# Chapter 0

# Review of set theory

Set theory plays a central role in the theory of probability. Thus, we will open this course with a quick review of those notions of set theory which will be used repeatedly.

## 0.1 Sets

Throughout this chapter, let $X$ be a set, which we will call the *whole space*. The elements of $X$ will often be called *points*. If $A$ is a subset of $X$ and $x$ is a point in $X$, then the notation

$$x \in A$$

means that $x$ belongs to $A$, i.e., it is one of the points in $A$. If $x$ does not belong to $A$, we write

$$x \notin A.$$

Whenever the identity of the whole space $X$ is clear, we will refer to subsets of $X$ simples as *sets*.

If $A$ and $B$ are sets, the notations

$$A \subset B \qquad \text{or} \qquad B \supset A$$

mean that $A$ is a subset of $B$, i.e., every point in $A$ is also a point in $B$. Two sets are called equal if they contain exactly the same set of points, i.e.,

$$A \subset B \qquad \text{and} \qquad B \subset A.$$

When considering the collection of all subsets of $X$, we always include the empty set, which is denoted by $\varnothing$. By definition,

$$\forall x \in X, \quad x \notin \varnothing.$$

For every set $A$, it always holds that

$$\varnothing \subset A \subset X.$$

We will often consider sets of sets. To avoid confusion, we will use the term a *collection* (אוסף) of sets, rather then a set of sets; however remember, a collection is nothing but a set. Thus, is $\mathcal{C}$ is a collection of sets and $A$ is a set, then $A \in \mathcal{C}$ means that the set $A$ is one of the elements of the collection of sets $\mathcal{C}$. The notation $A \subset \mathcal{C}$, on the other hand, is meaningless.

*Example*: Let $X$ be a whole space. The collection of all its subsets is denoted by $2^X$; it is called the *power set* of $X$ (קבוצת החזקה). This notation is suggestive, as if $X$ has finite cardinality $|X|$, then

$$|2^X| = 2^{|X|}.$$

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*Definition 0.1 A sequence $(A_n)$ of sets is called* increasing *(עולה) if*

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

*It is called* decreasing *(יורדת) if*

$$A_1 \supset A_2 \supset A_3 \supset \cdots.$$

*In either case it is called monotone.*

To denote sets, we will use curly brackets. For example, if $x, y \in X$, then

$$\{x, y\}$$

denotes the set whose only elements are $x$ and $y$. If $(x_k)_{k=1}^n$ is a finite sequence of distinct points in $X$, we will denote the set whose only elements are the elements of that sequence by

$$\{x_k \,:\, k = 1, \dots, n\}.$$

If $(x_k)$ is an infinite sequence of distinct points in $X$, we will denote the set whose only elements are the elements of that sequence by

$$\{x_k : k \in \mathbb{N}\}.$$

Suppose that for every $x \in X$, $\pi(x)$ is a proposition assuming values either True or False. Then,

$$\{x \in X : \pi(x)\}$$

denotes the set of all points in $X$ for which the proposition $\pi(x)$ is True. More generally, if $\pi_1(x), \pi_2(x), \dots$ is a sequence of propositions regarding the point $x$, then

$$\{x \in X : \pi_1(x), \pi_2(x), \dots\}$$

denotes the set of all points in $X$ for which all the propositions $\pi_k(x)$ are True.

*Example*: Let $X = \mathbb{N}$ and for $n \in \mathbb{N}$ let

$$\pi(n) = \begin{cases} \text{True} & n \text{ is even} \\ \text{False} & n \text{ is odd.} \end{cases}$$

Then,

$$\{n \in \mathbb{N} : \pi(n)\}$$

denotes the set of even integers. In many instances, we will use the more intuitive notation

$$\{n \in \mathbb{N} : n \text{ is even}\}.$$

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As a tautological, yet useful remark: every set $A$ can be expressed using the curly bracket notation as

$$A = \{x \in X : x \in A\}.$$

Finally, for every $x \in X$, $\{x\}$ is a subset of $X$, i.e., an element of $2^X$; such a subset is called a *singleton* (יחידון). Be careful to distinguish $x$ from $\{x\}$.

## 0.2 Unions and intersections

Let $\mathcal{C}$ be a collection of subsets of $X$. The subset of $X$ including all those points that belong to *at least one* set of the collection $\mathcal{C}$ is called the *union* (איחוד) of the collection, and it is denoted by

$$\bigcup \mathcal{C} \qquad \text{or} \qquad \bigcup \{A \,:\, A \in \mathcal{C}\}.$$

In the case where the collection $\mathcal{C}$ includes only two sets, e.g., $\mathcal{C} = \{A, B\}$, we simply write

$$\bigcup \mathcal{C} = A \cup B.$$

If

$$\mathcal{C} = \{A_i \,:\, i = 1, \dots, n\}$$

is a finite collection of sets, we write

$$\bigcup \mathcal{C} = \bigcup_{i=1}^{n} A_i.$$

If

$$\mathcal{C} = \{A_i \,:\, i \in \mathbb{N}\}$$

is a countable collection of sets, we write

$$\bigcup \mathcal{C} = \bigcup_{i=1}^{\infty} A_i.$$

Finally, if

$$\mathcal{C} = \{A_\gamma \,:\, \gamma \in \Gamma\}$$

is a non-countable collection of sets, where $\Gamma$ is an index set, we write

$$\bigcup \mathcal{C} = \bigcup_{\gamma \in \Gamma} A_\gamma.$$

By convention if $\mathcal{C}$ is an empty collection of sets, then

$$\bigcup \mathcal{C} = \varnothing.$$

It is easy to see that whenever $A \subset B$, then

$$A \cup B = B.$$

In particular, it always holds that

$$\varnothing \cup A = A \qquad \text{and} \qquad A \cup X = X.$$

Moreover, if $(A_n)$ is an increasing sequence of sets, then

$$\bigcup_{k=1}^{n} A_k = A_n.$$

Let $\mathcal{C}$ be a collection of subsets of $X$. The subset of $X$ including all those points that belong to *all* sets of the collection $\mathcal{C}$ is called the *intersection* (חיתוך) of the collection, and it is denoted by

$$\bigcap \mathcal{C} \qquad \text{or} \qquad \bigcap \{A \,:\, A \in \mathcal{C}\}.$$

In the case where the collection $\mathcal{C}$ includes only two sets, e.g., $\mathcal{C} = \{A, B\}$, we simply write

$$\bigcap \mathcal{C} = A \cap B.$$

If

$$\mathcal{C} = \{A_i \,:\, i = 1, \dots, n\}$$

is a finite collection of sets, we write

$$\bigcap \mathcal{C} = \bigcap_{i=1}^{n} A_i.$$

If

$$\mathcal{C} = \{A_i \,:\, i \in \mathbb{N}\}$$

is a countable collection of sets, we write

$$\bigcap \mathcal{C} = \bigcap_{i=1}^{\infty} A_i.$$

Finally, if

$$\mathcal{C} = \{A_\gamma \,:\, \gamma \in \Gamma\}$$

is a non-countable collection of sets, where $\Gamma$ is an index set, we write

$$\bigcap \mathcal{C} = \bigcap_{\gamma \in \Gamma} A_\gamma.$$

By convention if $\mathcal{C}$ is an empty collection of sets then

$$\bigcap \mathcal{C} = X.$$

It is easy to see that whenever $A \subset B$, then

$$A \cap B = A.$$

In particular, it always holds that

$$\emptyset \cap A = \emptyset \qquad \text{and} \qquad A \cap X = A.$$

Moreover, if $(A_n)$ is a decreasing sequence of sets, then

$$\bigcap_{k=1}^{n} A_k = A_n.$$

*Definition 0.2 Two sets A and B are called* disjoint *(זרים) if they have no points in common, i.e.,* $A \cap B = \emptyset$*. A* disjoint collection *of sets is a collection of sets such that every two distinct sets in this collection are disjoint.*

Since unions of disjoint collections of sets play an important role in the theory of probability, we will denote the union of a disjoint collection by $\uplus$. For example,

$$A \uplus B$$

denotes the union of the disjoint sets $A$ and $B$.

Unions and intersections satisfy the following important properties:

1. Both are commutative and associative.
2. Intersections are distributive over unions,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

3. Unions are distributive over intersections,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

## 0.3 Complements

Let $A$ be a set. It *complement* (משלים) is the set of points which do not belong to $A$,

$$A^c = \{x \,:\, x \notin A\}.$$

Complementation satisfies the following properties:

1. For all sets $A$,
$$A \cup A^c = X \qquad \text{and} \qquad A \cap A^c = \varnothing.$$
2. For all sets $A$, $(A^c)^c = A$.
3. $\varnothing^c = X$ and $X^c = \varnothing$.
4. If $A \subset B$ then $A^c \supset B^c$.
5. For every (possibly non-countable) collection $\mathcal{C}$ of sets,
$$\left(\bigcup\{A \,:\, A \in \mathcal{C}\}\right)^c = \bigcap\{A^c \,:\, A \in \mathcal{C}\},$$
and
$$\left(\bigcap\{A \,:\, A \in \mathcal{C}\}\right)^c = \bigcup\{A^c \,:\, A \in \mathcal{C}\}.$$

Let $A$ and $B$ be sets. The difference between $A$ and $B$ is the set of all those points that are in $A$ but not in $B$, namely,

$$A \setminus B = \{x \,:\, x \in A \quad \text{and} \quad x \notin B\} = A \cap B^c.$$

## 0.4 Limits

Let $(A_n)$ be a sequence of sets. The *superior limit* (גבול עליון) of this sequence is defined as the set of points that belong to infinitely many of those sets:

$$\limsup_n A_n = \{x \,:\, \text{for all } k \text{ there exists an } n \ge k \text{ such that } x \in A_n\}.$$

If $x \in \limsup_n A_n$ we say that $x \in A_n$ *infinitely often* (באופן נשנה).
The *inferior limit* (גבול תחתון) of this sequence is defined as the set of points that belong to all but finitely many of those sets:

$$\liminf_n A_n = \{x \,:\, \text{there exists } k \text{ such that } x \in A_n \text{ for all } n \ge k\}.$$

If $x \in \limsup_n A_n$ we say that $x \in A_n$ *eventually* (בסופו של דבר תמיד). Note that both superior and inferior limits always exist. Also, it always holds that

$$\liminf_n A_n \subset \limsup_n A_n.$$

In the event that the superior limit and the inferior limit of a sequence of sets coincide, we call this set the *limit* of the sequence, namely,

$$\lim_n A_n = \limsup_n A_n = \liminf_n A_n.$$

By definition, for $k \in \mathbb{N}$,

$$\bigcup_{n=k}^{\infty} A_n = \{x \,:\, \text{there exists an } n \geq k \text{ such that } x \in A_n\},$$

hence,

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \limsup_n A_n.$$

Likewise, for $k \in \mathbb{N}$,

$$\bigcap_{n=k}^{\infty} A_n = \{x \,:\, x \in A_n \text{ for all } n \geq k\},$$

hence,

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \liminf_n A_n.$$

*Proposition 0.1 If $(A_n)$ is a monotone sequence of sets, then it has a limit. Specifically, if $(A_n)$ is increasing then*

$$\lim_n A_n = \bigcup_{n=1}^{\infty} A_n.$$

*and if $(A_n)$ is decreasing then*

$$\lim_n A_n = \bigcap_{n=1}^{\infty} A_n.$$

*Proof*: Let $(A_n)$ be increasing. Then for every $k \in \mathbb{N}$,

$$\bigcup_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcap_{n=k}^{\infty} A_k.$$

Thus,

$$\limsup_n A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n,$$

and

$$\liminf_n A_n = \bigcup_{k=1}^{\infty} A_n.$$

Let $(A_n)$ be decreasing. Then for every $k \in \mathbb{N}$,

$$\bigcap_{n=k}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcup_{n=k}^{\infty} A_k.$$

Thus,

$$\limsup_n A_n = \bigcap_{k=1}^{\infty} A_k,$$

and

$$\liminf_n A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

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*Example*: Let $X$ be the set of integers, $\mathbb{N}$, and let

$$A_n = \begin{cases} \{2,4,6,\dots\} & n \text{ is even} \\ \{1,3,5,\dots\} & n \text{ is odd.} \end{cases}$$

Then,

$$\limsup_{n\to\infty} A_n = X \qquad \text{and} \qquad \liminf_{n\to\infty} A_n = \emptyset,$$

i.e., the sequence $(A_n)$ does not have a limit. ▲▲▲

*Example*: Let again $X = \mathbb{N}$ and let

$$A_k = \left\{k^j : \ j = 0,1,2,\dots\right\},$$

namely,

$$A_1 = \{1\} \qquad A_2 = \{1,2,4,\dots\} \qquad A_2 = \{1,3,9,\dots\},$$

etc. Then,

$$\lim_k A_k = \{1\}.$$

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## 0.5 Algebras and $\sigma$-algebras of sets

*Definition 0.3 A collection $\mathcal{C}$ of sets is called an* algebra of sets *(אלגברה של קבוצות) if it satifsies the following properties:*

1. *If $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$.*
2. *If $A, B \in \mathcal{C}$ the $A \cup B \in \mathcal{C}$.*
3. *$X \in \mathcal{C}$.*

*Proposition 0.2 Let $\mathcal{C}$ be an algebra of sets. Then,*

1. *$\varnothing \in \mathcal{C}$.*
2. *If $A_1, \dots, A_n \in \mathcal{C}$ then $\cup_{i=1}^n A_i \in \mathcal{C}$.*
3. *If $A_1, \dots, A_n \in \mathcal{C}$ then $\cap_{i=1}^n A_i \in \mathcal{C}$.*
4. *Let $A, B \in \mathcal{C}$. Then their difference $A \setminus B$ is in $\mathcal{C}$.*

*Proof*: Since $\mathcal{C}$ is closed under complementation and $X \in \mathcal{C}$,

$$\varnothing = X^c \in \mathcal{C}.$$

The second assertion follows by induction. The third assertion follows from the second assertion and de Morgan's Law,

$$\bigcap_{i=1}^n A_i = \left( \bigcup_{i=1}^n A_i^c \right)^c \in \mathcal{C}.$$

The fourth assertion holds because

$$A \setminus B = A \cap B^c.$$

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Thus, an algebra of sets is a collection of sets closed under finitely many set-theoretic operations of union, intersection and complementation.

*Example*: Let $A$ be a set. Then, the collection of sets

$$\mathcal{C} = \{\varnothing, A, A^c, X\}$$

is an algebra of sets. ▲ ▲ ▲

*Example*: The collection of all subsets $2^X$ is an algebra of sets. ▲ ▲ ▲

*Example*: Suppose that $X$ is an infinite set. The collection of all finite subsets of $X$ is not an algebra of sets. Neither is the collection of all infinite subsets of $X$. ▲ ▲ ▲

*Definition 0.4 A collection $\mathcal{C}$ of sets is called a* $\sigma$-algebra of sets *if it is an algebra of sets, and in addition, if $(A_n)$ is a sequence of sets in $\mathcal{C}$, then*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}.$$

*Proposition 0.3 Let $\mathcal{C}$ be a $\sigma$ algebra of sets. If $(A_n)$ is a sequence of sets in $\mathcal{C}$, then*

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{C}.$$

*Proof*: This is an immediate consequence of the identity

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c .$$

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That is, a $\sigma$-algebra of sets is closed with respect to *countably many* set-theoretic operations.

*Example*: The collection of all subsets $2^X$ is a $\sigma$-algebra of sets. ▲ ▲ ▲

*Proposition 0.4* *Let $\mathcal{C}_\gamma$, $\gamma \in \Gamma$ be a (possible non-countable) family of $\sigma$-algebras of sets. Then, their intersection,*

$$\mathscr{F} = \bigcap\{\mathcal{C}_\gamma \ : \ \gamma \in \Gamma\}$$

*is also a $\sigma$-algebra of sets.*

*Proof*: Let $(A_n)$ be a sequence of sets in $\mathscr{F}$. By definition, $A_n \in \mathcal{C}_\gamma$ for every $\gamma \in \Gamma$. Since $\mathcal{C}_\gamma$ is a $\sigma$-algebra, then

$$X \in \mathcal{C}_\gamma, \qquad A_n^c \in \mathcal{C}_\gamma \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}_\gamma.$$

This this holds for every $\gamma \in \Gamma$, then

$$X \in \mathscr{F}, \qquad A_n^c \in \mathscr{F} \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}.$$

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*Definition 0.5* *Let $\mathcal{C}$ be a non-empty collection of sets. The $\sigma$-algebra generated by $\mathcal{C}$ is the intersection of all those $\sigma$-algebras containing $\mathcal{C}$.*

$$\sigma(\mathcal{C}) = \bigcup\{\mathcal{A} \ : \ \mathcal{A} \textit{ is a } \sigma\textit{-algebra and } \mathcal{C} \subset \mathcal{A}\}.$$

*Since $2^X$ is a $\sigma$-algebra containing $\mathcal{C}$, this intersection is not empty. By Proposition 0.4, $\sigma(\mathcal{C})$ is a $\sigma$-algebra.*

*Example*: Let $A$ be a set. Then,

$$\sigma(\{A\}) = \{\varnothing, A, A^c, X\}.$$

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✎ *Exercise 0.1* Let $A, B$ be sets. Find $\sigma(\{A, B\})$.

## 0.6 Inverse functions

In calculus, you learned that a function $f : X \to Y$ is invertible only if it is both one-to-one and onto; in this case, we can define

$$f^{-1} : Y \to X.$$

An inverse function, however, can always be defined as a mapping between sets. For every function $f : X \to Y$ we can define

$$f^{-1} : 2^Y \to 2^X$$

by

$$f^{-1}(A) = \{x \in X \,:\, f(x) \in A\}, \qquad A \subset Y.$$

An important property of the inverse function $f^{-1}$ is that it preserves (commutes with) set-theoretic operations:

*Proposition 0.5* *Let $f : X \to Y$. Then,*

1. *For every $A \subset Y$*
$$(f^{-1}(A))^c = f^{-1}(A^c).$$
2. *If $A, B \subset Y$ are disjoint so are $f^{-1}(A), f^{-1}(B) \subset X$.*
3. *$f^{-1}(Y) = X$.*
4. *If $A_n \subset Y$ is a sequence of subsets, then*
$$f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(A_n).$$

*Proof*: Just follow the definitions. For example,

$$x \in f^{-1}(A) \qquad \text{iff} \qquad f(x) \in A,$$

hence

$$x \in (f^{-1}(A))^c \quad \text{iff} \quad f(x) \notin A \quad \text{iff} \quad f(x) \in A^c \quad \text{iff} \quad x \in f^{-1}(A^c).$$

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The fact that inverse functions commute with set-theoretic operations has the following implication. Let $X$ and $Y$ be sets and let $f : X \to Y$. Let $\mathscr{F} \subset 2^Y$ be a $\sigma$-algebra of subsets of $Y$, then

$$\{f^{-1}(A) \,:\, A \in \mathscr{F}\}$$

is a $\sigma$-algebra of subsets of $X$.