x \notin C$. Since C^c is open, there exists an r > 0, such that

$$\mathcal{B}_r(x) \subset C^c$$
,

and therefore

$$x_n \notin B_r(x) \quad \forall n \in \mathbb{N},$$

in contradiction to x being the limit of x_n .

2. A set containing the limits of all converging sequences is closed: Suppose that C has the property that every convergent sequence in C has its limit in C. We need to show that C is closed, i.e., that C^c is open. Suppose that C^c is not open. Then,

$$\exists x \in C^c$$
 such that $\forall r > 0$ $B_r(x) \notin C^c$,

and equivalently,

$$\exists x \notin C$$
 such that $\forall r > 0$ $B_r(x) \cap C \neq \emptyset$.

In particular,

$$\exists x \notin C$$
 such that $\forall n \in \mathbb{N} \quad \exists x_n \in B_{1/n}(x) \cap C$.

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Then, x_n is a sequence in C converging to $x \notin C$, as

$$\lim_{n\to\infty}d(x_n,x)\leq\lim_{n\to\infty}\frac{1}{n}=0.$$

This contradicts the assumption, hence we conclude that *C* is closed.

Examples:

(a) Consider $\mathbb{X} = C([a,b])$ endowed with the max-norm and let

$$A = \{ f \in C([a,b]) : f \text{ is differentiable} \}.$$

A is not closed because there are sequences of differentiable functions converging (uniformly) to a non-differentiable limit. A is neither open because for every $f \in A$ and every r > 0, there are non-differentiable functions in $\mathcal{B}_r(f)$.

(b) Let (X, d) be an arbitrary metric space and let x_n be a sequence converging to x. Then

$$A = \{x_n : n \in \mathbb{N}\} \cup \{x\}$$

is closed because every converging sequence in A is either eventually constant or converges to x. In either case, the limit is an element of A.

Exercise 1.13 Prove the above assertion: if x_n is a sequence converging to x, then

$$A=\left\{x_n\ :\ n\in\mathbb{N}\right\}\cup\left\{x\right\}$$

is a closed set.

As explained above, sets in metric spaces are not necessarily either open or closed. Every set, however, has a largest open set contained in it, and a smallest closed set containing it.

Definition 1.28 Let $A \subset \mathbb{X}$ be a set in a metric space. Its **interior** (פנים), A° , is the union of all the open subsets of A,

$$A^{\circ} = \bigcup \{U : U \text{ is open and } U \subset A\}$$

(it is the largest open subset of A). Its **closure** (סגור), \overline{A} , is the intersection of all the closed sets that include A,

$$\overline{A} = \bigcap \{C : C \text{ is closed and } A \subset C\}$$

(it is the smallest closed set containing A).

Comment: Recall the notion of an interior point in a set. Likewise, for $A \subset \mathbb{X}$, we call $x \in \mathbb{X}$ an **exterior point** (נקודה היצונית) of A if there exists an r > 0 such that $\mathcal{B}_r(x) \subset A^c$. It is easy to verify that the interior of a set is the union of all its interior points, whereas the closure of a set is the union of all the points that are not exterior to it.

Proposition 1.29 The interior of a set is open and the closure of a set is closed.

Proof: Any union of open sets is open and any intersection of closed sets is closed.

Also,

Proposition 1.30 If U is open and C is closed, then

$$U^{\circ} = U$$
 and $\overline{C} = C$.

Proof: Immediate from the definitions of U° being the maximal open set contained in U and \overline{C} being the minimal closed set containing C.

Proposition 1.31 For every set $A \subset \mathbb{X}$,

$$A^{\circ} \subset A \subset \overline{A}$$
.

Proof: This is immediate, as A° is a union of sets all contained in A and \overline{A} is an intersection of sets all containing A.

Example: Let $X = \mathbb{R}$. Then,

$$\mathbb{Q}^{\circ} = \emptyset$$
 and $\overline{\mathbb{Q}} = \mathbb{R}$.

Indeed, $\mathbb Q$ does not contain any open segment, and since every open set is a union of open segments, $\mathbb Q$ does not contain any non-empty open set. The second statement can be justified as follows. Let C be a closed set containing $\mathbb Q$. Since every point in $\mathbb R$ is a limit of a sequence in $\mathbb Q$, and since we have seen that closed sets contain all their limit points, we conclude that $C = \mathbb R$, i.e., $\mathbb R$ is the only closed set containing $\mathbb Q$.

S *Exercise 1.14* Prove that for every $A \subset \mathbb{X}$

$$(A^{\circ})^c = \overline{A^c}$$
 and $(A^c)^{\circ} = (\overline{A})^c$.

The following are immediate consequences of the definitions of the interior and the closure:

Lemma 1.32 Let (X,d) be a metric space and let $A \subset X$.

- 1. If $U \subset A$ is open, then $U \subset A^{\circ}$.
- 2. If $A \subset C$ and C is closed, then $\overline{A} \subset C$.

Corollary 1.33 For every set A, the open sets contained in A coincide with the open sets contained in A° and the closed sets containing A coincide with the closed sets containing \overline{A} . As a result, the operations of interior and closure are idempotent. For every set A,

$$(A^{\circ})^{\circ} = A^{\circ}$$
 and $\overline{\overline{A}} = \overline{A}$.

Proof: The first part is an immediate corollary of the lemma. For the second part,

$$A^{\circ} = \bigcup \{ U \text{ is open } : U \subset A \}$$

= $\bigcup \{ U \text{ is open } : U \subset A^{\circ} \}$
= $(A^{\circ})^{\circ}$,

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and

$$\overline{A} = \bigcap \{C \text{ is closed } : A \subset C\}$$

= $\bigcap \{C \text{ is closed } : \overline{A} \subset C\}$
= $\overline{\overline{A}}$.

We have seen that the property of being closed is related to limits being in that set. The following proposition relates the closure of a set to limits:

Proposition 1.34 The closure of a set A is the set of points that are limits of sequences in A,

$$\overline{A} = \{x \in \mathbb{X} : \exists x_n \in A \text{ such that } x_n \to x\}.$$

Proof: Define

$$B = \{x \in \mathbb{X} : \exists x_n \in A \text{ such that } x_n \to x\}.$$

We need to show that $\overline{A} = B$, i.e., that $\overline{A} \subset B$ and $B \subset \overline{A}$.

Let $x \in B$. By definition, there exists a sequence $x_n \in A$ converging to x. Since $A \subset \overline{A}$, x_n is a sequence in \overline{A} . Since \overline{A} is closed, $x \in \overline{A}$, implying that $B \subset \overline{A}$.

In the other direction, we first assert that B is closed. Let (b_n) be a sequence in B converging to $x \in \mathbb{X}$. By the definition of B, there exists for every $n \in \mathbb{N}$ a sequence in A converging to b_n , and in particular, an $a_n \in A$, such that

$$d(b_n,a_n)<\frac{1}{n}.$$

It follows that

$$d(x,a_n) \le d(x,b_n) + d(b_n,a_n) \to 0.$$

By definition, $x \in B$, and by Proposition 1.27, B is closed.

Clearly, every point in A belongs to B (take the trivial sequence). Thus, B is a closed set containing A, and by Lemma 1.32, $\overline{A} \subset B$.

Example: Let $\mathbb{X} = \mathbb{R}^2$ and consider the set

$$A = \{(\cos 2\pi q, \sin 2\pi q) : q \in \mathbb{Q}\}.$$

_

Then $\overline{A} \subset \mathbb{S}^1$, because every point

$$(\cos 2\pi x, \sin 2\pi x) \in \mathbb{S}^1$$

is a limit of points in A. On the other hand, since S^1 is closed and contains A, $\overline{A} \subset S^1$, from which we deduce that $\overline{A} = \mathbb{S}^1$.

Definition 1.35 The boundary (שפה) ∂A of a set A is defined as

$$\partial A = \overline{A} \setminus A^{\circ}$$
.

Proposition 1.36 The boundary of a set is closed.

Proof: The boundary is the intersection of two closed sets, \overline{A} , and $(A^{\circ})^{c}$, hence it is closed.

Example: In any normed space,

$$\partial \mathcal{B}_r(x) = \hat{\mathcal{B}}_r(x) \setminus \mathcal{B}_r(x)$$

$$= \{ y : ||y - x|| \le r \} \setminus \{ y : ||y - x|| < r \}$$

$$= \mathcal{S}_r(x).$$

This is not true in a general metric space. For example, in the case of a discrete space,

$$\partial \mathcal{B}_1(x) = \overline{\mathcal{B}_1(x)} \setminus \mathcal{B}_1(x) = \emptyset \neq \mathcal{S}_1(x),$$

where we used the fact that every set is both open and closed.

Example: Consider the space $\mathbb{X} = C([0,1])$ endowed with the maximum norm, and consider the set

$$A = \{ f \in C([0,1]) : f \text{ is strictly decreasing} \}.$$

We will obtain its interior, its closure and its boundary.

Set

$$B = \{ f \in C([0,1]) : f \text{ is decreasing} \}.$$

Let $f_n \in B$ and $f_n \to f$. Then, f must be decreasing as well, otherwise if x < y and f(x) < f(y), then it must hold that for n sufficiently large, $f_n(x) < f_n(y)$, which is a contradiction. Thus, B contains all it limit, and therefore it is closed.

Next, we claim that every element of B is a sequence of elements of A. Given $f \in B$, set

$$f_n(x) = f(x) - x/n.$$

Then,

$$d(f_n, f) = \max_{0 \le x \le 1} x/n = 1/n,$$

and for every x < y,

$$f_n(x) = f(x) - x/n > f(y) - y/n = f_n(y),$$

i.e., $f_n \in A$. Since B is closed, and since every point in B is a limit of a sequence in A, it follows that

$$B \subset \overline{A}$$
.

On the other hand, since $A \subset B$ and B is closed, it follows that $\overline{A} \subset B$, namely,

$$\overline{A} = B$$
.

Let now $f \in A$ and let r > 0. Since f is continuous, there exists a pair of points x < y, such that f(y) > f(x) - r/2. Let $g \in C([0, 1])$ be any function satisfying

$$g(y) - g(x) = r/2$$
 and $||g||_{\infty} < r/2$.

Then,

$$f + g \in \mathcal{B}_r(f)$$
,

however

$$(f+g)(x) < f(y) + r/2 + g(y) - r/2 = (f+g)(y),$$

i.e., $f + g \notin A$. Thus, for every $f \in A$ and every r > 0, the open ball $\mathcal{B}_r(x)$ contains elements not in A, namely, A does not have any non-empty open subset. We conclude that

$$A^{\circ} = \emptyset$$
.

Finally,

$$\partial A = \overline{A} \setminus A^{\circ} = B.$$

Exercise 1.15 Let A, B be subsets of a metric space (\mathbb{X}, d) . Prove that

- 1. $\overline{A} \cup \overline{B} = \overline{A \cup B}$.
- 2. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
- 3. If A is open then $(\partial A)^{\circ} = \emptyset$.
- 4. A is closed if and only if $\partial A \subset A$.
- 5. A is open if and only if $\partial A \cap A = \emptyset$.

 \otimes Exercise 1.16 Let A be a set in a metric space (\mathbb{X}, d) . Show that its boundary ∂A coincides with the set of points x, such that every open ball with center at x intersects both A and A^c .

Consider now a metric space (\mathbb{X}, d) . Any $Y \subset \mathbb{X}$ is a metric space with the induced metric. Note, however, that a set $A \subset Y$ can be open in the metric space (Y, d), but closed in the metric space (\mathbb{X}, d) (e.g., the set Y itself is always open in (Y, d), but could be closed in (\mathbb{X}, d)).

Proposition 1.37 Let (X, d_X) be a metric space and let $Y \subset X$; we denote the induced metric on Y by d_Y . A set $A \subset Y$ is open in (Y, d_Y) if and only if it is the intersection of Y and a set that is open in (X, d_X) . The same applies if we replace "open" by "closed".

Proof: By Proposition 1.7, every ball in Y is the intersection of Y and a ball in X. Suppose that A is open in Y. Then, for every $a \in A$, there exists an r(a) > 0, such that $\mathcal{B}_{r(a)}^{Y}(a) \subset Y$. It follows that

$$A = \bigcup_{a \in A} \mathcal{B}_{r(a)}^{Y}(a) = \bigcup_{a \in A} (Y \cap \mathcal{B}_{r(a)}^{\mathbb{X}}(a)) = Y \cap \left(\bigcup_{a \in A} \mathcal{B}_{r(a)}^{\mathbb{X}}(a)\right),$$

proving that A is the intersection of Y and a set open in \mathbb{X} .

Conversely, let

$$A = Y \cap B$$
.

where B is open in \mathbb{X} . Since B is a union of open balls in \mathbb{X} , A is a union of open balls in Y.

Definition 1.38 Let (X,d) be a metric space. A set $A \subset X$ is called **dense** (צפוף) if

$$\overline{A} = \mathbb{X}$$
.

Example: The rationals \mathbb{Q} are dense in \mathbb{R} because $\overline{\mathbb{Q}} = \mathbb{R}$.

Definition 1.39 A metric space (\mathbb{X} , d) is called **separable** (ספרבילי) if it contains a countable dense set. That is, there exists a countable set $\{x_n: n \in \mathbb{N}\} \subset \mathbb{X}$ such that

$$\mathbb{X}=\overline{\{x_n\,:\,n\in\mathbb{N}\}}.$$

Example: \mathbb{R} is separable because \mathbb{Q} is countable and $\overline{\mathbb{Q}} = \mathbb{R}$.

A A

X Exercise 1.17 Show that every subspace (Y,d) of a separable metric space (X,d) is separable.

Simple Exercise 1.18 (a) Prove that in a normed space,

$$(\hat{\mathcal{B}}_r(a))^\circ = \mathcal{B}_r(a)$$
 and $\overline{\mathcal{B}_r(a)} = \hat{\mathcal{B}}_r(a)$.

(b) Prove that this is not necessarily true in a general metric space.

X Exercise 1.19 A point a in a metric space X is called an **accumulation point** (נקודת הצטברות) of a set A if in any open ball centered at a there exists a point of A (excluding a!). Prove that a is an accumulation point of A if and only if there exists a sequence of distinct points in A that converges to a.

 \bigcirc Exercise 1.20 Let A be a set in a metric space. The set of its accumulation points is called its **derived set** (קבוצת הנגזרת) and is denoted by A^* . Prove that A^* is closed.

 \mathbb{Z} \mathcal{E} \mathcal{E}

 $\$ Exercise 1.23 Prove that ℓ_{∞} , the space of infinite sequence with the metric induced by the norm

$$||x||_{\infty} = \sup_{n} |x_n|$$

is not separable. Hint: use Cantor's diagonalization technique.

1.3 Continuous functions on metric spaces

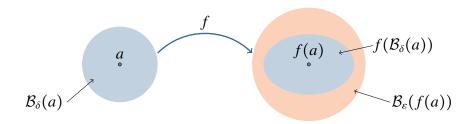
Metric spaces are an abstraction generalizing the real line. Abstractions, such as metric spaces, groups, rings, etc., never live on their own. They are always associated with **mappings** (morphisms, in the language of categories) between instances of the same abstraction, which preserve certain properties (e.g., groups and homomorphisms). In the first calculus courses we learned about continuous mappings (i.e., functions) $\mathbb{R} \to \mathbb{R}$. This concept will be generalized into continuous mappings between metric spaces.

1.3.1 Definitions and basic properties

Definition 1.40 Let (\mathbb{X},d) and (\mathbb{Y},ρ) be two metric spaces. A mapping $f:\mathbb{X}\to\mathbb{Y}$ is said to be **continuous** (רציך) at a point $a\in\mathbb{X}$ if for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$d(x,a) < \delta$$
 implies $\rho(f(x), f(a)) < \varepsilon$.

That is, f is continuous at a if for every open ball $\mathcal{B}_{\varepsilon}(f(a)) \subset \mathbb{Y}$ there exists an open ball $\mathcal{B}_{\delta}(a) \subset \mathbb{X}$ whose image under f is contained in $\mathcal{B}_{\varepsilon}(f(a))$. A function is called continuous if it is continuous at all points.



Examples:

- (a) Every constant function is continuous.
- (b) Let (\mathbb{X}, d) be an arbitrary metric space and let $a \in \mathbb{X}$ be an arbitrary point. The function $f: x \mapsto d(x, a)$ is a continuous mapping $f: (\mathbb{X}, d) \to (\mathbb{R}, |\cdot|)$ because for every $\varepsilon > 0$ take $\delta = \varepsilon$. Then $d(x, z) < \delta$ implies that

$$|f(x)-f(z)|=|d(x,a)-d(z,a)|\leq d(x,z)<\varepsilon.$$

(c) Every function defined on a discrete space and whose range is any metric space is continuous. Indeed, for every $\varepsilon > 0$ take $\delta = 1/2$. Then,

$$f(\mathcal{B}_{\delta}(a)) = f(\{a\}) = \{f(a)\} \subset \mathcal{B}_{\varepsilon}(f(a)).$$

Proposition 1.41 (Sequential continuity (דיפות סדרתית)) Let $f:(\mathbb{X},d) \to (\mathbb{Y},\rho)$ be a function between two metric spaces. Then f is continuous at a if and only if for every sequence x_n in \mathbb{X}

$$\lim_{n\to\infty} x_n = a \qquad implies \qquad \lim_{n\to\infty} f(x_n) = f(a).$$

(This notion of continuity is attributed to Heine, unlike the ε - δ notion, which is attributed to Cauchy.)

Proof:

1. Suppose that f is continuous at a and let x_n be a sequence in \mathbb{X} that converges to a. Since f is continuous at a,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad x \in \mathcal{B}_{\delta}(a) \quad \text{implies} \quad f(x) \in \mathcal{B}_{\varepsilon}(f(a)).$$

Since $x_n \to a$,

$$\forall \delta > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n > N \quad x_n \in \mathcal{B}_{\delta}(a).$$

Putting both assertions together,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n > N \quad f(x_n) \in \mathcal{B}_{\varepsilon}(f(a)),$$

i.e.,
$$f(x_n) \rightarrow f(a)$$
.

2. Suppose that $x_n \to a$ implies $f(x_n) \to f(a)$. Suppose that f were not continuous at a: then

$$\exists \varepsilon > 0$$
 such that $\forall n \in \mathbb{N}$ $\exists x_n : d(x_n, a) < \frac{1}{n}$ and $\rho(f(x_n), f(a)) \ge \varepsilon$.

The sequence x_n converges to a, however the sequence $f(x_n)$ does not converge to f(a), contradicting the assumption.

Comment: In the more general context of topological spaces, continuity always implies sequential continuity, but the converse is not true.

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Examples:

(a) Let $(\mathbb{X}, \|\cdot\|)$ be a normed space and let $B = \mathcal{B}_1(0)$ be the unit ball centered at the origin. The **indicator function** (הפונקציה המציינת)

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

is not continuous. We will show it by constructing a converging sequence $x_n \to x$ for which $f(x_n) \not\to f(x)$.

Let x be a unit vector; by definition it is not in B, i.e., f(x) = 0. Take the sequence

$$x_n = \left(1 - \frac{1}{n}\right) x \in B.$$

Then,

$$\lim_{n\to\infty} x_n = x \qquad \text{however} \qquad \lim_{n\to\infty} f(x_n) = 1 \neq f(x).$$

(b) Let $\mathbb{X} = C([a, b])$. For $x_0 \in [a, b]$ define the function $E : \mathbb{X} \to \mathbb{R}$ by

$$E(f) = f(x_0)$$

(*E* is called an **evaluation map**). Then, *E* is continuous. Indeed, for every $f \in \mathbb{X}$ and every sequence $\mathbb{X} \ni f_n \to f$,

$$\lim_{n\to\infty} d(f_n, f) = \lim_{n\to\infty} \max_{a\le x\le b} |f_n(x) - f(x)| = 0.$$

In particular,

$$\lim_{n\to\infty} |E(f_n) - E(f)| = \lim_{n\to\infty} |f_n(x_0) - f(x_0)| = 0.$$

(c) Let $\mathbb{X} = C([-1,1])$ and let $A \subset \mathbb{X}$ be the subset of functions that are differentiable on [-1,1]. Consider the function $D:A \to \mathbb{R}$ defined by

$$D(f) = f'(0).$$

Then, D is not continuous (with A is endowed with the subspace metric). Indeed, take the sequence of functions

$$f_n(x) = \frac{1}{n}\sin nx.$$

and set f = 0. Then

$$\lim_{n \to \infty} d(f_n, f) = \lim_{n \to \infty} \max_{x \in [-1, 1]} |f_n(x) - f(x)| = 0,$$

whereas

$$|D(f_n) - D(f)| = 1 + 0.$$

(d) Every function $f: \mathbb{X} \to \mathbb{R}^n$ is continuous if and only if each of its component is a continuous function $\mathbb{X} \to \mathbb{R}$; this is because a sequence converges in \mathbb{R}^n if and only if each of its components converges.

The following theorem provides alternative characterizations to the continuity of functions between metric spaces. In fact, these definitions are more general in the sense that they remain valid in (non-metric) topological spaces (i.e., spaces that are only endowed with a structure of open sets).

Theorem 1.42 Let $f: (\mathbb{X}, d) \to (\mathbb{Y}, \rho)$ be a function between two metric spaces. Then the following three statements are equivalent:

- 1. f is continuous.
- 2. The pre-image of every set open in \mathbb{Y} is a set open in \mathbb{X} (f pulls back (לאחור) open sets into open sets).
- 3. The pre-image of every set closed in \mathbb{Y} is a set closed in \mathbb{X} (f pulls back closed sets into closed sets).

Proof:

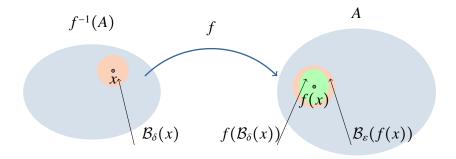
1. The equivalence between Property 2 and Property 3 follows from the commutativity between the pre-images and set-theoretic operations. Thus, if the

pre-image of every set open in $\mathbb Y$ is a set open in $\mathbb X$ and $C \subset \mathbb Y$ is closed in $\mathbb Y$, then

$$f^{-1}(C) = (f^{-1}(C^c))^c$$

is closed in \(\mathbb{Y} \). The other direction is obtained similarly.

2. We next show that Property 1 implies Property 2. Let $A \subset \mathbb{Y}$ be an open set, and let $x \in f^{-1}(A)$. Since A is open and $f(x) \in A$, there exists a ball $\mathcal{B}_{\varepsilon}(f(x)) \subset A$. Since f is continuous, there exists a ball $\mathcal{B}_{\delta}(x)$ whose image under f is a subset of $\mathcal{B}_{\varepsilon}(f(x))$, i.e., $\mathcal{B}_{\delta}(x) \subset f^{-1}(A)$, which implies that $f^{-1}(A)$ is open.



3. It remains to show that Property 2 implies Property 1. Let $x \in \mathbb{X}$ be an arbitrary point; by assumption, for every $\varepsilon > 0$ the pre-image of the set $\mathcal{B}_{\varepsilon}(f(x))$ is open in \mathbb{X} , and includes the point x, i.e., there exists an open ball $\mathcal{B}_{\delta}(x) \subset f^{-1}(\mathcal{B}_{\varepsilon}(f(x)))$, or, $f(\mathcal{B}_{\delta}(x)) \subset \mathcal{B}_{\varepsilon}(f(x))$ which implies the continuity of f at x.

Comment: Continuous function not necessarily map open sets into open sets. For example the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f: x \mapsto 0$ maps the open set (0,1) into the closed set $\{0\}$. Functions that map open sets into open sets are called **open maps** (העחקות סגורות) (and likewise, functions that map closed sets into closed sets are called **closed maps** (העחקות סגורות)). Open maps, on the other hand are not necessarily continuous.

Definition 1.43 (Function composition) Let $f: (\mathbb{X}_1, d_1) \to (\mathbb{X}_2, d_2)$ and $g: (\mathbb{X}_2, d_2) \to (\mathbb{X}_3, d_3)$ be functions between metric spaces. Their composition $g \circ f$ is a function $(\mathbb{X}_1, d_1) \to (\mathbb{X}_3, d_3)$ defined by

$$(g \circ f)(x) = g(f(x)).$$

Proposition 1.44 (Composition of continuous mappings) The composition of continuous mappings is a continuous mapping.

Proof: Let $f: (\mathbb{X}_1, d_1) \to (\mathbb{X}_2, d_2)$ and $g: (\mathbb{X}_2, d_2) \to (\mathbb{X}_3, d_3)$ be continuous functions between metric spaces. The proof is immediate if we use the "topological interpretation" and observe that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$
.

Then, for every open $A \subset \mathbb{X}_3$, $g^{-1}(A)$ is open in \mathbb{X}_2 and $f^{-1}(g^{-1}(A))$ is open in \mathbb{X}_1 , proving that $g \circ f$ is continuous.

Alternative proof using Heine's characterization: Let $x_n \to a$ in \mathbb{X}_1 . Then, $f(x_n) \to f(a)$ in \mathbb{X}_2 and $g(f(x_n)) \to g(f(a))$ in \mathbb{X}_3 .

Lemma 1.45 Let (X,d) be a metric space and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous. Then, the map $h: X \to \mathbb{R}^2$ defined by

$$h(x) = (f(x), g(x))$$

is continuous.

Proof: This is an immediate consequence of sequential continuity. Let $x_n \to a$. Then,

$$d((f(x_n),g(x_n)),(f(a),g(a))) = \sqrt{(f(x_n)-f(a))^2 + (g(x_n)-g(a))^2}$$

$$\leq 2|f(x_n)-f(a)| + 2|g(x_n)-g(a)|$$

$$\to 0.$$

Corollary 1.46 For continuous functions $f, g : \mathbb{X} \to \mathbb{R}$, the functions $f \pm g$, and $f \cdot g$ are continuous. Furthermore, f/g is continuous at all points where $g \neq 0$.

Proof: Let $h: \mathbb{X} \to \mathbb{R}^2$ be defined as in Lemma 1.45. Let $s_1, s_2, s_3: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$s_1(x, y) = x + y$$
 $s_2(x, y) = x - y$ and $s_3(x, y) = xy$.

The maps s_1, s_2, s_2 are continuous, because, for example, as we know from elementary calculus, if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$. By Proposition 1.44, the map

$$s_1 \circ h(x) = f(x) + g(x)$$

is continuous, and the same argument holds for subtraction and multiplication. To prove the continuity of division, it suffices to show that the mapping $x \to 1/x$ is continuous at all points where $x \ne 0$ (which we know from elementary calculus).

Proposition 1.47 Let (\mathbb{X},d) and (\mathbb{Y},ρ) be metric spaces and let $f:\mathbb{X}\to\mathbb{Y}$ be continuous. Let $\mathbb{X}'\subset\mathbb{X}$ be non-empty. Then,

$$f|_{\mathbb{X}'}:\mathbb{X}'\to\mathbb{Y}$$

is continuous, where X' is endowed with the subspace metric.

Proof: Since f is continuous, for every $A \subset \mathbb{Y}$,

$$f^{-1}(A) = \{x \in \mathbb{X} : f(x) \in A\}$$

is open in \mathbb{X} . By Proposition 1.37,

$$f|_{\mathbb{X}'}^{-1}(A) = \{x \in \mathbb{X}' : f(x) \in A\} = \mathbb{X}' \cap \{x \in \mathbb{X} : f(x) \in A\} = \mathbb{X}' \cap f^{-1}(A)$$

is open in \mathbb{X}' , thus proving that $f|_{\mathbb{X}'}^{-1}$ is continuous.

The following proposition generalizes a theorem you proved in first year calculus for the case of real-valued functions:

Proposition 1.48 Let (X,d) and (Y,ρ) be metric spaces and let $f_n: X \to Y$ be a sequence of continuous functions satisfying

$$\lim_{n\to\infty}\sup_{x\in\mathbb{X}}\rho(f_n(x),f(x))=0,$$

where $f: \mathbb{X} \to \mathbb{Y}$. Then, f is also continuous.

Proof: Let $x_k \to x$ in \mathbb{X} . By Heine's characterization, we need to show that

$$\lim_{k\to\infty}\rho(f(x_k),f(x))=0.$$

For every k and every n,

$$\rho(f(x_k), f(x)) \le \rho(f(x_k), f_n(x_k)) + \rho(f_n(x_k), f_n(x)) + \rho(f_n(x), f(x))$$

$$\le 2 \sup_{z \in \mathbb{X}} \rho(f_n(z), f(z)) + \rho(f_n(x_k), f_n(x)).$$

Letting $k \to \infty$, using the fact that all the f_n are continuous,

$$\limsup_{k\to\infty} \rho(f(x_k),f(x)) \le 2 \sup_{z\in\mathbb{X}} \rho(f_n(z),f(z)).$$

Since this holds for every n, and since the right-hand side tends to zero as $n \to \infty$ we obtain the desired result.

 $\textcircled{Exercise 1.24 Let } A \subset \mathbb{X}$. The **indicator function** (פונקציה מציינת) of the the set A is defined as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Show that the points of discontinuity of χ_A coincide with ∂A .

Exercise 1.25 Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function.

(a) Prove that f is continuous if and only if its graph

$$\{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$$

is closed in \mathbb{R}^2 .

(b) Show why this is not generally true if f is not bounded.

S Exercise 1.26 Let F,G be sets in a metric space \mathbb{X} and let \mathbb{Y} be a metric space. Given two functions $f:F\to\mathbb{Y}$ and $g:G\to\mathbb{Y}$ such that f=g on $F\cap G$, we define the function

$$(f \wedge g)(x) = \begin{cases} f(x) & x \in F \\ g(x) & x \in G. \end{cases}$$

- 1. Prove that if F, G are closed in \mathbb{X} and f, g are continuous, then $f \wedge g$ is continuous.
- 2. Prove that this is also correct if F, G are open in \mathbb{X} .
- 3. Show that this is not necessarily the case if F is open and G is closed.

1.3.2 Functions from \mathbb{R}^k to \mathbb{R}^m

Functions between vector spaces, and in particular, functions between \mathbb{R}^k to \mathbb{R}^m for some $k, m \in \mathbb{N}$ play a prominent role in analysis. Unless otherwise specified, we will always assume that those spaces are endowed with the Euclidean metric.

Comments:

(a) A function $f: \mathbb{R}^k \to \mathbb{R}^m$ gets for input a k-tuple of real numbers and returns an m-tuple of real numbers. We can represent it as

$$f(x_1,...,x_k) = (f_1(x_1,...,x_k),...,f_m(x_1,...,x_k)).$$

(b) The **graph** of a function $f: \mathbb{X} \to \mathbb{Y}$ between two sets is the set

Graph
$$(f) = \{(x, f(x)) : x \in X\} \subset X \times Y.$$

In particular, for $\mathbb{X} = \mathbb{R}^k$ and $\mathbb{Y} = \mathbb{R}^m$,

Graph
$$(f) = \{(x, f(x)) : x \in \mathbb{R}^k\} \subset \mathbb{R}^k \times \mathbb{R}^m = \mathbb{R}^{k+m}.$$

- (c) When m = 1, the function is called a **scalar function** (פונקציה סקלרית). Its graph is a surface in \mathbb{R}^{m+1} .
- (d) When k = 1, the function

$$f(t) = (f_1(t), \ldots, f_m(t)).$$

is called a **path** (מסילה). You could think of the parameter t as "time".

Example: Consider the path $f:[0,2\pi] \to \mathbb{R}^2$ given by

$$f(t) = (\cos t, \sin t).$$

Its image is the unit circle. Note that if the domain is $[0, 4\pi]$ or the whole of \mathbb{R} , the image is still the unit circle. What about its graph? It is the set

$$(t,\cos t,\sin t)\subset\mathbb{R}^3$$
,

which is a helix (סלילס) whose axis is along the x-axis.

Proposition 1.49 A function $f: \mathbb{R}^k \to \mathbb{R}^m$ is continuous if and only if each of its m components $f_i: \mathbb{R}^k \to \mathbb{R}$ is continuous.

Proof: Denote by π_i the projection, $\pi_i : \mathbb{R}^m \to \mathbb{R}$,

$$\pi_j(y_1,\ldots,y_m)=y_j, \qquad j=1,\ldots,m.$$

Each π_i is a continuous function, because if $A \subset \mathbb{R}$ is open, then

$$\pi_j^{-1}(A) = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{j-1 \text{ times}} \times A \times \underbrace{\mathbb{R} \times \dots \mathbb{R}}_{m-j \text{ times}}$$

is open in \mathbb{R}^m .

Now suppose that $f: \mathbb{R}^k \to \mathbb{R}^m$ is continuous; then $f_j = \pi_j \circ f$ is continuous as a composition of continuous functions. Conversely, suppose that all the f_j are continuous, and let $x^n = (x_1^n, \dots, x_k^n) \in \mathbb{R}^k$ converge to $x \in \mathbb{R}^k$. Then

$$\lim_{n\to\infty} f_j(x^n) = f_j(x) \qquad \text{for all } j=1,\ldots,m,$$

hence

$$\lim_{n\to\infty} (f_1(x^n), \dots, f_m(x^n)) = (f_1(x), \dots, f_m(x)).$$

Example: Any linear function $A: \mathbb{R}^k \to \mathbb{R}^m$ is continuous because

$$(A(x_1,\ldots,x_k))_j=\sum_{i=1}^k a_{ji}x_i,$$

which is clearly continuous.

1.4 Compactness

1.4.1 Definition and basic properties

In this section, we study an important property that certain metric spaces (or subsets of) possess: compactness. As we will see, compactness, is a certain measure of "smallness".

Definition 1.50 (Compactness) A metric space (\mathbb{X} , d) is called **compact** (קומפקטי) if every sequence x_n has a convergent subsequence. A subset $K \subset \mathbb{X}$ is called compact if the subspace (K,d) is compact (i.e., if every sequence in K has a subsequence that converges to a limit in K).

Comment: The compactness of a set requires two conditions: (i) that every sequence has a converging subsequence, and (ii) that the limit belongs to that set. The second conditions is superfluous when the set is the entire space.

Comment: Note that the compactness of a set is a property pertinent only to the set, whereas openness/closedness is property of a set in relation with the entire space.

Exercise 1.27 Let (\mathbb{X},d) be a metric space and let $Y \subset \mathbb{X}$ with the subspace metric d_Y . Prove that a set $A \subset Y$ is compact as a subset of Y if and only if it is compact as a subset of \mathbb{X} .

Exercise 1.29 Consider the space $\{0,1\}^{\mathbb{N}}$ (infinite binary sequences) with the metric

$$d(x,y) = 2^{\min\{i : x_i \neq y_i\}}.$$

(First prove that it is indeed a metric.) Prove that this space is compact. Hint: show first that convergence in this space amounts to component-by-component convergence.

Proposition 1.51 A finite union of compact sets is compact.

Proof: Given a sequence $x_n \in \bigcup_{i=1}^n K_i = A$, with K_i compact, it must have an infinite subsequence in at least one of the K_i 's. Since this K_i is compact, there is a sub-subsequence converging in K_i hence in A.

Examples:

(a) Every singleton in a metric space is compact.

- (b) Every metric space having a finite number of elements is compact.
- (c) The Bolzano-Weierstraß theorem implies that a closed segment on the line is compact.

Definition 1.52 A set A in a metric space (X,d) is called α -discrete if

$$\forall x \neq y \in A$$
 $d(x,y) \geq \alpha$.

Lemma 1.53 In a metric space (\mathbb{X}, d) , a set containing an infinite α -discrete set for some $\alpha > 0$ is not compact.

Proof: By definition, one can construct an infinite sequence $x_n \in A$, such that

$$\forall m \neq n$$
 $d(x_m, x_n) \geq \alpha$.

If x_n has a subsequence x_{n_k} converging to a limit x, then there exist $k \neq \ell$ such that

$$d(x_{n_k},x)<\frac{\alpha}{2}$$
 and $d(x_{n_\ell},x)<\frac{\alpha}{2}$.

Hence,

$$d(x_{n_k},x_{n_\ell}) \leq d(x_{n_k},x) + d(x,x_{n_\ell}) < \alpha,$$

which is a contradiction.

Comment: The other direction is not true: a set may fail to contain an α -discrete set for every $\alpha > 0$ and yet not be compact. The segment $(0,1) \subset \mathbb{R}$ is an example.

 \bigcirc *Exercise 1.30* True of false: if a set *A* in a metric space contains a subset $B \subset A$ which is not compact, then *A* is not compact.

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Proposition 1.54 A compact set in a metric space is both bounded and closed.

Proof: Closedness: Let K be a compact set in a metric space (\mathbb{X}, d) . By definition, every sequence $x_n \in K$ has a subsequence that converges to a point in K. In particular, if x_n is a sequence converging to a point $x \in \mathbb{X}$, then by the uniqueness of the limit, $x \in K$. It follows from Proposition 1.27 that K is closed.

Boundedness: Suppose, by contradiction, that K was not bounded. We will show that it contains a 1-discrete sequence, hence it is not compact. Take an arbitrary point x_1 . Since K is not bounded, there exists a point $x_2 \in K \setminus \mathcal{B}_1(x_1)$. Inductively, there exists for every n a point

$$x_n \in K \setminus \bigcup_{j=1}^{n-1} \mathcal{B}_1(x_j),$$

forming a 1-discrete sequence in *K*.

Comment: One often confuses between "compact" and "bounded and closed". In general, the implication is only uni-directional, although we will soon see situations in which there is indeed a correspondence between the two. An elementary counter-example is the segment (0,1), which is closed (as a space) and bounded, but not compact.

Proposition 1.55 If (X,d) is a compact metric space, then every closed subset of X is compact.

Proof: Let K be closed in \mathbb{X} and let $x_n \in K$. Since \mathbb{X} is compact, x_n has a subsequence x_{n_k} that converges to some limit $x \in \mathbb{X}$. Since K is closed, $x \in K$, hence K is compact.

Corollary 1.56 In a compact metric space closedness and compactness are the same.

Supercise 1.31 Prove that in a metric space in which all the closed balls are compact, the compact sets coincide with the sets that are closed and bounded.

Definition 1.57 Let (X,d) be a metric space with $A,B \subset X$ and $a \in X$. We define the distance between a point and a set, and the distance between two sets as follows:

$$d(a,A) = \inf_{x \in A} d(a,x)$$
 and $d(A,B) = \inf_{x \in A, y \in B} d(x,y).$

(Note that we write inf because we can't guarantee the existence of a minimum.)

Proposition 1.58 Let $A, B \subset \mathbb{X}$ be nonempty, closed disjoint sets, such that at least one of them is compact. Then d(A, B) > 0.

Example: A non-example: let $\mathbb{X} = \mathbb{R}^2$ with $A = \{(x, 1/x) : x > 0\}$ and $B = \{(x, 0) : x \ge 0\}$. The sets A, B are non-empty, disjoint and closed (why?). Yet, d(A, B) = 0. There is no contradiction because neither A nor B is compact. $\triangle \triangle$

Proof: Suppose that *A* is compact and *B* is closed. Assume, by contradiction, that d(A, B) = 0. Then, there exist sequences $x_n \in A$, $y_n \in B$, such that

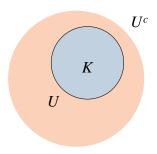
$$\lim_{n\to\infty}d(x_n,y_n)=0.$$

Since A is compact, the sequence x_n has a subsequence x_{n_k} with limit $x \in A$, Then,

$$d(x, y_{n_k}) \le d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) \to 0,$$

from which follows that $y_{n_k} \to x$. Since *B* is closed, $x \in B$, contradicting the disjointness of *A* and *B*.

Corollary 1.59 If U is open in (\mathbb{X},d) and $K \subset U$ is compact, then U^c is closed and disjoint from K, from which follows that $d(K,U^c) > 0$. In particular, since $\partial U \subset U^c$, $d(K,\partial U) > 0$.



Exercise 1.32 For a countable sequence metric spaces (\mathbb{X}_n, d_n) define the product space

$$X = \prod_{n=1}^{\infty} X_n = \{(x_1, x_2, \dots,) : x_1 \in X_1, x_2 \in X_2, \dots\}$$

with the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Prove that if each \mathbb{X}_n is a compact space, then the product space \mathbb{X} is compact. Hint: think of Cantor's diagonalization.

1.4.2 Topological characterizations of compactness

In this section we will see equivalent characterizations of compactness (which in fact are the definitions in the more general context of topological spaces).

Definition 1.60 Let (\mathbb{X},d) be a metric space. A **covering** (כיסוי) of \mathbb{X} is a collection of subsets $\{A_{\alpha}: \alpha \in I\}$, whose union is \mathbb{X} . A covering is called a **finite** covering if the number of sets A_{α} is finite. It is called an **open covering** (בחנות) if all the A_{α} are open sets (in \mathbb{X}).

Let $\{A_{\alpha}\}$ be a covering of \mathbb{X} . One may ask whether we could still cover \mathbb{X} with a smaller collection of subsets. Consider for example the case where $\mathbb{X} = \mathbb{R}$; the countable collection of subsets

$$\{(n-1,n+1):n\in\mathbb{Z}\}$$

is an open covering of \mathbb{R} . The omission of any of those (n-1,n+1) would fail to cover the point x = n; that is, this open covering does not have a strict subcovering, not to speak of a finite sub-covering. As the main theorem below shows, compactness can be defined alternatively as "every open covering can be reduced to a finite sub-covering".

Lemma 1.61 Let (\mathbb{X},d) be a compact metric space, and let \mathscr{F} an open covering of \mathbb{X} (\mathscr{F} is a collection of open subsets of \mathbb{X}). Then, there exists an $\varepsilon > 0$ such that every open ball of radius ε is contained in one of the sets in \mathscr{F} . That is,

$$\exists \varepsilon > 0$$
 such that $\forall x \in \mathbb{X}$ $\exists A \in \mathscr{F}$ such that $\mathcal{B}_{\varepsilon}(x) \subset A$.

Comment: Given an open covering, the supremum over all such ε is called the **Lebesgue number** (מספר לְבֶּג) of the covering.

Proof: We need to prove that

$$\exists n \in \mathbb{N} \quad \forall x \in \mathbb{X} \quad \exists A \in \mathscr{F} \quad \text{such that} \qquad \mathcal{B}_{1/n}(x) \subset A.$$

Suppose that the desired property does not hold. That is,

$$\forall n \in \mathbb{N} \quad \exists x_n \in \mathbb{X} \quad \text{such that} \qquad \forall A \in \mathscr{F} \quad \mathcal{B}_{1/n}(x_n) \notin A.$$

Since \mathbb{X} is compact, the sequence x_n has a subsequence x_{n_k} converging to some $x \in \mathbb{X}$. Since \mathscr{F} is a covering of \mathbb{X} , $x \in A$ for some $A \in \mathscr{F}$. Moreover, since A is open,

$$\exists m \in \mathbb{N}, \qquad \mathcal{B}_{1/m}(x) \subset A.$$

Since x is a partial limit of x_n ,

$$\exists n > 2m, \qquad x_n \in \mathcal{B}_{1/2m}(x).$$

Thus, for every $y \in \mathcal{B}_{1/n}(x_n)$,

$$d(y,x) \le d(y,x_n) + d(x_n,x) < \frac{1}{n} + \frac{1}{2m} < \frac{1}{m},$$

i.e., $y \in \mathcal{B}_{1/m}(x)$, which implies that

$$\mathcal{B}_{1/n}(x_n) \subset \mathcal{B}_{1/m}(x) \subset A \in \mathscr{F},$$

in contradiction with our assumption.

Definition 1.62 A metric space (\mathbb{X},d) is called **totally-bounded** (חסום כליל) if for every $\varepsilon > 0$, \mathbb{X} can be covered by a finite set of open balls of radius ε . That is,

$$\forall \varepsilon > 0 \quad \exists x_1, \ldots, x_n \quad such \ that \qquad \bigcup_{i=1}^n \mathcal{B}_{\varepsilon}(x_i) = \mathbb{X}.$$

Proposition 1.63 Let (X,d) be a compact metric space. Then, it is totally-bounded.

Proof: Suppose that the space was not totally bounded. Then, there exists an $\varepsilon > 0$ such that no finite collection of balls of radius ε covers \mathbb{X} . As a result, we can construct inductively an ε -discrete sequence, proving that \mathbb{X} is not compact.

Lemma 1.64 Let x_n be a sequence in a metric space (\mathbb{X},d) . If x_n doesn't have a converging subsequence, then the set

$$\{x_n: n \in \mathbb{N}\}$$

is closed.

Proof: Denote $A = \{x_n : n \in \mathbb{N}\}$. Let $x \notin A$. Since x is not a partial limit of the sequence x_n , it is *not* true that

$$\forall \varepsilon > 0 \quad \forall n_0 \in \mathbb{N} \quad \exists k > n_0 \quad \text{such that} \quad d(x_k, x) < \varepsilon.$$

That is,

$$\exists \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \text{such that} \quad \forall k > n_0 \quad d(x_k, x) \ge \varepsilon.$$

Since $x \notin A$, there exists an

$$r = \min\{d(x_n, x) : n \le n_0\} > 0.$$

Then,

$$\forall n \le n_0$$
 $d(x_n, x) > r$ and $\forall n > n_0$ $d(x_n, x) > \varepsilon$,

i.e., $\mathcal{B}_{\min(r,\varepsilon)}(x) \subset A^c$, proving that A^c is open, i.e., that A is closed.

Theorem 1.65 (Heine-Borel) The following three properties of a metric space (\mathbb{X},d) are equivalent:

- 1. X is compact.
- 2. Every open covering of X has a finite sub-covering.
- 3. Every collection of closed sets $\{F_{\alpha}\}_{{\alpha}\in I}$ has the property that if the intersection of every finite sub-collection is not empty, then the intersection of the entire collection is not empty.

Proof: We start by proving that Property 2 implies Property 3. Let $\{F_{\alpha}: \alpha \in I\}$ be a collection of closed sets, hence $\{F_{\alpha}^{c}: \alpha \in I\}$ is a collection of open sets. If Property (2) holds, then

$$\bigcup_{\alpha \in I} F_{\alpha}^{c} = \mathbb{X} \quad \Longrightarrow \quad \exists \text{ finite } J \subset I \text{ such that } \bigcup_{\alpha \in J} F_{\alpha}^{c} = \mathbb{X}.$$

By duality.

$$\left(\bigcup_{\alpha\in I}F_{\alpha}^{c}\right)^{c}=\varnothing\quad\Longrightarrow\;\exists\;\text{finite}\;J\subset I\;\text{such that}\;\left(\bigcup_{\alpha\in J}F_{\alpha}^{c}\right)^{c}=\varnothing.$$

Using de Morgan's laws,

$$\bigcap_{\alpha \in I} F_{\alpha} = \emptyset \implies \exists \text{ finite } J \subset I \text{ such that } \bigcap_{\alpha \in J} F_{\alpha} = \emptyset.$$

Recalling that $A \Rightarrow B$ is the same as $B^c \Rightarrow A^c$,

$$\forall \text{ finite } J \subset I \quad \bigcap_{\alpha \in J} F_\alpha \neq \varnothing \quad \Longrightarrow \quad \bigcap_{\alpha \in I} F_\alpha \neq \varnothing.$$

We next show that Property 3 implies Property 1. Suppose that Property 3 holds, and let x_n be a sequence. We need to show that it has a converging subsequence. Suppose, by contradiction, that it doesn't, and consider the sets

$$A_n = \{x_k : k \ge n\}.$$

By Lemma 1.64, the sets A_n are closed; they are also decreasing, and have have the property that every finite sub-collection has a non-empty intersection, as for every finite set $J \subset \mathbb{N}$,

$$\bigcap_{n\in J} A_n = A_{\max(J)}.$$

By Property 3, there exists an

$$x \in \bigcap_{n=1}^{\infty} A_n$$
.

This means that x repeats infinitely many times in the sequence x_k , hence x is a partial limit, contradicting the assumption.

It remains to show that Property 1 implies Property 2. Let \mathbb{X} be compact and let \mathscr{U} be an open covering. By Lemma 1.61 there exists an $\varepsilon > 0$ such that every open ball of radius ε is contained in an element of \mathscr{U} , namely,

$$\forall x \in \mathbb{X} \quad \exists V_x \in \mathscr{U} : \mathcal{B}_{\varepsilon}(x) \subset V_x.$$

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By Proposition 1.63, \mathbb{X} is totally-bounded, i.e., we can cover \mathbb{X} with a finite number of open balls of radius ε ,

$$\exists \{x_i\}_{i=1}^n : \bigcup_{i=1}^n \mathcal{B}_{\varepsilon}(x_i) = \mathbb{X}.$$

Then,

$$\bigcup_{i=1}^n V_{x_i} = \mathbb{X},$$

i.e., \mathcal{U} has a finite subcover of \mathbb{X}

Comment: The term compactness was introduced by Maurice Fréchet in 1906. It was primarily introduced in the context of metric spaces in the sense of "sequential compactness". The "covering compact" definition has become more prominent because it allows us to consider general topological spaces, and many of the old results about metric spaces can be generalized to this setting. This generalization is particularly useful in the study of function spaces, many of which are not metric spaces.

One of the main reasons for studying compact spaces is because they are in some ways very similar to finite sets. In other words, there are many results which are easy to show for finite sets, the proofs of which carry over with minimal change to compact spaces.

Example: A non-example: the space $\mathbb{X} = (0,1)$ is not compact, yet for every $\varepsilon > 0$ we can cover (0,1) with a finite number of segments of size ε (this space is totally-bounded). However, the open covering of (0,1),

$$\mathscr{F} = \{(1/(n+2), 1/n) : n \in \mathbb{N}\},\$$

does not have a finite sub-covering.

The Heine-Borel theorem refers to compact metric spaces. An analogous theorem applies for compact sets in metric spaces:

Theorem 1.66 (Heine-Borel for sets) A set K in a metric space \mathbb{X} is compact if and only if for every collection of open sets $\{A_{\alpha}\}_{\alpha\in I}$, such that $K\subset\bigcup_{\alpha\in I}A_{\alpha}$, there exists a finite sub-collection $\{A_{\alpha}\}_{\alpha\in J}$, such that $K\subset\bigcup_{\alpha\in I}A_{\alpha}$.

Service 1.33 Prove it.

Recall that a set $A \subset \mathbb{X}$ is called **dense** (צפון) in \mathbb{X} if $\overline{A} = \mathbb{X}$ (every non-empty open set in \mathbb{X} intersects A), and that a metric space is called **separable** (ספרבילי) if is contains a dense countable subset.

Example: \mathbb{R} is separable because it contains the countable dense set \mathbb{Q} . So is any \mathbb{R}^n , which contains the countable dense set \mathbb{Q}^n .

Proposition 1.67 A compact metric space is separable.

Proof: Let \mathbb{X} be compact. Then, it is totally bounded, hence it can be covered by a finite number of open balls of size 1,

$$\exists x_1^1, \dots, x_{n_1}^1$$
 such that $\mathbb{X} = \bigcup_{i=1}^{n_1} \mathcal{B}_1(x_i^1)$.

Likewise, every $k \in \mathbb{N}$,

$$\exists x_1^k, \dots, x_{n_k}^k$$
 such that $\mathbb{X} = \bigcup_{i=1}^{n_k} \mathcal{B}_{1/k}(x_i^k)$.

Consider the countable set,

$$A = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \{x_i^k\}.$$

The set A is dense in \mathbb{X} , because for every $x \in \mathbb{X}$ and every $\varepsilon > 0$, take $k > 1/\varepsilon$. Since

$$\mathbb{X} = \bigcup_{i=1}^{n_k} \mathcal{B}_{1/k}(x_i^k),$$

there exists an x_i^k such that $d(x_i^k, x) < 1/k < \varepsilon$.

TA material 1.2 The **Hausdorff distance** (also called Pompeiu-Hausdorff distance) measures how far two subsets of a metric space are from each other. It turns the set of non-empty compact subsets of a metric space into a metric space in its own right.

 $\textcircled{\mathbb{Z}}$ *Exercise 1.34* Let (\mathbb{X}, d) be a metric space in which the closed-and-bounded sets are compact (like \mathbb{R}^n). Prove that for any $a \in \mathbb{X}$ and $F \subset \mathbb{X}$ non-empty and closed, there exists a point $x \in F$ such that

$$d(a,x) = d(a,F).$$

 \cong *Exercise 1.35* Show that the finite product space of compact spaces is compact: Let $(\mathbb{X}_1, d_1), \ldots, (\mathbb{X}_n, d_n)$ be metric spaces and consider the product space

$$\mathbb{X} = \prod_{i=1}^n \left\{ (x_1, \dots, x_n) : x_j \in \mathbb{X}_j \,\forall j \right\},\,$$

with

$$d(x,y) = \max_{i=1,\dots,n} d_i(x_i,y_i).$$

(1) Prove that (X, d) is a metric space. (2) Prove that if the (X_i, d_i) are compact then (X, d) is compact.

1.4.3 Continuous functions on compact sets

Proposition 1.68 (Continuity and compactness) Continuous functions between metric spaces map compact sets into compact sets.

Proof: Let $f : \mathbb{X} \to \mathbb{Y}$ be continuous and let $K \subset \mathbb{X}$ be a compact set. We need to show that $f(K) \subset \mathbb{Y}$ is compact. Take a sequence y_n in f(K). By the definition of f(K),

$$\forall y_n \in f(K)$$
 $\exists x_n \in K$ such that $f(x_n) = y_n$.

Since *K* is compact, x_n has a subsequence x_{n_k} converging to a limit $x \in K$,

$$\lim_{k\to\infty}x_{n_k}=x\in K.$$

Since *f* is continuous,

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x) \in f(K),$$

hence f(K) is compact.

Corollary 1.69 Suppose that \mathbb{X} is compact and $f : \mathbb{X} \to \mathbb{Y}$ is continuous. Then, the image of every closed set is closed.

Proof: Let $A \subset \mathbb{X}$ be closed. By Corollary 1.55 it is compact. By Proposition 1.68 f(A) is compact, and by once again Corollary 1.55, it is closed.

The following theorem is a generalization of a well-known theorem in elementary calculus:

Proposition 1.70 (Extremal values in compact spaces) A continuous function from a compact metric space (equivalently, from a compact set in a metric space) into \mathbb{R} assumes a minimum and a maximum.

Proof: Let $f: \mathbb{X} \to \mathbb{R}$ with \mathbb{X} compact. By the previous proposition, the set $f(\mathbb{X}) \subset \mathbb{R}$ is compact, hence bounded and closed. Since it is bounded, it has a finite supremum, and since it is closed this supremum must belong to the set, i.e., it is a maximum.

(Note the advantage of working with abstract concepts: we prove powerful theorems without having to deal with problem-specific details.)

Definition 1.71 (Uniform continuity) A mapping $f: (\mathbb{X},d) \to (\mathbb{Y},\rho)$ is called uniformly continuous (מירה שוה) if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad : \quad \forall \{x,y \, : \, d(x,y) < \delta\} \quad : \quad \rho(f(x),f(y)) < \varepsilon.$$

That is, f is uniformly continuous if it is continuous, and the same δ can be chosen for all points. The function $\delta(\varepsilon)$ (with $\delta(\varepsilon)$ being the supremum of all δ 's) is called the **continuity modulus** (מודולוס הרציפות) of the function.

Definition 1.72 (Hölder and Lipschitz continuity) A mapping $f: (\mathbb{X}, d) \to (\mathbb{Y}, \rho)$ is called **Hölder continuous** of order α if there exists a real number k > 0 such that for every $a, b \in \mathbb{X}$,

$$\rho(f(a),f(b)) \le k [d(a,b)]^{\alpha}.$$

The mapping is called **Lipschitz continuous** (ליפשיץ) if it is Hölder continuous of order $\alpha = 1$.

 \cong *Exercise 1.37* Let $F \subset \mathbb{X}$ be a subset in a metric space. Prove that the function $\mathbb{X} \to \mathbb{R}$ defined by $x \mapsto d(x, F)$ is Lipschitz continuous.

Theorem 1.73 A continuous function on a compact metric space (equivalently, on a compact set in a metric space) is uniformly continuous.

Proof: Let $f: (\mathbb{X}, d) \to (\mathbb{Y}, \rho)$ be continuous with \mathbb{X} compact. Suppose that f were not uniformly continuous. Then,

$$\exists \varepsilon > 0 : \forall \delta > 0 \quad \exists x, y : d(x, y) < \delta \quad \text{and} \quad \rho(f(x), f(y)) \ge \varepsilon.$$

In particular,

$$\exists \varepsilon > 0 : \forall n \in \mathbb{N} \quad \exists x_n, y_n : d(x_n, y_n) < 1/n \quad \text{and} \quad \rho(f(x_n), f(y_n)) \ge \varepsilon.$$

Since \mathbb{X} is compact, there exists a subsequence x_{n_k}, y_{n_k} such that $x_{n_k} \to x$ and $y_{n_k} \to y$. For every n,

$$d(x,y) \le d(x,x_{n_k}) + d(x_{n_k},y_{n_k}) + d(y_{n_k},y),$$

hence,

$$d(x,y) \leq \liminf_{k\to\infty} (d(x,x_{n_k}) + d(x_{n_k},y_{n_k}) + d(y_{n_k},y)) = 0,$$

from which follows that x = y. By the continuity of the metric ρ ,

$$\lim_{k \to \infty} \rho(f(x_{n_k}), f(y_{n_k})) = \rho(f(x), f(x)) = 0,$$

which is a contradiction.

1.4.4 Homeomorphisms and isometries

In this section we study notions of equivalences between metric spaces:

Definition 1.74 (Homeomorphism) A mapping $f: (\mathbb{X}, d) \to (\mathbb{Y}, \rho)$ between metric spaces is called a **homeomorphism** (הומאומורפיזם) if it is one-to-one, onto, and both f and f^{-1} are continuous, i.e., $A \subset \mathbb{X}$ is open if and only if $f(A) \subset Y$ is open. Two metric spaces are called **homeomorphic** if there exists a homeomorphism between them.

Comment: It is easy to see that being homeomorphic is an **equivalence relation** between metric spaces: it is symmetric, reflexive and transitive. In fact, homeomorphism is a topological concept (only requires the definition of open sets), and is also called **topological equivalence**.

Examples:

- (a) The identity map from $(\mathbb{R}^n, \|\cdot\|_p)$ to $(\mathbb{R}^n, \|\cdot\|_q)$ is a homeomorphism, because open balls with respect to the $\|\cdot\|_p$ metric are open sets (although not balls) with respect to the $\|\cdot\|_q$ metric, and vice versa.
- (b) Every two open segments (a,b) and (c,d) on the line are homeomorphic, because the linear mapping $x \to c + (d-c)(x-a)/(b-a)$ is a homeomorphism.
- (c) Every open segment (a,b) is homeomorphic to the entire real line because it is homeomorphic to the open segment $(-\pi/2, \pi/2)$ and the latter is homeomorphic to the real line though the homeomorphism $x \mapsto \tan x$.

An homeomorphism is before all an equivalence between sets, i.e., a bijection; it identifies points in one space with points in another space. The additional property of being a homeomorphism is summarized by the following proposition:

Proposition 1.75 A homeomorphism $f : \mathbb{X} \to \mathbb{Y}$ preserves the topological properties of openness, closedness and compactness. That is, $A \subset \mathbb{X}$ is open (resp. closed or compact) if and only if $f(A) \subset \mathbb{Y}$ is open (resp. closed or compact).

Proof: This is an immediate corollary on the fact that the pre-image of an open set under f or f^{-1} is open and the image of a compact set under f or f^{-1} is compact.

Proposition 1.76 If f is a continuous, one-to-one mapping from a compact metric space (\mathbb{X}, d) *onto an (arbitrary) metric space* (\mathbb{Y}, ρ) *, then f is a homeomorphism.*

Proof: Since f is one-to-one and onto, f^{-1} is well-defined, and it only remains to prove that it is continuous. By Theorem 1.42, it suffices that the pre-image under f^{-1} of a closed set in \mathbb{X} is closed in \mathbb{Y} , namely, that f maps closed sets into closed sets. This follows from Corollary 1.69.

How does one show that two metric spaces are **not** homeomorphic? In principle, we should show that there does not exist a homeomorphism between them. We can't of course check all mappings. The idea is to show that the two spaces differ in a **topological invariant**—a property that is preserved by homeomorphisms.

Examples:

- (a) \mathbb{R} is not homeomorphic to a discrete space because in a discrete space all compact sets are finite, whereas this not true in \mathbb{R} . Thus, the image of [0,1] in the discrete space cannot be compact.
- (b) (0,1) is not homeomorphic to [0,1] because the first is not compact whereas the second is.
- (c) The unit square $[0,1] \times [0,1]$ is not homeomorphic to [0,1]. Both spaces are compact, and it requires more subtle arguments to show it.
- (d) \mathbb{R}^n is not homeomorphic to \mathbb{R}^m for $m \neq n$. This is a very deep theorem (**invariance of domain**, Brouwer 1912), which is hard to prove and beyond the scope of this course (requires advanced topology).

A homeomorphism preserves topological properties of sets but it does not necessarily preserve metric properties such that boundedness. **Metric equivalence** is a particular case of topological equivalence:

Definition 1.77 Two metric spaces (\mathbb{X},d) and (\mathbb{Y},ρ) are called **isometric** (איזומטריים) if they are homeomorphic, and there exists a homeomorphism $f:\mathbb{X}\to\mathbb{Y}$ which is distance-preserving: for all $x,y\in\mathbb{X}$,

$$\rho(f(x), f(y)) = d(x, y).$$

Comment: Like homeomorphism, isometry is an equivalence relation between metric space.

™ Exercise 1.38

- 1. Show that every distance-preserving function from \mathbb{X} *onto* \mathbb{Y} is an homeomorphism. Give an example to show that the converse is not true.
- 2. Prove that every isometry from \mathbb{R} onto itself is of the form $x \mapsto \pm x + a$.
- Exercise 1.39 Show that every distance-preserving function from a compact space into itself is onto.
- $\$ \mathbb{E} xercise 1.40 Let $f:(0,1) \to \mathbb{R}$ be continuous. Show that f can be extended into a continuous function on [0,1] iff f is uniformly continuous on (0,1).
- Exercise 1.41 Prove that the following spaces are homeomorphic (using the natural metrics):
 - 1. \mathbb{R}^2 .
 - 2. The unit disc: $D^2 = \{(x, y) : x^2 + y^2 < 1\}$.
 - 3. The unit square: $Sq^2 = \{(x, y) : \max(|x|, |y|) < 1\}.$
 - 4. The unit sphere in \mathbb{R}^3 less one point, e.g. $S^2 \setminus \star = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \setminus \{(0, 0, 1)\}.$

1.4.5 Compact sets in normed spaces

We have seen that a compact set in a metric space is always closed and bounded. What about the opposite: is every closed and bounded set compact? The general answer is no. In this section, we will examine this question in the particular context of normed spaces.

Theorem 1.78 (Bolzano-Weirestraß in \mathbb{R}^m) In \mathbb{R}^m (endowed with the Euclidean metric) closed and bounded sets are compact.

Proof: Let $K \subset \mathbb{R}^m$ be closed and bounded. Since it is bounded, there exists an M > 0, such that for every $x = (x^1, \dots, x^m) \in K$,

$$d(x,0) = \left(\sum_{i=1}^{m} |x^{i}|^{2}\right)^{1/2} < M.$$

In particular,

$$|x^i| < M$$
 $\forall i = 1, \ldots, m.$

Let now $x_n = (x_n^1, \dots, x_n^m)$ be a sequence in K. Consider the sequence of real numbers x_n^1 , formed by the first component. Since this sequence is bounded, x_n has by the Bolzano-Weierstraß theorem a subsequence x_{n_k} such that $x_{n_k}^1$ has a limit x^1 . Consider now the second component $x_{n_k}^2$ of that subsequence: once again, this is a bounded sequence in \mathbb{R} , hence there exists a sub-subsequence $x_{n_{k_\ell}}$, such that $x_{n_{k_\ell}}^2$ converges to a limit x^2 . We proceed component-by-component, until we obtain a subsequence, let's denote it y_n , such that each component y_n^i converges to a limit x^i . It follows that $y_n \to x = (x^1, \dots, x^m)$. Since K is closed, $x \in K$, hence K is compact.

Corollary 1.79 The closed unit ball $\hat{\mathcal{B}}_1(0)$ in \mathbb{R}^m is compact, and so is the unit sphere $\mathcal{S}_1(0)$.

Definition 1.80 Let V be a vector space and let $\|\cdot\|$ and $\|\cdot\|'$ be norms on V. The norms are called **equivalent** (שקולות) if there exist constants $c_1, c_2 > 0$, such that for all $v \in V$,

$$c_1||v|| \le ||v||' \le c_2||v||.$$

Another way to state it is that the mapping $\mathrm{Id}:(V,\|\cdot\|)\to(V,\|\cdot\|')$ is Lipschitz and so is its inverse (it is **bi-Lipschitz**).

Proposition 1.81 Let $\|\cdot\|$ and $\|\cdot\|'$ by equivalent norms on V. Then the identity

$$\mathrm{Id}: (V, \|\cdot\|) \to (V, \|\cdot\|')$$

is a homeomorphism mapping bounded sets to bounded sets.

Proof: By symmetry, it suffices to prove that Id is continuous and maps bounded sets to bounded sets. Let $x_n \to x$ with respect to the $\|\cot\|$ -norm. Then,

$$\|\delta(x_n) - \mathrm{Id}(x)\|' = \|x_n - x\|' \le c_2 \|x_n - x\| \to 0,$$

i.e., Id is continuous. Furthermore, let $A \in \mathcal{B}_r(0)$, i.e., for all $x \in A$,

$$||x|| < r$$
.

Then,

$$||x||' \le c_2 ||x|| < c_2 r$$

i.e.,

$$\operatorname{Id}(A) \subset \mathcal{B}'_{c_2r}(0).$$

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Theorem 1.82 All the norms on \mathbb{R}^m are equivalent.

Proof: Since the equivalence of norms is an equivalence relation, it suffices to show that all the norms are equivalent to the Euclidean norm. Let $\|\cdot\|$ be a norm on \mathbb{R}^m . Every vector $x \in \mathbb{R}^m$ can be written as

$$x = \sum_{i=1}^{m} x_i e_i,$$

where e_i is the *i*-th unit vector. Using the triangle inequality and the Cauchy-Schwarz inequality,

$$||x|| \le \sum_{i=1}^{m} |x_i| ||e_i|| \le \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \left(\sum_{i=1}^{m} ||e_i||^2\right)^{1/2} \le \left(\sum_{i=1}^{m} ||e_i||^2\right)^{1/2} ||x||_2.$$

In the other direction, write

$$||x|| = \left\| \frac{x}{\|x\|_2} \right\| ||x||_2 \ge (\inf\{||y|| : ||y||_2 = 1\}) ||x||_2.$$

We will be done if we prove that $\inf\{\|y\| : \|y\|_2 = 1\} > 0$. Since the function $x \mapsto \|x\|$ is continuous (every norm is continuous) and since the unit sphere with respect of the Euclidean norm is compact, the infimum attains a minimum. This minimum cannot be zero by the positivity of the norm.

Corollary 1.83 Let $\|\cdot\|$ be a norm on \mathbb{R}^m . Then, all the closed and bounded sets are compact.

Proof: Let $A \subset \mathbb{R}^m$ be closed and bounded. Since the identity map is a homeomorphism from $(\mathbb{R}^m, \|\cdot\|)$ to $(\mathbb{R}^m, \|\cdot\|_2)$ mapping bounded sets to bounded sets, it follows that $\mathrm{Id}(A) = A$ is closed and bounded in $(\mathbb{R}^m, \|\cdot\|_2)$, hence compact. Finally, using again the fact that the identity map is a homeomorphism, A is compact in $(\mathbb{R}^m, \|\cdot\|)$.

Proposition 1.84 Let $(V, \|\cdot\|)$ be a normed space of finite dimension m. Then, it is isometric to \mathbb{R}^m endowed with some norm.

Proof: Let a_i be a basis for V and consider the linear map $T: V \to \mathbb{R}^m$, given by

$$T\left(\sum_{i=1}^m x_i a_i\right) = (x_1, \ldots, x_m).$$

This map is invertible and in fact a homeomorphism. For $x \in \mathbb{R}^m$ define

$$||x||_m = ||T^{-1}(x)||.$$

Clearly, $||x||_m \ge 0$ with equality if and only if $T^{-1}(x) = x = 0$. For homogeneity,

$$\|\alpha x\|_m = \|T^{-1}(\alpha x)\| = \|\alpha T^{-1}(x)\| = |\alpha| \|T^{-1}(x)\| = |\alpha| \|x\|_m.$$

Finally, for $x, y \in \mathbb{R}^m$,

$$||x+y||_m = ||T^{-1}(x+y)|| = ||T^{-1}(x) + T^{-1}(y)|| \le ||T^{-1}(x)|| + ||T^{-1}(y)|| = ||x||_m + ||y||_m.$$

It follows that $\|\cdot\|_m$ is a norm on \mathbb{R}^m . By construction, T is an isometry, since for every $u, v \in V$,

$$||u-v|| = ||T^{-1}T(u-v)|| = ||T(u-v)||_m = ||Tu-Tv||_m.$$

Corollary 1.85 Let V be a finite-dimensional vector space and let $\|\cdot\|$ and $\|\cdot\|'$ be norms on V. Then they are equivalent.

Proof: It follows from the fact that $(V, \|\cdot\|)$ are $(V, \|\cdot\|')$ are isometric to \mathbb{R}^m endowed with two norms and that every two norms on \mathbb{R}^m are equivalent.

Corollary 1.86 Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. Then, all the closed and bounded sets are compact.

Proof: It follows from the fact that $(V, \| \cdot \|)$ is isometric to a normed space in which this property holds.

Example: If V is an infinite-dimensional vector space, then, not every two norms on V are equivalent. Take for example the vector space C([0,1]) and consider the sequence of elements $f_n(x) = x^n$. Then,

$$\lim_{n\to\infty} \|f_n - 0\|_1 = \lim_{n\to\infty} \int_0^1 |x^n - 0| \, dx = 0,$$

however

$$\lim_{n\to\infty} ||f_n - 0||_{\infty} = \lim_{n\to\infty} \sup_{0 \le x \le 1} |x^n - 0| = 1,$$

so the sequence converges to zero in one norm and not in the other, implying that the norms are not equivalent. \triangle \triangle

Example: If $(V, \| \cdot \|)$ is an infinite-dimensional normed space, then, closed and bounded sets are not necessarily compact. Take for example the unit ball in $(C[0,1], \| \cdot \|_{\infty})$. For every $n \in \mathbb{N}$ denote

$$I_n = [1/(n+1), 1/n],$$

and let f_n be a sequence of non-negative continuous functions satisfying $||f_n||_{\infty} = 1$, vanishing outside I_n and assuming the value 1 is an interior point of I_n . Then for every $n \neq n$,

$$||f_n-f_m||_{\infty}=1,$$

i.e., f_n is a 1-discrete sequence, hence does not have a converging subsequence.



1.4.6 Convergence of sequences of functions

This section deals with sequences of functions between metric spaces.

Definition 1.87 (Pointwise and uniform convergence) A sequence f_n of functions from (\mathbb{X},d) to (\mathbb{Y},ρ) is said to **converge pointwise** (מתכנסות נקודתית) to a limit function f if for every $x \in \mathbb{X}$,

$$\lim_{n\to\infty} f_n(x) = f(x),$$

i.e.,

$$\forall x \in \mathbb{X} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \forall n > N \quad \rho(f_n(x), f(x)) < \varepsilon.$$

It is said to **converge uniformly** (מתכנסות במידה שווה) if N can be chosen independently of x; that is,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \forall n > N \quad \forall x \in \mathbb{X} \quad \rho(f_n(x), f(x)) < \varepsilon.$$

Comment: Pointwise convergence amounts to

$$\forall x \in \mathbb{X}$$
 $\lim_{n \to \infty} \rho(f_n(x), f(x)) = 0,$

whereas uniform convergence amounts to

$$\lim_{n\to\infty}\sup_{x\in\mathbb{X}}\rho(f_n(x),f(x))=0.$$

Proposition 1.88 (Cauchy criterion for uniform convergence) Let (\mathbb{X},d) be a metric spaces and let $f_n: \mathbb{X} \to \mathbb{R}$ be a sequence of functions (not necessarily continuous) satisfying the following condition:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad such that \quad \forall m, n > N \quad \sup_{x \in \mathbb{X}} |f_n(x) - f_m(x)| < \varepsilon.$$

Then, there exists a function $f: \mathbb{X} \to \mathbb{R}$, such that $f_n \to f$ uniformly.

Proof: We start by showing that the sequence f_n converges pointwise. Fix x. Then,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall m, n > N \quad |f_n(x) - f_m(x)| < \varepsilon.$$

It follows from Cauchy's criterion for sequences in \mathbb{R} that the sequence $f_n(x)$ has a limit, which we denote by f(x).

It remains to prove that the convergence is uniform. Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every m, n > N and for every $x \in \mathbb{X}$,

$$|f_n(x)-f_m(x)|<\frac{\varepsilon}{2}.$$

Thus, for every $x \in \mathbb{X}$ and m, n > N,

$$|f_n(x) - f(x)| \le |f_n(x) - f_m(x)| + |f_m(x) - f(x)|,$$

from which follows, letting $m \to \infty$, that

$$|f_n(x)-f(x)|\leq \frac{\varepsilon}{2}<\varepsilon.$$

Thus,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \forall n > N \quad \forall x \in \mathbb{X} \qquad |f_n(x) - f(x)| < \varepsilon,$$

which completes the proof.

Examples:

- (a) The functions between the discrete spaces \mathbb{N} and $\{0,1\}$ defined by $f_n(j) = \delta_{j,n}$ converge to the constant function $f: j \mapsto 0$ pointwise but not uniformly. In fact, one can show that a sequence of functions f_n into a discrete space converges uniformly to f if and only if f_n coincides with f eventually.
- (b) The sequence of functions $f_n : [0,1) \to [0,1)$ (with the standard metric) defined by $f_n(x) = x^n$ converges to f(x) = 0, pointwise, but not uniformly, because for every n,

$$\sup_{0 \le x < 1} |f_n(x) - f(x)| = \sup_{0 \le x < 1} |x^n - 0| = 1.$$

(c) Consider the following sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} n^2 x & 0 < x < 1/n \\ 2n - n^2 x & 1/n < x < 2/n \\ 0 & \text{otherwise.} \end{cases}$$

It converges pointwise to f = 0, but not only it does not converges uniformly, we even have

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\lim_{n\to\infty}n=\infty.$$

Consider the sequence of functions $f_n(x) = x^n$ on the closed interval [0, 1]. Although each of the f_n is continuous, this sequence converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1. \end{cases}$$

The following theorem shows that this can't happen if the convergence is uniform.

Theorem 1.89 Let f_n be a sequence of continuous functions from (\mathbb{X}, d) to (\mathbb{Y}, ρ) , converging uniformly to a limit f; then, the limit is continuous. Moreover, if all the f_n are uniformly continuous on \mathbb{X} , then f is uniformly continuous on \mathbb{X} .

Proof: Fix a point $x \in \mathbb{X}$ and let x_m be a sequence converging to x. For every pair of indexes n and m,

$$\rho(f(x_m), f(x)) \leq \rho(f(x_m), f_n(x_m)) + \rho(f_n(x_m), f_n(x)) + \rho(f_n(x), f(x))$$

$$\leq 2 \sup_{y \in \mathbb{X}} \rho(f(y), f_n(y)) + \rho(f_n(x_m), f_n(x)).$$

Let $\varepsilon > 0$ be given. Because the f_n converge to f uniformly, we can choose n sufficiently large such that the first term is less than $\varepsilon/2$. Having fixed n, there exists an N such that for all m > N, the second term is less than $\varepsilon/2$ (since f_n is

continuous). To conclude, for all $\varepsilon > 0$ there exists an N such that for all m > N, $\rho(f(x_m), f(x)) < \varepsilon$, i.e., f is continuous.

Assume next that the f_n are uniformly continuous. For $x, y \in \mathbb{X}$ we write

$$\rho(f(x), f(y)) \le \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y))$$

$$\le \sup_{z \in \mathbb{X}} \rho(f(z), f_n(z)) + \rho(f_n(x), f_n(y)).$$

Since f_n converges uniformly to f, given $\varepsilon > 0$, there exists an n such that the first term is less than $\varepsilon/2$. Since this f_n is uniformly continuous, there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $\rho(f_n(x), f_n(y)) < \varepsilon/2$, i.e., implies $\rho(f(x), f(y)) < \varepsilon$.

The importance of uniform convergence is exemplified in the following proposition:

Proposition 1.90 Let (\mathbb{X},d) *and* (\mathbb{Y},ρ) *be metric spaces and let* $f_n, f: \mathbb{X} \to \mathbb{Y}$. *If* f_n *are continuous and converge uniformly to* f, *then for every sequence* x_n *in* \mathbb{X} ,

$$x_n \to x$$
 implies $f_n(x_n) \to f(x)$.

Example: A non-example: the sequence $f_n(x) = x^n$ defined on [0, 1] converges pointwise to the function

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1. \end{cases}$$

Consider the sequence $x_n = 1 - 1/n$, which converges to 1. Then,

$$\lim_{n \to \infty} f_n(x_n) = \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq f(1).$$

Proof: We have

$$\rho(f_n(x_n), f(x)) \le \rho(f_n(x_n), f(x_n)) + \rho(f(x_n), f(x)) \le \sup_{y \in \mathbb{X}} \rho(f_n(y), f(y)) + \rho(f(x_n), f(x)).$$

The first term on the right hand side converges to zero by the uniform convergence of f_n , whereas the second term converges to zero by the continuity of f (which results from the previous proposition).

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Definition 1.91 Let f_n be a sequence of functions from a metric space (\mathbb{X},d) into \mathbb{R} (in fact, we only need the target to be a normed space). The **series** (שור) $\sum_{n=1}^{\infty} f_n$ is said to converge uniformly to S if the sequence of partial sums,

$$S_n = \sum_{k=1}^n f_k$$

converges uniformly to S.

Theorem 1.92 (Weierstraß' M-test) Let f_n be a sequence of functions from a metric space (\mathbb{X}, d) into \mathbb{R} . Suppose that there exists a sequence of real numbers M_n such that

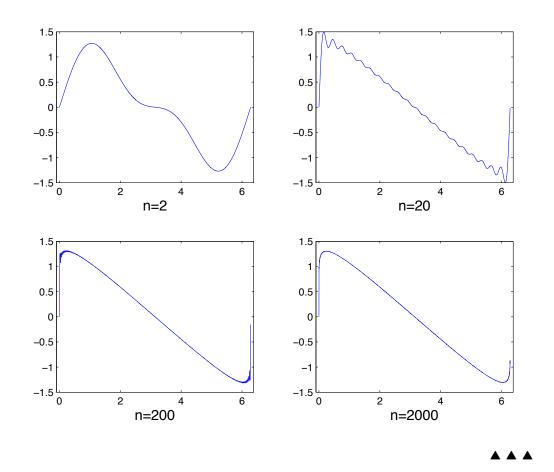
$$\sup_{x\in\mathbb{X}}|f_n(x)|\leq M_n,$$

and that the series $\sum_{n=1}^{\infty} M_n$ converges. Then, the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on \mathbb{X} . In particular, if the f_n are continuous, so is the series.

Example: The figure below shows the functions

$$S_n(x) = \sum_{k=1}^n \frac{\sin kx}{k^{1.1}}$$

for n = 2, 20, 200, 2000. By the Weierstraß M-test, the sequence S_n converges uniformly on $[0, 2\pi]$. Moreover, since each of the partial sums is a uniformly continuous function, the limit is uniformly continuous.



Proof: For every m > n,

$$\sup_{x \in \mathbb{X}} |S_m(x) - S_n(x)| = \sup_{x \in \mathbb{X}} \left| \sum_{k=n+1}^m f_k(x) \right| \le \sup_{x \in \mathbb{X}} \sum_{k=n+1}^m |f_k(x)|$$

$$\le \sum_{k=n+1}^m M_k \le \sum_{k=n+1}^\infty M_k.$$

Since the right-hand side is the tail of a convergent series, there exists for every $\varepsilon > 0$ an $N \in \mathbb{N}$, such that for all m, n > N,

$$\sup_{x\in\mathbb{X}}|S_m(x)-S_n(x)|<\varepsilon,$$

hence S_n converges uniformly by Cauchy's criterion.

Proposition 1.93 Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions converging uniformly on [a,b] to a limit f. Then,

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Comment: We restrict ourselves to functions $\mathbb{R} \to \mathbb{R}$ because we have so far only learned about integrals of real-valued univariate functions.

Proof: The integrals exists because the f_n and their limit f are continuous (Theorem 1.89), hence Riemann integrable. By the linearity of the integral, for every n,

$$\left| \int_{a}^{b} f(x) dx - \int_{a}^{b} f_{n}(x) dx \right| = \left| \int_{a}^{b} (f(x) - f_{n}(x)) dx \right|$$

$$\leq \int_{a}^{b} |f(x) - f_{n}(x)| dx$$

$$\leq \int_{a}^{b} \sup_{a \leq y \leq b} |f(y) - f_{n}(y)| dx$$

$$= (b - a) \sup_{a \leq y \leq b} |f(y) - f_{n}(y)|.$$

Since f_n converges to f uniformly, the right hand side tends to zero.

Theorem 1.89 provides a criterion for the limit of a sequence of continuous functions to be continuous (respectively, uniformly continuous). What about differentiability? Under what conditions is the limit of a sequence of differentiable functions differentiable?

Theorem 1.94 Let $f_n:(a,b) \to \mathbb{R}$ be a sequence of differentiable functions such that (i) the sequence $f_n(x)$ converges at at least one point $c \in (a,b)$, and (ii) the sequence of derivatives, f'_n converges uniformly on (a,b) to a limit function g. Then,

- 1. The sequence f_n converges uniformly on (a,b) to a limit f.
- 2. f is differentiable with f' = g.

Comments:

- (a) We need the first condition, because the non-convergent sequence $f_n(x) = n$ satisfies the second condition with g = 0.
- (b) Since we have not assumed the f_n to be *continuously* differentiable, we can't assume g to be continuous, and not even integrable. This prevents us from defining $f(x) = \int_a^x g(y) dy$, and then show that f_n converges to f uniformly. This will "cost us" in technical complications. Providing a simpler proof for the case where the f_n are continuously differentiable is left as an exercise (see below).

Proof: We first prove the uniform convergence of f_n based on Cauchy's criterion. For every m, n, we use the mean-value theorem to obtain

$$|f_m(x) - f_n(x)| \le |(f_m - f_n)(x) - (f_m - f_n)(c)| + |f_m(c) - f_n(c)|$$

= $|x - c||(f'_m - f'_n)(\xi_{mn})| + |f_m(c) - f_n(c)|,$

for some ξ_{mn} between x and c. Hence,

$$\sup_{x} |f_m(x) - f_n(x)| \le (b - a) \sup_{\xi} |f'_m(\xi) - f'_n(\xi)| + |f_m(c) - f_n(c)|.$$

Since f'_n converges uniformly on (a,b), given $\varepsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $m, n > N_1$,

$$\sup_{\xi} |f'_m(\xi) - f'_n(\xi)| < \frac{\varepsilon}{2(b-a)}.$$

Likewise, since $f_n(c)$ converges, there exists an $N_2 \in \mathbb{N}$ such that for all $m, n > N_2$,

$$|f_m(c)-f_n(c)|<\frac{\varepsilon}{2}.$$

Thus, for all $m, n > \max(N_1, N_2)$,

$$\sup_{x} |f_m(x) - f_n(x)| < \varepsilon,$$

i.e., f_n satisfies Cauchy's criterion for uniform convergence; we denote the limit by f. Since the f_n are differentiable, and in particular continuous, f is continuous.

It remains to show that f' = g. Let x be an arbitrary point, and set

$$\varphi(h) = \begin{cases} \frac{1}{h} [f(x+h) - f(x)] & h \neq 0 \\ g(x) & h = 0, \end{cases}$$

which is defined on some neighborhood I of zero. We will be done if we prove that φ is continuous at zero.

How can we show that a function is continuous? For example, by showing that it is the uniform limit of a sequence of continuous functions. Define the sequence of functions on I,

$$\varphi_n(h) = \begin{cases} \frac{1}{h} [f_n(x+h) - f_n(x)] & h \neq 0 \\ f'_n(x) & h = 0. \end{cases}$$

These functions are, by definition, continuous at zero, and we already know that they converge *pointwise* to φ . To show that they converge uniformly, we show that they satisfy Cauchy's criterion: if $h \neq 0$ then

$$|\varphi_n(h) - \varphi_m(h)| = \frac{1}{h} |(f_n - f_m)(x+h) - (f_n - f_m)(x)|$$

$$\leq \sup_{\xi \in \mathbb{X}} |f'_n(\xi) - f'_m(\xi)|,$$

whereas at h = 0,

$$|\varphi_n(0) - \varphi_m(0)| = |f'_n(0) - f'_m(0)| \le \sup_{\xi \in \mathbb{X}} |f'_n(\xi) - f'_m(\xi)|.$$

Thus, given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all m, n > N,

$$\sup_{h} |\varphi_n(h) - \varphi_m(h)| \le \sup_{\xi \in \mathbb{X}} |f'_n(\xi) - f'_m(\xi)| < \varepsilon,$$

proving that φ_n satisfies Cauchy's criterion, hence converges to φ uniformly. This concludes the proof.

 \otimes Exercise 1.43 Let $f_n:(a,b) \to \mathbb{R}$ be a sequence of continuously differentiable functions such that (i) the sequence $f_n(x)$ converges at at least one point $c \in (a,b)$, and (ii) the sequence of derivatives, f'_n converges uniformly on (a,b) to a limit function g. Prove that

$$\lim_{n\to\infty} (f_n(x) - f_n(c)) = \lim_{n\to\infty} \int_c^x f_n'(y) \, dy = \int_c^x g(y) \, dy.$$

Conclude then that f_n has a limit, which we denote by f, and that f' = g.

Proposition 1.93 and Theorem 1.94 can be reformulated in terms of convergent series:

Corollary 1.95 (1) If the series of continuous real-valued functions, $\sum_{n=1}^{\infty} f_n$, converges uniformly on an interval [a,b], then

$$\int_a^b \left(\sum_{n=1}^\infty f_n(x)\right) dx = \sum_{n=1}^\infty \int_a^b f_n(x) dx.$$

(2) Let f_n be a sequence of differentiable functions on (a,b) such that the series of derivatives converges uniformly on (a,b), and there exists a point $c \in (a,b)$ at which the series $\sum_{n=1}^{\infty} f_n$ converges, then the series converges uniformly on (a,b) and

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f'_n(x).$$

We have seen that the uniform convergence of continuous functions implies a continuous limit. What about the other direction? Does the fact that a (pointwise) converging sequence of continuous functions have a continuous limit imply that the convergence is uniform? The following theorem show that the answer may be affirmative under certain additional conditions.

Theorem 1.96 (Dini) Let f_n be a sequence of continuous real-valued functions defined on a compact metric space (\mathbb{X},d) , and converging pointwise to a function f, which is also continuous. If for every $x \in \mathbb{X}$ the sequence $f_n(x)$ is monotone, then f_n converges to f uniformly.

Comment: There is no requirement that $f_n(x)$ and $f_n(y)$ be monotone in the same direction.

Sumption does not hold.

Proof: Suppose, by contradiction, that the convergence was not uniform. Then, it is not true that

$$\forall \varepsilon > 0 \quad \exists k \in \mathbb{N} : \forall n > k \ \forall x \in \mathbb{X} \quad |f_n(x) - f(x)| < \varepsilon,$$

i.e.,

$$\exists \varepsilon > 0 : \forall k \in \mathbb{N} \quad \exists n_k > k \ \exists x_k \in \mathbb{X} : |f_{n_k}(x_k) - f(x_k)| \ge \varepsilon,$$

Since the sequence $f_n(x)$ is monotone for every x, it follows that

$$|f_m(x_k) - f(x_k)| \ge |f_{n_k}(x_k) - f(x_k)| \ge \varepsilon \quad \forall m \le n_k.$$

Putting it together,

$$\exists \varepsilon > 0 : \forall k \in \mathbb{N} \quad \exists n_k > k \ \exists x_k \in \mathbb{X} : \forall m \le n_k \quad |f_m(x_k) - f(x_k)| \ge \varepsilon,$$

We can state it differently,

$$\exists \varepsilon > 0 : \forall k \in \mathbb{N} \quad \exists n_k > k \ \exists x_k \in \mathbb{X} : \forall m \ |f_m(x_k) - f(x_k)| \ge \varepsilon \text{ for large enough } k.$$

Since \mathbb{X} is compact, the sequence x_k has a converging subsequence x_{k_l} with limit x. Since f_m and f are continuous, the limit $l \to \infty$ may be taken, yielding

$$\exists \varepsilon > 0 : \forall m \in \mathbb{N} \quad |f_m(x) - f(x)| \ge \varepsilon.$$

This violate the assumption that f_m converges to f pointwise.

Proof: Here is another proof that uses the "covering version" of compactness. Since the sequence f_n converges pointwise and monotonically to f, then there exists for every $\varepsilon > 0$ and every point x a number $N(x, \varepsilon)$, such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$
 $\forall n > N(x, \varepsilon).$

Moreover, since both f_n and f are continuous, there exists a $\delta(x, \varepsilon) > 0$ such that

$$|f(y)-f(x)|<rac{arepsilon}{3} \quad ext{ and } \quad |f_{N(x,arepsilon)}(y)-f_{N(x,arepsilon)}(x)|<rac{arepsilon}{3} \qquad \forall y\in \mathcal{B}_{\delta(x,arepsilon)}(x).$$

The union of all $\mathcal{B}_{\delta(x,\varepsilon)}(x)$ is an open covering of \mathbb{X} . Since \mathbb{X} is compact, there exists a finite sub-covering,

$$\mathbb{X} = \bigcup_{i=1}^m \mathcal{B}_{\delta(x_i,\varepsilon)}(x_i).$$

Let now $N = \max_{N(x_i,\varepsilon)}$ (here we exploit the finiteness). Since every $y \in \mathbb{X}$ is contained in one of these balls, say the k-th ball, then for all n > N (here we exploit the monotonicity),

$$|f_n(y) - f(y)| \le |f_n(y) - f_n(x_k)| + |f_n(x_k) - f(x_k)| + |f(x_k) - f(y)| < \varepsilon,$$

which prove uniform convergence.

Corollary 1.97 Let f_n be a sequence of non-negative, real-valued function defined on a compact metric space (\mathbb{X},d) . If the sequence $\sum_{n=1}^{\infty} f_n$ converges pointwise and its limit is continuous, then it converges uniformly on \mathbb{X} .

1.4.7 The Arzela-Ascoli theorem

For a compact metric space K, the space C(K) of real-valued functions over K is a normed space with respect to

$$||f|| = \max_{x \in K} |f(x)|,$$

where the maximum exists by Proposition 1.70. Convergence in this space is uniform convergence, as

$$d(f,g) = ||f - g|| = \max_{x \in K} |f(x) - g(x)|.$$

We have already seen that closed and bounded sets in C(K) are not necessarily compact. It turns out that if closedness and boundedness are supplemented with another property, sets in C([0,1]) are compact.

Definition 1.98 (Equicontinuity) Let K be a compact metric space and let A be a collection of functions in C(K). This collection is said to be equicontinuous (רציפה במידה אחידה) if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \ : \ \forall f \in A \quad \forall x,y \ : \ d(x,y) < \delta \qquad |f(x) - f(y)| < \varepsilon.$$

That is, all the $f \in A$ are uniformly continuous, and the same modulus of continuity $\delta(\varepsilon)$ applies for all $f \in A$.

Theorem 1.99 (Arzelà-Ascoli) A closed and bounded set $A \subset C(K)$ is compact if and only if it is equicontinuous.

Proof: Suppose that A equicontinuous. Since it is bounded, there exists an M such that

$$A \subset \mathcal{B}_M(0)$$
,

i.e., for all $f \in A$, ||f|| < M, or

$$\sup_{f \in A} \left(\max_{x \in K} |f(x)| \right) < M.$$

Consider a sequence of functions $f_n \in A$. We need to show that it has a subsequence converging uniformly.

Let x_n be a sequence of points which is dense in K; such a sequence exists by Proposition 1.67 (compact spaces are separable). The sequence of real numbers $f_n(x_1)$ is bounded, hence there exists a subsequence $f_n^{(1)}$ of f_n converging at x_1 . Take then the sequence $f_n^{(1)}(x_2)$; by the same argument, there exists a subsequence $f_n^{(2)}$ of $f_n^{(1)}$ converging at x_2 (and by inheritance also at x_1). We proceed inductively, constructing a sequence of subsequences, $f_n^{(k)}$, which for given k converge at x_1, \dots, x_k .

The following picture is helpful:

$$f_1$$
 f_2 ... f_n ... $f_n^{(1)}$ $f_2^{(1)}$... $f_n^{(1)}$... converges at x_1 $f_1^{(2)}$ $f_2^{(2)}$... $f_n^{(2)}$... converges at x_1, x_2 ... $f_n^{(k)}$... $f_n^{(k)}$... $f_n^{(k)}$... converges at x_1, \dots, x_k .

Every row is a sequence of functions which is (i) a subsequence of the previous row, and (ii) converges at a finite number of points.

Consider now the "diagonal" sequence $f_n^{(n)}$. Since it is a subsequence of $f_n^{(1)}$ it converges at x_1 . By the same argument, it converges at every point x_k , i.e., it converges pointwise at a dense set of points.

So far, we have not used equicontinuity. We will use it now to show that $f_n^{(n)}$ converges uniformly on K. Let $\varepsilon > 0$ be given, and take $\delta > 0$ according to the definition of equicontinuity. Consider the collection of open sets

$$\mathcal{U} = \{\mathcal{B}_{\delta}(x_k) : k \in \mathbb{N}\}.$$

Since x_k is dense in K, \mathcal{U} is an open cover of K (every point in K it at a distance less than δ from one of the points x_k). Since K is compact, there exists a finite sub-cover,

$$K \subset \{\mathcal{B}_{\delta}(x_{k_j}) : j = 1, \ldots, p\}.$$

As the sequence $f_n^{(n)}(x_k)$ converges for all k, there exists for each j an N_j , such that for every $m, n > N_j$,

$$|f_n^{(n)}(x_{k_i})-f_m^{(m)}(x_{k_i})|<\varepsilon.$$

Finally, as there are only finitely many N_i , we may set

$$N = \max_{j=1}^p N_j.$$

Let $x \in K$. Since the finite sets of open balls cover K, there exists a j such that

$$x \in \mathcal{B}_{\delta}(x_{k_i}).$$

For every m, n > N,

$$|f_{n}^{(n)}(x) - f_{m}^{(m)}(x)| \leq |f_{n}^{(n)}(x) - f_{n}^{(n)}(x_{k_{j}})| + |f_{n}^{(n)}(x_{k_{j}}) - f_{m}^{(m)}(x_{k_{j}})| + |f_{m}^{(m)}(x_{k_{j}}) - f_{m}^{(m)}(x)| < \varepsilon + \varepsilon + \varepsilon,$$

where in the first and the third term we used the definition of δ . It follows that for every $\varepsilon > 0$ there exists an N such that for all m, n > N

$$||f_n^{(n)}-f_m^{(m)}||_{\infty}<\varepsilon.$$

By Cauchy's criterion, the sequence $f_n^{(n)}$ converges uniformly on K (and its limit is continuous). Finally, since A is closed, the limit is in A, hence A is compact.

We leave the other direction as an exercise.

S Exercise 1.45 Let $K \subset \mathbb{X}$ be compact. Show that if a set $A \subset C(K)$ is compact, then it is equicontinuous.

Comment: Note that we used the closedness only in the last step. Without closenedess, we still have that if $A \subset C(K)$ is bounded and equicontinuous, then every sequence in A has a subsequence converging uniformly; the limit is in C(K) but not necessarily in A.

Comment: A classical application of the Arzela-Ascoli theorem is the local existence proof of solutions of ordinary differential equations.

Example: Consider a sequence of differentiable functions $f_n : [a,b] \to \mathbb{R}$, that are uniformly bounded, $||f||_{\infty} \le M_1$ and have a uniformly bounded derivative, $||f'_n||_{\infty} \le M_2$. Then the sequence has a subsequence that converges uniformly.

This is because the set of differentiable functions that satisfy those conditions is bounded and equicontinuous, as

$$|f_n(x)-f_n(y)|\leq M_2|x-y|.$$

Exercise 1.46 1. Let f and f_n be continuous mappings from a compact metric space \mathbb{X} into a metric space \mathbb{Y} . Show that if

$$x_n \to x \qquad \Rightarrow \qquad f_n(x_n) \to f(x)$$

for all $x \in \mathbb{X}$ and sequences $x_n \in \mathbb{X}$, then f_n converges uniformly to f.

2. Show, by a counter-example, that this is not true if X is not compact.

Exercise 1.47 Let f_n be a sequence of real-valued functions uniformly bounded (i.e., $f_n(x) \leq M$) and Riemann integrable on [a,b]. Define for $x \in [a,b]$:

$$F_n(x) = \int_a^x f_n(y) \, dy.$$

Show that the sequence (F_n) has a subsequence that converges uniformly on [a,b].

1.4.8 The cantor set and the Cantor function

TA material 1.3

1.5 Completeness

1.5.1 Definitions and properties

Definition 1.100 (Cauchy sequence) Let (\mathbb{X},d) be a metric space. A sequence of points x_n in \mathbb{X} is called a **Cauchy sequence** (סדרת קושי) if there exists for every $\varepsilon > 0$ an N such that $d(x_n, x_m) < \varepsilon$ for all m, n > N.

Comment: A converging sequence is always a Cauchy sequence; if $x_n \to x$, then there exists for every $\varepsilon > 0$ an N such that for every n > N, $d(x_n, x) < \varepsilon/2$. It follows that for all m, n > N,

$$d(x_n, x_m) \le d(x_n, x) + d(x_m, x) < \varepsilon.$$

On the other hand, a Cauchy sequence does not necessarily converge. Take the sequence $x_n = 1/n$ in (0,1]. Then,

$$d(x_n, x_m) < 1/\min(m, n) \to 0,$$

but the sequence has no limit in this space.

The following proposition should be familiar from your first calculus class:

Proposition 1.101 If a Cauchy sequence has a converging subsequence then the whole sequence converges.

Proof: Suppose that the Cauchy sequence x_n has a subsequence x_{n_k} that converges to x. Let $\varepsilon > 0$ be given. By the Cauchy property,

$$\exists N : \forall m, n > N \qquad d(x_m, x_n) < \varepsilon/2,$$

and by the definition of the limit,

$$\exists K : \forall k > K \qquad d(x_{n_k}, x) < \varepsilon/2.$$

Let k be such that k > K and $n_k > N$. Then, for every n > N,

$$d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2.$$

Definition 1.102 (Complete metric space) A metric space is called **complete** (שלם) if every Cauchy sequence in \mathbb{X} converges.

—18h(2019)—

Examples:

- (a) \mathbb{R} is a complete metric space (elementary calculus course).
- (b) \mathbb{Q} is not a complete metric space because the sequence of finite decimal approximations of $\sqrt{2}$ is a Cauchy sequence but it has no limit in \mathbb{Q} .
- (c) Completeness should not be confused with closedness. The metric space $(\mathbb{Q}, |\cdot|)$ is closed (as every metric space).
- (d) If (X, d) is a complete metric space and $Y \subset X$, then (Y, d) is a complete metric space if and only if Y is closed in X.
- (e) Every compact metric space is complete (a consequence of Proposition 1.101).
- (f) A complete normed space is called a **Banach space**; a complete inner product space is called a **Hilbert space**.

Proposition 1.103 If K is a compact metric space then C(K) is a complete metric space.

Proof: We have already seen that C(K) is a normed space, hence a metric space. Let f_n be a Cauchy sequence in C(K), i.e.,

$$\forall \varepsilon > 0 \quad \exists N : \forall m, n > N \quad \sup_{x \in K} |f_n(x) - f_m(x)| < \varepsilon.$$

This is precisely Cauchy's criterion for uniform convergence, that is, f_n converges uniformly, and its limit is in C(K).

Comment: Completeness is **not** a topological invariant: it is not conserved under a homeomorphism, as shows the example of $f(x) = \tan x$ which is a homeomorphism between the incomplete space $(-\pi/2, \pi/2)$ and the complete space \mathbb{R} . In fact, the Cauchy property of a sequence is not a topological invariant.

 $\textcircled{\mathbb{Z}}$ Exercise 1.48 Prove that a metric space (\mathbb{X}, d) is complete if and only if for every decreasing sequence of closed balls,

$$\hat{\mathcal{B}}_{r_1}(x_1) \supset \hat{\mathcal{B}}_{r_2}(x_2) \supset \cdots \supset \hat{\mathcal{B}}_{r_n}(x_n) \supset \ldots$$

such that $r_n \to 0$, we have

$$\bigcap_{n=1}^{\infty} \hat{\mathcal{B}}_{r_n}(x_n) \neq \varnothing.$$

Exercise 1.49 This exercise shows that the condition $r_n \to 0$ in the previous exercise was necessary. Let $\mathbb{X} = \mathbb{N}$, and define

$$d(m,n) = \begin{cases} 0 & m=n \\ 1 + \frac{1}{m+n} & m \neq n. \end{cases}.$$

- 1. Show that d is a metric on \mathbb{X} .
- 2. Show that X is complete with respect to this metric.
- 3. Show that every point in \mathbb{X} is isolated, i.e., for every $n \in \mathbb{N}$ there exists an r_n such that $\mathcal{B}_{r_n}(x_n) = \{x_n\}$.
- 4. Show that the sequence of closed balls $B_n = \hat{\mathcal{B}}_{1+1/2n}(n)$ is decreasing, i.e., $B_{n+1} \subset B_n$, and yet their countable intersection is empty.

X Exercise 1.50 Let (X,d) be a metric space and $D \subset X$ a dense set. Show that X is complete if and only if every Cauchy sequence in D converges in X.

 $\textcircled{\mathbb{Z}}$ Exercise 1.51 Show that if \mathbb{X} and \mathbb{Y} are metric spaces, \mathbb{Y} is complete, $D \subset \mathbb{X}$ is a dense subset of \mathbb{X} , and $f; D \to \mathbb{Y}$ is a uniformly continuous function, then f has a unique extension which is a function on the whole \mathbb{X} . That is, that there exists a unique continuous function, $\tilde{f}: \mathbb{X} \to \mathbb{Y}$, such that $\tilde{f}|_{D} = f$. Show, furthermore, that the uniform continuity of f is necessary.

Supercise 1.52 Prove that the following spaces are Banach spaces:

1. The space ℓ_{∞} of infinite sequences with the norm

$$||x||_{\infty} = \sup_{n} |x_n|.$$

2. The space ℓ_1 of infinite sequences with the norm

$$||x||_1 = \sum_{n=1}^{\infty} |x_n|.$$

TA material 1.4 Every non-complete metric space has a canonical completion (just as \mathbb{Q} has a canonical completion yielding \mathbb{R}). We formulate this as a theorem:

Theorem 1.104 (Completion of metric spaces) Let (X,d) be a (non-complete) metric space. Then there exists a complete metric space (\hat{X}, D) , such that

- (a) There exists a one-to-one map $\iota: \mathbb{X} \to \hat{\mathbb{X}}$, which is distance preserving.
- (b) The image of ι is dense in $\hat{\mathbb{X}}$.

A metric space $(\hat{\mathbb{X}}, D)$ satisfying those conditions is called a **completion** (השלמה) of (\mathbb{X}, d) . Finally, the completion is unique modulo an isomorphism; that is, if (\mathbb{Y}, ρ) is a completion of (\mathbb{X}, d) , then (\mathbb{Y}, ρ) and $(\hat{\mathbb{X}}, D)$ are isometric.

Proof: The first task is to identify the natural candidate for the completion $(\hat{\mathbb{X}}, D)$. Since the "holes" in (\mathbb{X}, d) are detected by Cauchy sequences that do not converge, we would like to define $\hat{\mathbb{X}}$ to be the "set of all limits of Cauchy sequences in \mathbb{X} ". The problem of course is there there are no such limits... Thus, the only plausible candidates for the elements of the completion would be the Cauchy sequences themselves.

So let $\hat{\mathbb{X}}$ be temporarily the set of all Cauchy sequences in \mathbb{X} . How would we then define the distance between two Cauchy sequences (x_n) and (y_n) ? The natural choice is

$$D((x_n),(y_n)) = \lim_{n\to\infty} d(x_n,y_n).$$

We need to verify that the limit exists. Note that

$$|d(x_n, y_n) - d(x_m, y_m)| \le |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)|$$

$$\le d(y_n, y_m) + d(x_n, x_m).$$

Since both (x_n) and (y_n) are Cauchy sequence, the sequence of real numbers $(d(x_n, y_n))$ is a Cauchy sequence, hence converges.

The nest step is to show that D is a metric on the space of Cauchy sequences. Symmetry is obvious. For the triangle inequality, let (x_n) , (y_n) and (z_n) be Cauchy sequences in \mathbb{X} , then

$$D((x_n), (y_n)) = \lim_{n \to \infty} d(x_n, y_n)$$

$$\leq \lim_{n \to \infty} d(x_n, z_n) + \lim_{n \to \infty} d(z_n, y_n)$$

$$= D((x_n), (z_n)) + D((z_n), (y_n)).$$

Remains positivity; D is clearly non-negative and $D((x_n), (x_n)) = 0$. It is however not strictly positive. Take any two different sequence (x_n) and (y_n) converging to the same limit a. Both are Cauchy sequences, however

$$D((x_n),(y_n)) = \lim_{n\to\infty} d(x_n,y_n) \le \lim_{n\to\infty} d(x_n,a) + \lim_{n\to\infty} d(a,y_n) = 0.$$

It follows that *D* is a **pseudo-metric** on the space of Cauchy sequences.

On the first week of this course we saw that every pseudo-metric space can be turned into a metric space by defining an equivalence relation in X,

$$(x_n) \sim (y_n)$$
 if and only if $D((x_n), (y_n)) = 0$.

For a sequence (x_n) , we denote its equivalence class by $[(x_n)]$. Then, we redefine $\hat{\mathbb{X}}$ to be the space of all equivalence classes of Cauchy sequences in \mathbb{X} , endowed with the metric

$$D([(x_n)],[(y_n)]) = \lim_{n\to\infty} d(x_n,y_n).$$

We have seen that this definition does not depend on the choice of representatives; that is, if $[(x_n)] = [(z_n)]$ and $[(y_n)] = [(w_n)]$, then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, w_n).$$

We finally have a *bone fide* metric space which we will show to be a completion of (\mathbb{X}, d) . For $x \in \mathbb{X}$ we define

$$\iota(x) = [(x)],$$

where [(x)] stands for the equivalence class of the constant sequence (x, x, ...). This map is distance preserving as

$$D([(x)],[(y)]) = \lim_{n\to\infty} d(x,y) = d(x,y).$$

In particular, it is one-to-one. It remains to prove that its image is dense in $(\hat{\mathbb{X}}, D)$. Let $[(x_n)]$ be an element of $\hat{\mathbb{X}}$, and consider the sequence of elements of $\hat{\mathbb{X}}$, $\iota(x_k)$ (it is a sequence of equivalence classes of constant sequences). Then,

$$D([(x_n)],\iota(x_k)) = \lim_{n\to\infty} d(x_n,x_k).$$

Since (x_n) is a Cauchy sequence,

$$\lim_{k\to\infty}\lim_{n\to\infty}d(x_n,x_k)=0,$$

proving that the image of ι is dense in $\hat{\mathbb{X}}$.

The next step is to show that $(\hat{\mathbb{X}}, D)$ is complete. Let $[(x_n^k)]$ be a Cauchy sequence (in k) of elements of $\hat{\mathbb{X}}$. That is, for every $\varepsilon > 0$ there exists a K such that for every $k_1, k_2 > K$,

$$D([(x_n^{k_1})],[(x_n^{k_2})]) = \lim_{n\to\infty} d(x_n^{k_1},x_n^{k_2}) < \varepsilon.$$

Consider the sequence $y_n = x_n^n$. It is easy to see that it is a Cauchy sequence in \mathbb{X} ; furthermore,

$$D([(x_n^k)],[(y_n)]) = \lim_{n\to\infty} d(x_n^k,x_n^n) \le \varepsilon.$$

This proves that

$$\lim_{k\to\infty}[(x_n^k)]=[(y_n)],$$

i.e., $(\hat{\mathbb{X}}, D)$ is complete.

It remains to show that the completion is unique modulo isometries. Let (\mathbb{Y}, ρ) be a complete metric space such that there exists a distance preserving map $J : \mathbb{X} \to \mathbb{Y}$ whose image is dense of \mathbb{Y} . Consider the map

$$A: \hat{\mathbb{X}} \ni \operatorname{Image}(\iota) \to \operatorname{Image}(\jmath) \in \mathbb{Y}$$

defined by $A = J \circ \iota^{-1}$. This is a distance preserving map from a dense set in \mathbb{X} to a dense set in \mathbb{Y} . It can be extended to an isometry $A : \hat{\mathbb{X}} \to \mathbb{Y}$. Let's show it with simpler notations: for $\alpha \in \hat{\mathbb{X}}$ there exists a sequence $\alpha_n \in \operatorname{Image}(\iota)$ converging to α . Since A is distance-preserving, $A(\alpha_n)$ is a Cauchy sequence in \mathbb{Y} , and since \mathbb{Y} is complete we may define

$$A(\alpha) = \lim_{n \to \infty} A(\alpha_n).$$

A is onto \mathbb{Y} as every element $\beta \in \mathbb{Y}$ is a limit of elements $A(\alpha_n)$. It follows that $A: \hat{\mathbb{X}} \to \mathbb{Y}$ is distance-preserving and onto, hence an isometry.

1.5.2 Contractive mapping theorem

Definition 1.105 Let \mathbb{X} be a set and let $f: \mathbb{X} \to \mathbb{X}$. A point $x \in \mathbb{X}$ is called a **fixed** point (נקודת שבת) of f if f(x) = x.

Definition 1.106 Let (\mathbb{X},d) be a metric space. A mapping $f:\mathbb{X}\to\mathbb{X}$ is called contractive (העחקה מכווצח) if there exists a real number $\lambda<1$ such that for all $x,y\in\mathbb{X}$

$$d(f(x), f(y)) \le \lambda d(x, y).$$

Lemma 1.107 A contractive mapping is continuous.

Proof: Let $f: X \to X$ be contractive with parameter $\lambda < 1$. Let $x_n \to x$. Then,

$$\lim_{n\to\infty} d(f(x_n), f(x)) \le \lim_{n\to\infty} \lambda d(x_n, x) = 0.$$

Theorem 1.108 (Contractive mapping theorem) Let (X, d) be a complete metric space. If $f: X \to X$ is contractive, then it has a unique fixed point $x \in X$.

Proof: The proof is constructive: let x_0 be an arbitrary point in \mathbb{X} and consider the sequence x_n defined inductively by

$$x_{n+1} = f(x_n).$$

We first show that x_n is a Cauchy sequence. Note that

$$d(x_{n+1},x_n) = d(f(x_n),f(x_{n-1})) \le \lambda d(x_n,x_{n-1}),$$

and by induction, $d(x_{n+1}, x_n) \le \lambda^n d(x_1, x_0)$. For n > m,

$$d(x_n, x_m) \leq \sum_{k=m}^{n-1} \lambda^k d(x_1, x_0) \leq \frac{\lambda^m}{1 - \lambda} d(x_1, x_0).$$

Thus, given $\varepsilon > 0$, choose N such that $\lambda^N/(1-\lambda) d(x_1,x_0) < \varepsilon$ and for all n, m > N,

$$d(x_n, x_m) < \varepsilon$$
.

Since \mathbb{X} is complete it follows that x_n converges to a limit, which we denote x.

Taking the $n \to \infty$ limit of the relation $x_{n+1} = f(x_n)$, using the continuity of f, we obtain that x is a fixed point of f.

Finally, if x, y are two distinct fixed points of f, then

$$d(x,y) = d(f(x), f(y)) \le \lambda d(x,y) < d(x,y),$$

which is a contradiction.

Example: Let p > 1. What is the value of the following expression:

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}.$$

By this expression, we mean the limit of the sequence $x_0 = p$, $x_{n+1} = \sqrt{p + x_n}$. To show that this limit exists, we use the contractive mapping theorem. First, we claim that the mapping

$$x \mapsto \sqrt{p+x} \equiv \Phi(x)$$

maps the interval K = [0, 2p] into itself. Indeed, the mapping is monotonic, and $2p \mapsto \sqrt{3p} \le 2p$. Thus, $\Phi : K \to K$, where K is a complete metric space.

We then show that this mapping is contractive: for $x, y \in K$,

$$\Phi(x) - \Phi(y) = \Phi'(\xi)(x - y),$$

where $\Phi'(\xi) = 1/2\sqrt{p+\xi} < 1/2$, and ξ is a point between x and y. It follows that

$$|\Phi(x) - \Phi(y)| \le \frac{1}{2}|x - y|,$$

i.e., the mapping is contractive. Thus x_n converges to the unique fixed point, satisfying $x = \sqrt{p+x}$, i.e.,

$$x = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p}.$$

Example: Suppose we want to solve the nonlinear equation f(x) = 0, where $f: \mathbb{R} \to \mathbb{R}$ is continuous. Consider the following iterations,

$$x_{n+1} = x_n - \alpha f(x_n) \equiv \Phi(x_n),$$

with x_0 chosen arbitrarily, and α is to be determined. Then, for every $x, y \in \mathbb{R}$,

$$|\Phi(x) - \Phi(y)| = |1 - \alpha f'(\xi)||x - y|,$$

where ξ is a point between x and y. If f'(x) has the property that $0 < \lambda_1 \le f'(x) \le \lambda_2 < M$, then we can choose $\alpha = 2/M$, so that

$$|1 - \alpha f'(\xi)| \le \max(|1 - 2\lambda_1/M|, |1 - 2\lambda_2/M|) < 1,$$

so that the mapping is contractive and the iterations converge.



1.5.3 Newton's method for root finding

TA material 1.5 Let $f : \mathbb{R} \to \mathbb{R}$. In many applications, one is interested is computing roots of f, i.e., points $r \in \mathbb{R}$, where f(r) = 0. Unless f is particularly simple, this cannot be done by analytical means. One resorts then to numerical methods.

Suppose that f is differentiable, and that x_0 is "close" to a root r; furthermore, assume that $f'(x_0) \neq 0$. The tangent to f at the point x_0 is the line

$$P(x) = f(x_0) + f'(x_0)(x - x_0).$$

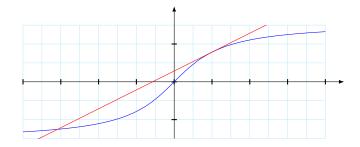
It seems natural to approximate the root of f by the root of P; the latter can be calculated analytically,

$$x=x_0-\frac{f(x_0)}{f'(x_0)}.$$

While nothing guarantees that x is a root of f, this suggested the following approach; start with some initial guess x_0 , and then generate a sequence

$$x_{n+1} = \Phi(x_n)$$
 where $\Phi(x) = x - \frac{f(x)}{f'(x)}$

with the hope that it converges to a root of f. This algorithm is called the **Newton-Raphson scheme**.



Example: Suppose that you want to evaluate \sqrt{R} using just the four operations of arithmetics. The goal then is to find a root of the function

$$f(x) = x^2 - R.$$

The Newton-Raphson scheme is

$$x_{n+1} = \Phi(x_n)$$
 where $\Phi(x) = x - \frac{x^2 - R}{2x} = \frac{x}{2} + \frac{R}{2x}$.

Below we show the first eight iterates for R = 2 starting with $x_0 = 1$. The digits in bold are correct. See how fast this sequence converges to $\sqrt{2}$.

 $x_0 = 1$

 $x_1 = 1.5$

 $x_3 =$ **1.41421**56862745098039215686274509803921568627450980392156862745

 $x_4 =$ **1.41421356237**46899106262955788901349101165596221157440445849057

 $x_5 =$ **1.41421356237309504880168**96235025302436149819257761974284982890

 $x_6 = \textbf{1.41421356237309504880168872420969807856967187537} \\ 72340015610125$

 $x_7 = 1.4142135623730950488016887242096980785696718753769480731766796$

Theorem 1.109 Let $f: \mathbb{R} \to \mathbb{R}$ be continuously twice differentiable. Let r be a root of f such that $f'(r) \neq 0$. Then, there exists a $\delta > 0$ such that the Newton-Raphson scheme converges to r for every $x_0 \in \hat{\mathcal{B}}_{\delta}(r)$.

Proof: We will use the contractive mapping theorem, showing that there exists a δ . > 0 such that

$$\Phi: \hat{\mathcal{B}}_{\delta}(r) \to \hat{\mathcal{B}}_{\delta}(r)$$

is contractive. From this will follows that x_n converges to a fixed point r of Φ , i.e.,

$$r = r - \frac{f(r)}{f'(r)},$$

i.e., r is a root of f.

Note that

$$\Phi'(x) = -\frac{f(x)f''(x)}{(f'(x))^2}.$$

Since f(r) = 0, $f'(r) \neq 0$ and f, f', f'' are continuous, there exists a $\delta > 0$, such that $|\Phi'(x)| < 1/2$ for all $x \in \hat{\mathcal{B}}_{\delta}(r)$. For such x,

$$|\Phi(x)-r|=|\Phi(x)-\Phi(r)|\leq \sup_{\xi\in\hat{\mathcal{B}}_{\delta}(r)}|\Phi'(\xi)||x-r|\leq \frac{\delta}{2},$$

i.e., $\Phi: \hat{\mathcal{B}}_{\delta}(r) \to \hat{\mathcal{B}}_{\delta}(r)$. Moreover, for $x, y \in \hat{\mathcal{B}}_{\delta}(r)$,

$$|\Phi(x) - \Phi(y)| = \sup_{\xi \in \hat{\mathcal{B}}_{\delta}(r)} |\Phi'(\xi)| |x - y| \le \frac{1}{2} |x - y|,$$

proving that the mapping is contractive.

We can say even more: for every $x \in \hat{\mathcal{B}}_{\delta}(r)$ there exists a $\xi \in (x, r)$ such that

$$|\Phi(x) - r| = |\Phi'(\xi)| |x - r| \le \left| \frac{f(\xi) - f(r)}{\xi - r} \right| \left| \frac{f''(\xi)}{(f'(\xi))^2} \right| |x - r|^2,$$

from which follows that

$$\lim_{n\to\infty}\frac{|x_{n+1}-r|}{|x_n-r|^2}=\left|\frac{f''(r)}{f'(r)}\right|.$$

Thus, eventually, there exists an M such that

$$|x_{n+1} - r| \le M |x_n - r|^2$$

which explains why each error is of the order of the previous error squared (this is called a **quadratic convergence rate**).

1.5.4 Existence of solutions of ODEs

We will now use the contractive map theorem to prove the fundamental theorem of existence of solutions to ordinary differential equations. In fact, we shall prove a simplified version of the general theorem in order to avoid some technical complications.

Theorem 1.110 (Picard, Existence and uniqueness of solutions to ODEs) Let $F: \mathbb{R}^2 \to \mathbb{R}$ be continuous, bounded with constant M and Lipschitz continuous in its second argument with constant L, i.e.,

$$|F(x,y)| \le M$$
 and $|F(x,y_1) - F(x,y_2)| \le L|y_1 - y_2|$.

Then, there exists for every $(x_0, y_0) \in \mathbb{R}^2$ a $\delta > 0$ and a unique function $y \in C([x_0, x_0 + \delta])$ satisfying the **differential equation**,

$$y'(x) = F(x, y(x))$$

subject to the initial condition (חנאי התחלה) $y(x_0) = y_0$.

Proof: Set,

$$\delta < \min(1/M, 1/L),$$

and consider the set of functions

$$K = \{ y \in C([x_0, x_0 + \delta]) : \|y - y_0\|_{\infty} \le 1 \},$$

which is a closed subset of $C([x_0, x_0+\delta])$ (hence complete in the subspace metric). Consider now the mapping

$$\Phi: C([x_0, x_0 + \delta]) \to C([x_0, x_0 + \delta]),$$

defined by

$$(\Phi(y))(x) = y_0 + \int_{x_0}^x F(t, y(t)) dt.$$

We first show that Φ maps K into itself: if $y \in K$, then for every $x \in [x_0, x_0 + \delta]$,

$$|(\Phi(\gamma))(x) - y_0| = \left| \int_{x_0}^x F(t, y(t)) dt \right| \le \int_{x_0}^x |F(t, y(t))| ds \le M\delta < 1,$$

i.e., $\Phi(y) \in K$.

We next show that Φ is contractive: let $y, z \in K$. Then, for every $x \in [x_0, x_0 + \delta]$,

$$|(\Phi(y))(x) - (\Phi(z))(x)| = \left| \int_{x_0}^x (F(t, y(t)) - F(t, z(t))) dt \right|$$

$$\leq \int_{x_0}^x |F(t, y(t)) - F(t, z(t))| dt$$

$$\leq L \int_{x_0}^x |y(t) - z(t)| dt$$

$$\leq L \|y - z\|_{\infty} \delta,$$

Taking the supremum over all x,

$$\|\Phi(y) - \Phi(z)\|_{\infty} \le (L\delta)\|y - z\|_{\infty},$$

proving that Φ is indeed contractive.

It follows from the contractive mapping theorem that Φ has a unique fixed point, $y \in K$, i.e., for every $x \in [x_0, x_0 + \delta]$,

$$y(x) = y_0 + \int_{x_0}^x F(t, y(t)) dt.$$

It remains to show that such a y solves the differential equation. Indeed, y is differentiable, and

$$y'(x) = F(x, y(x)),$$

along with the initial condition $y(x_0) = y_0$.

Comment: Under the conditions of this version of Picard's theorem, a solution for the initial value problem can be obtained for all $x \in \mathbb{R}$, by continuing the solution over δ -intervals.

1.5.5 Baire's category theorem (not taught in 2019)

The subject of this section is the classification of "large sets" in a metric space. We have already seen the following definition:

Definition 1.111 Let (X,d) be a metric space. A set $A \subset X$ is called **dense** (צפופה) if every non-empty open set $U \subset X$ intersects A. This condition is equivalent to the condition that

$$\overline{A} = \mathbb{X}$$
.

That is, every point in \mathbb{X} is a limit of a sequence in A.

—19h(2019)—

We can say that a dense set is "almost everywhere", but in many respects, dense sets can be "small". For example, two dense sets may have an empty intersection (e.g., \mathbb{Q} and $\sqrt{2} + \mathbb{Q}$ in \mathbb{R}). The rationals and the irrationals are another example of dense sets in \mathbb{R} that do not intersect. It turns out that this could not have happened if we had added the additional requirement that the sets be open.

First, a lemma:

Lemma 1.112 A metric space (\mathbb{X}, d) is complete if and only if every decreasing sequence of closed balls, $\hat{\mathcal{B}}_{r_n}(x_n)$ with $r_n \to 0$ has a non-empty intersection.

Proof: Suppose first that X is complete and let

$$\hat{\mathcal{B}}_{r_1}(x_1) \supset \hat{\mathcal{B}}_{r_2}(x_2) \supset \ldots,$$

with $r_n \to 0$. Taking $y_n \in \hat{\mathcal{B}}_{r_n}(x_n)$ we obtain a Cauchy sequence. Since the space is complete, the sequence converges, $y_n \to y$. Fix n; for every $k \ge n$, $y_k \in \hat{\mathcal{B}}_{r_n}(x_n)$; since the balls are closed, $y \in \hat{\mathcal{B}}_{r_n}(x_n)$ for every n.

Conversely, suppose that the non-empty intersection property holds. Let x_n be a Cauchy sequence. For every $n \in \mathbb{N}$ let

$$\varepsilon_n = \sup_{m \geq n} d(x_m, x_n).$$

Since the sequence x_n is a Cauchy sequence, it follows that $\varepsilon_n \to 0$. Then, we can take a subsequence n_k of natural number, such that

$$\sum_{k=1}^{\infty} \varepsilon_{n_k} < \infty.$$

Define

$$r_k = \sum_{j\geq k} \varepsilon_{n_j}$$
.

Then, for every k,

$$\hat{\mathcal{B}}_{r_{k+1}}(x_{n_{k+1}}) \subset \hat{\mathcal{B}}_{r_k}(x_{n_k}).$$

This is because $r_k = r_{k+1} + \varepsilon_{n_k}$ and $d(x_{n_{k+1}}, x_{n_k}) \le \varepsilon_{n_k}$. By assumption, there exists an

$$x \in \bigcap_{n=1}^{\infty} \hat{\mathcal{B}}_{r_k}(x_{n_k}).$$

Since $d(x_{n_k}, x) \le \varepsilon_{n_k} \to 0$, x is a partial limit of the sequence x_n . We have already seen on several occasions that if a Cauchy sequence has a partial limit, then it converges.

Theorem 1.113 (Baire) Let X be a complete metric space. If $U_n \subset X$ is a sequence of open and dense sets, then

$$Y = \bigcap_{n=1}^{\infty} U_n$$

is dense.

Proof: Let $\mathcal{B}_r(x)$ be some open ball. We need to show that it intersects Y. The idea is to construct a decreasing sequence of closed balls. Since U_1 is dense, it intersects $\mathcal{B}_r(x)$ and the intersection is open. Hence, there exists a close ball,

$$\hat{\mathcal{B}}_{r_1}(x_1) \subset U_1 \cap \mathcal{B}_r(x).$$

Similarly, since U_2 is dense, it intersects $\mathcal{B}_{r_1}(x_1)$ and the intersection is open. Hence there exists a closed ball,

$$\hat{\mathcal{B}}_{r_2}(x_2) \subset B_{r_1}(x_1) \cap U_2 \subset U_2 \cap U_1 \cap \mathcal{B}_r(x).$$

We proceed inductively, constructing

$$\hat{\mathcal{B}}_{r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \cap U_n \subset U_n \cap \cdots \cap U_1 \cap \mathcal{B}_r(x).$$

Moreover, we may always let $r_n \to 0$. By the lemma, there exists a point in the intersection of all those closed balls, which belongs to $Y \cap \mathcal{B}_r(x)$.

It should be noted that a countable intersection of open sets is not necessarily open. We have the following definition:

Definition 1.114 Let (X,d) be a metric space. A set $Y \subset X$ is called a G_{δ} -set if it can be represented as a countable intersection of open sets.

Comment:

1. Every open set U is of class G_{δ} as it is its own countable intersection.

2. Every closed set *C* is of class G_{δ} as

$$C = \bigcap_{n=1}^{\infty} U_n, \qquad U_n = \bigcup_{x \in C} \mathcal{B}_{1/n}(x).$$

(Explain!)

3. There are G_{δ} -sets that are neither open nor closed, for example

$$\{-1\} \cup (0,1) \subset \mathbb{R}$$
.

4. The requirement that the intersection be countable is crucial; every set can be represented as an intersection of open sets,

$$A = \bigcap_{z \notin A} \{z\}^c,$$

so that this classification would be of no interest.

5. It is not yet clear that there exist sets that are not of class G_{δ} .

Corollary 1.115 If (X,d) is a complete metric space, then the class of dense G_{δ} -sets is closed under countable intersections.

Proof: Let Y_n be dense G_{δ} -sets. Then, for each n there exist open $U_{n,k}$, such that

$$Y_n = \bigcap_{k=1}^{\infty} U_{n,k}.$$

Clearly, the $U_{n,k}$ must be dense. By Baire's theorem,

$$\bigcap_{n=1}^{\infty} Y_n = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_{n,k},$$

is dense, and by definition it is also of class G_{δ} .

Thus, the dense G_{δ} -sets can be viewed as large sets, since even when intersected countably many times with sets of that class, they remain dense.

Corollary 1.116 \mathbb{Q} is not of class G_{δ} in \mathbb{R} .

Proof: Since \mathbb{Q} is countable, we can write its elements as a sequence (q_n) . Then,

$$\mathbb{Q}^c = \bigcap_{n=1}^{\infty} \{q_n\}^c,$$

is both dense and of class G_{δ} . \mathbb{Q} is dense. If it were also of class G_{δ} its intersection with \mathbb{Q}^c must have been dense, but it is actually empty.

We are finally in measure to characterize "large sets".

Definition 1.117 Let (\mathbb{X},d) be a complete metric space. A set $Y \subset \mathbb{X}$ is called residual (שמנה) if it contains a dense G_{δ} -set.

Corollary 1.118 Let (X,d) be a complete metric space. Let R be the collection of all residual sets. Then, R satisfies the following properties:

- 1. $\mathbb{X} \in \mathcal{R}$ and $\emptyset \notin \mathcal{R}$.
- 2. If $A \in \mathcal{R}$ and $A \subset B$ then $B \in \mathcal{R}$.
- 3. R is closed under countable intersections.

The notion of "large sets" has a dual notion of "small sets", which we now review briefly:

Definition 1.119 Let (X,d) be a metric space.

- 1. A set $Y \subset \mathbb{X}$ is called **nowhere dense** (דלילה) if its closure has an empty interior (i.e., its closure does not contain open balls).
- 2. A set $Y \subset \mathbb{X}$ is called of class F_{ε} if it is a countable union of closed sets.
- 3. A set Y ⊂ X is called of the **first category** if it is a countable union of nowhere dense sets. A set which is not of the first category is called of the **second category**.

Proposition 1.120 Y is nowhere dense if and only it its complement contains an open dense set.

Proof:

$$Y$$
 is nowhere dense \iff $(\overline{Y})^{\circ} = \varnothing$
 \iff $((\overline{Y})^{\circ})^{c} = \mathbb{X}$
 \iff $(\overline{Y})^{c} = \mathbb{X}$
 \iff $(Y^{c})^{\circ} = \mathbb{X}$
 \iff $(Y^{c})^{\circ}$ is dense
 \iff Y^{c} contains an open dense set.

Proposition 1.121 A is an F_{ε} -set if and only if A^c is a G_{δ} -set.

Proof: Immediate from duality.

Proposition 1.122 Y is of the first category if and only if Y^c is residual.

Proof: Let *Y* be of first category. By definition,

$$Y=\bigcup_{n=1}^{\infty}A_n,$$

where A_n are nowhere dense. Hence,

$$Y^c = \bigcap_{n=1}^{\infty} A_n^c,$$

where A_n^c contain an open dense set, i.e., Y^c is residual.

Conversely, let Y^c be residual, i.e., Y^c contains a dense G_{δ} -set,

$$Y^c \supset \bigcap_{n=1}^{\infty} B_n$$
,

where the B_n are open and dense. It follows that

$$Y\subset\bigcup_{n=1}^{\infty}B_{n}^{c},$$

where the B_n^c are nowhere dense, hence

$$Y = \bigcup_{n=1}^{\infty} (B_n^c \cap Y)$$

is of the desired form.

Proposition 1.123 Let (X,d) be a complete metric space and let M denote the collection of sets of the first category. Then,

- 1. $\emptyset \in \mathcal{M}$ and $\mathbb{X} \notin \mathcal{M}$.
- 2. If $A \in \mathcal{M}$ and $B \subset A$ then $A \in \mathcal{M}$.
- 3. M is closed under countable union.

1.5.6 Semicontinuity and the continuity of pointwise limits

Suppose that $f_n \in C([a,b])$ converges *pointwise* to a function f. As we know, the limit is not necessarily continuous. A question is "how bad" can the limit be? We will that the set of points in which f is continuous is residual in [a,b]. To prove it, we'll first study the notion of **semicontinuity** (רציפות למחצה), which is interesting in its own.

Definition 1.124 Let (X,d) be a metric space. A function $f: X \to \mathbb{R}$ is called **lower-semicontinuous** if

$$f^{-1}((t,\infty))$$
 is open

for all $t \in \mathbb{R}$.

Proposition 1.125 Lower-semicontinuity is equivalent to either of the following two characterizations:

1. for every $a \in \mathbb{X}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) > f(a) - \varepsilon \qquad \forall x : d(x, a) < \delta.$$

2. For every sequence $x_n \to a$,

$$f(a) \leq \liminf_{n \to \infty} f(x_n).$$

Proof: (i) Suppose that for every *t*,

$$\{x \in \mathbb{X} : f(x) > t\}$$
 is open.

Let $a \in \mathbb{X}$. For every $\varepsilon > 0$,

$$f(a) > f(a) - \varepsilon$$
.

Hence,

$$a \in \{x \in \mathbb{X} : f(x) > f(a) - \varepsilon\}.$$

Since the latter is open, there exists a open ball $\mathcal{B}_{\delta}(a)$ such that for every $x \in \mathcal{B}_{\delta}(a)$,

$$f(x) > f(a) - \varepsilon$$
.

(ii) Suppose that the second condition holds and let $x_n \to a$. For every $\varepsilon > 0$ and n large enough,

$$f(x_n) > f(a) - \varepsilon$$
.

Letting $n \to \infty$,

$$\liminf_{n\to\infty} f(x_n) \ge f(a) - \varepsilon.$$

Since this holds for every $\varepsilon > 0$ we obtain the desired result.

(iii) Suppose that the sequential condition holds and consider the set

$$A = \{x : f(x) > t\}.$$

Suppose that this set was not open. It would imply the existence of $a \in A$ such that for every n there exists an $x_n \in \mathcal{B}_{1/n}(a)$ for which

$$f(x_n) \leq t$$
.

Then, $x_n \to a$ and

$$\liminf_{n\to\infty} f(x_n) \le t < f(a),$$

which is a contradiction.

Example: For $\mathbb{X} = \mathbb{R}$ and a < b,

$$f(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

is lower-semicontinuous, as easily verified by the sequential characterization, or by noting that

$$f^{-1}((0,t)) = \begin{cases} \emptyset & 1 < t \\ (a,b) & 0 < t \le 1 \\ \mathbb{R} & t \le 0. \end{cases}$$

A A A

Theorem 1.126 Let (\mathbb{X},d) be a metric space and let $f:\mathbb{X}\to\mathbb{R}$ be lower-semicontinuous. Then, the set of points in which f is continuous is a dense G_{δ} -set (hence residual).

Proof: As a first step, suppose that f is bounded from above. Define

$$C_n = \{a \in \mathbb{X} : \exists r > 0, |f(x) - f(y)| < 1/n \qquad \forall x, y \in \mathcal{B}_r(a)\}.$$

 C_n is an open set for all n, because suppose that $a \in C_n$ and that for every k there exists $x_k \in \mathcal{B}_{1/k}(a)$ which is not in C_n . This would imply that for every k there exists sequences $y_\ell^k, z_\ell^k \in \mathcal{B}_{1/\ell}(x_k)$ such that

$$|f(y_{\ell}^k) - f(z_{\ell}^k)| \ge 1/n.$$

Now, $x_k \to a$ and $y_\ell^k, z_\ell^k \to x_k$. Consider the "diagonal" sequences y_k^k, z_k^k ; both converges to a and

$$|f(y_k^k)-f(z_k^k)|\geq 1/n,$$

contradicting the fact that $a \in C_n$.

Then,

$$C \equiv \bigcap_{n=1}^{\infty} C_n = \{ a \in \mathbb{X} : \forall n \ \exists r > 0, \ |f(x) - f(y)| < 1/n \qquad \forall x, y \in \mathcal{B}_r(a) \}$$
$$\subset \{ a \in \mathbb{X} : \forall n \ \exists r > 0, \ |f(x) - f(a)| < 1/n \qquad \forall x \in \mathcal{B}_r(a) \},$$

is the set of continuity points of f. Since C_n is open for every n, it suffices to show that C_n is dense.

Let $\mathcal{B}_r(a)$ be an arbitrary open ball; we need to show that it intersects C_n . Let

$$M = \sup_{x \in \mathcal{B}_r(a)} f(x),$$

which is finite by assumption. Since f is lower-semicontinuous, the set

$$f^{-1}((M-1/n,\infty)) = \{x \in \mathbb{X} : f(x) > M-1/n\}$$

is open. Moreover, by the definition of M as the supremum of f in $\mathcal{B}_a(r)$,

$$\mathcal{B}_r(a) \cap f^{-1}((M-1/n,\infty)) \neq \emptyset$$
,

and the intersection is open. Hence, there exists an open ball

$$\mathcal{B}_{\rho}(b) \subset \mathcal{B}_{r}(a) \cap f^{-1}((M-1/n,\infty)).$$

For every $x, y \in \mathcal{B}_{\rho}(b)$,

$$M-1/n < f(x), f(y) \le M,$$

hence $b \in C_n$, proving that C_n is dense.

Remains the case where f is not bounded. For this, we note that that (i) if f is lower-semicontinuous and g is continuous, the $g \circ f$ is lower-semicontinuous, and (ii) if f is any function and g is continuous, then every continuity point of f is a continuity point of $g \circ f$. Since $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ is continuous,

$$F(x) = \tan^{-1}(f(x))$$

is bounded, lower-semicontinuous, and its continuity points coincides with those of f. Since the continuity points of F are residual, so are the continuity points of f.

We now turn to consider pointwise limits of continuous functions. First a lemma:

Lemma 1.127 Let (X,d) be a metric space. Let $f_n \in C(X)$ be a sequence of continuous functions. Suppose that

$$f(x) = \sup_{n} f_n(x)$$

is finite for every x. Then, f is lower-semicontinuous.

Proof: For every $t \in \mathbb{R}$,

$$\{x : f(x) > t\} = \{x : \sup_{n} f_{n}(x) > t\}$$
$$= \{x : \exists n, f_{n}(x) > t\}$$
$$= \bigcup_{n=1}^{\infty} \{x : f_{n}(x) > t\},$$

and since f_n are continuous, the latter is a union of open sets.

Comment: The lemma would still hold if f_n were only lower-semicontinuous and the collection of functions was non-countable.

Comment: There exists a dual notion of **upper-semicontinuity**, which requires $f^{-1}((-\infty,t))$ to be open for every $t \in \mathbb{R}$. It can be shown similarly to above that

Proposition 1.128 Upper-semicontinuity is equivalent to either of the following two characterizations:

1. for every $a \in \mathbb{X}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) < f(a) + \varepsilon$$
 $\forall x : d(x,a) < \delta$.

2. For every sequence $x_n \to a$,

$$f(a) \ge \limsup_{n \to \infty} f(x_n).$$

Theorem 1.129 Let (X, d) be a complete metric space and let $f_n \in C(X)$ converge pointwise to f. Then, the continuity points of f form a residual set.

Proof: For every *n* define

$$F_n(x) = \sup_{k \ge n} f_k(x).$$

By the previous lemma, the F_n 's are lower-semicontinuous. Moreover, F_n is a decreasing sequence, converging pointwise to f (the lim sup equals the limit).

Let

$$E_n = \{\text{continuity points of } F_n\}.$$

This is a dense G_{δ} -set, hence

$$E = \bigcap_{n=1}^{\infty} E_n$$

is also a dense G_{δ} -set.

Let $x \in E$ and let $\varepsilon > 0$. Since $F_n(x) \to f(x)$, there exists an n such that

$$F_n(x) < f(x) + \frac{\varepsilon}{2}.$$

Since $x \in E_n$ is a continuity point of F_n , there exists an r such that

$$F_n(y) < F_n(x) + \frac{\varepsilon}{2}$$
 $\forall y \in \mathcal{B}_r(x)$.

Hence, for every $y \in \mathcal{B}_r(x)$,

$$f(y) \le F_n(y) < F_n(x) + \frac{\varepsilon}{2} < f(x) + \varepsilon.$$

In other words, there exists a dense G_{δ} -set E, such that

$$\forall x \in E, \quad \forall \varepsilon > 0 \quad \exists r > 0, \qquad f(y) < f(x) + \varepsilon \quad \forall y \in \mathcal{B}_r(x).$$

Now, we can proceed symmetrically, replacing lower-semicontinuity with uppersemicontinuity. We infer the existence of a dense G_{δ} -set F, such that

$$\forall x \in F$$
, $\forall \varepsilon > 0$ $\exists r > 0$, $f(y) > f(x) - \varepsilon$ $\forall y \in \mathcal{B}_r(x)$.

The intersection $E \cap F$ is a dense G_{δ} -set, containing all the continuity points of f.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable. It derivative f' is not necessarily continuous, however, since

$$f'(x) = \lim_{n \to \infty} \frac{f(x+1/n) - f(x)}{1/n}$$

is a pointwise limit of continuous functions, it follows that the continuity points of f' are a residual set. \blacktriangle \blacktriangle