2. Banach spaces

2.1 Definitions and examples

We start by defining what a Banach space is:

Definition 2.1 A **Banach space** is a complete, normed, vector space.

Comment 2.1 Completeness is a metric space concept. In a normed space the metric is

$$d(x,y) = \|x-y\|.$$

Note that this metric satisfies the following "special" properties:

① The underlying space is a vector space.
② **Homogeneity**: $d(\alpha x, \alpha y) = |\alpha| d(x,y)$.
③ **Translation invariance**: $d(x+z, y+z) = d(x,y)$.

Conversely, every metric satisfying these three conditions defines a norm:

$$\|x\| = d(x,0).$$

Exercise 2.1 Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. A series $\sum_{n=1}^{\infty} x_n$ is **absolutely convergent** if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Prove that a normed space is a Banach space (i.e., complete) if and only if every absolutely convergent series is convergent. ▪

Definition 2.2 An injection $f : \mathscr{X} \hookrightarrow \mathscr{Y}$ (i.e., one-to-one) between two normed spaces $\mathscr{X}$ and $\mathscr{Y}$ is called an **norm-preserving** if

$$(\forall x \in \mathscr{X}) \quad \|Tf(x)\| = \|x\|.$$

If the image of f is $\mathscr{Y}$ then the two spaces are called **isometric** and f is called an **isometry** (which is an isomorphism in the category of metric spaces).

Theorem 2.1 — Completion theorem. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. There exists a Banach space $(\mathscr{B}, \|\cdot\|)$ and a linear norm-preserving map, $T : \mathscr{X} \to \mathscr{B}$, whose image is dense in $\mathscr{B}$. Furthermore, $\mathscr{B}$ is unique up to an isometry.

Proof. The proof is practically identical to the proof for Hilbert spaces. Define $\mathscr{B}$ to be the space of all Cauchy sequences in $\mathscr{X}$, modulo the equivalence relation $x \sim y$ if $\lim(x_n - y_n) = 0$. Denote the equivalence class of a Cauchy sequence x by $[x]$.

We endow $\mathscr{B}$ with a norm:

$$\|\alpha\| = \lim_{n\to\infty} \|x_n\|,$$

where $[(x_n)] = \alpha$. This norm is well-defined because

$$|\|x_n\| - \|x_m\|| \le \|x_n - x_m\|.$$

If (x_n) is a Cauchy sequence so is $(\|x_n\|)$, and the latter converges by the completeness of $\mathbb{R}$. It is also easy to see that the norm of $\alpha \in \mathscr{B}$ is independent of the representing sequence $(x_n) \subset \mathscr{X}$.

The mapping $T : \mathscr{X} \to \mathscr{B}$ is defined by

$$Tx = [(x, x, \dots)].$$

It is obviously norm-preserving. The rest of the proof continues as for Hilbert spaces. ■

▪ **Example 2.1** Let $\mathscr{X} = \mathbb{C}^n$. For $p \ge 1$ we define:

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

It is easy to see that $\|\cdot\|$ satisfies the positivity and homogeneity conditions. To see that it satisfies also the triangle inequality we need to derive first a number of classical inequalities. ▪

Lemma 2.2 — Young's inequality. Let $p > 1$ and set $q = p/(p-1)$. For every $a, b \in \mathbb{C}$:

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}$$

Proof. Note that $1/p + 1/q = 1$. Since $(-\log)$ is a convex function then for every $\alpha, \beta > 0$:

$$-\log\left(\frac{\alpha}{p} + \frac{\beta}{q}\right) \le -\frac{1}{p}\log\alpha - \frac{1}{q}\log\beta = -\log(\alpha^{1/p}\beta^{1/q}).$$

It follows that:

$$\frac{\alpha}{p} + \frac{\beta}{q} \ge \alpha^{1/p}\beta^{1/q}.$$

Setting $\alpha = |a|^p$ and $\beta = |b|^q$ we recover the desired result. ∎

Proposition 2.3 — Hölder inequality. Let $p > 1$ and set $q = p/(p-1)$. For every $x, y \in \mathbb{C}^n$:

$$\sum_{i=1}^{n} |x_i||y_i| \le \|x\|_p \|y\|_q.$$

Proof. Using Young's inequality term-by-term:

$$\frac{\sum_{i=1}^n |x_i y_i|}{\|x\|_p \|y\|_q} = \sum_{i=1}^{n}\left(\frac{|x_i|}{\|x\|_p}\right)\left(\frac{|y_i|}{\|y\|_q}\right) \le \frac{1}{p}\sum_{i=1}^{n}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \frac{1}{q}\sum_{i=1}^{n}\left(\frac{|y_i|}{\|y\|_q}\right)^q = \frac{1}{p} + \frac{1}{q} = 1.$$

∎

Proposition 2.4 — Minkowski inequality. Let $p > 1$. For every $x, y \in \mathbb{C}^n$:

$$\|x + y\|_p \le \|x\|_p + \|y\|_p.$$

Proof. For every $i \in \{1, \ldots, n\}$, it follows from the triangle inequality that

$$|x_i + y_i|^p = |x_i + y_i|\,|x_i + y_i|^{p-1} \le |x_i|\,|x_i + y_i|^{p-1} + |y_i|\,|x_i + y_i|^{p-1}.$$

Summing over i, and using Hölder's inequality:

$$\|x + y\|_p^p \le (\|x\|_p + \|y\|_p)\left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{1/q},$$

where $q = p/(p-1)$. Noting that $q(p-1) = p$ and $1/q = (p-1)/p$:

$$\|x+y\|_p^p \le (\|x\|_p + \|y\|_p)\,\|x+y\|_p^{p-1},$$

which completes the proof. ■

Corollary 2.5 For every $p \ge 1$, $\|\cdot\|_p$ is a norm.

■ **Example 2.2** For $\mathscr{X} = \mathbb{C}^n$ we define

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

It is easy to see that this is indeed a norm. It is a classical exercise to show that

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

■

■ **Example 2.3** The above examples can be extended to space of infinite sequences, which are denoted by ℓ_p, $1 \le p \le \infty$. ■

The ℓ_p spaces are "generic normed" spaces. The cases of $p \ne \infty$ and $p = \infty$ are inherently different for the following reason:

Proposition 2.6 The spaces ℓ_p, $1 \le p < \infty$ are separable.

Proof. The subset of rational sequences that only have a finite number of non-zero entries is countable and dense. ■

Proposition 2.7 The space ℓ_∞ is not separable.

Proof. Consider the set A of elements in ℓ_∞ whose entries only assume the values 0 and 1. This set is uncountable (it is isomorphic to $[0,1]$), and the distance between every two elements in A is 1. Let B be a dense subset of ℓ_∞. Every $x \in A$ has a $\xi(x) \in B$ such that

$$\|x - \xi(x)\|_\infty < 1/4.$$

By the triangle inequality, $x_1 \ne x_2$ implies $\xi(x_1) \ne \xi(x_2)$, hence $\xi : A \to B$ is injective, which implies that B is uncountable. ■

■ **Example 2.4** Consider the linear subspace of $c \subset \ell_\infty$ that consists of all converging sequences. This subspace is closed, hence it is a Banach space. The subspace c_0 of c that consists of sequences that converge to zero is also a Banach space. ■

■ **Example 2.5** Let $B(S)$ be the set of all bounded functions on some set S. It is a vector space with respect to pointwise addition and scalar multiplication. It is made into a normed space with respect to

$$\|f\| = \sup_{x \in S} |f(x)|.$$

Convergence with respect to this norm is called **uniform convergence** (**התכנסות במידה שווה**). ■

Proposition 2.8 Let S be any set. Then, $B(S)$ is a Banach space. (Please note that this conclusion requires no structure on the set S.)

Proof. Let $f_n \in B(S)$ be a Cauchy sequence; we need to show that it converges uniformly. Fix $x \in S$. Then:

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|,$$

which implies that $(f_n(x))$ is a Cauchy sequence, hence converges. Denote the limit of $f_n(x)$ by $f(x)$; we will show that $f_n \to f$ in $B(S)$.

For every ε there exists an N such that for every $m, n > N$,

$$\|f_n - f_m\| < \varepsilon.$$

Thus, for every x,

$$|f_n(x) - f(x)| \le \varepsilon,$$

i.e., f_n converges uniformly to f. ■

The next example adds structure to the previous one:

■ **Example 2.6** Let K be a Hausdorff topological space, and let $C(K) \subset B(K)$ be the subspace of bounded continuous functions with the norm inherited from $B(K)$. Since and uniform limits of continuous functions are continuous, then $C(K)$ is a closed subspace of $B(K)$ and hence a Banach space. ■

As of now K can be any Haussdorf topological space. More structure on K affects the properties of the Banach space $C(K)$ as shown in the following theorem:

Theorem 2.9 Let K be a compact Hausdorff topological space. The Banach space $C(K)$ is separable if and only if K is metrizable (i.e., one can define on K a metric that induces the same topology).

Proof. Suppose first that K is metrizable with metric d. Compact spaces are separable, so let x_n be a dense sequence in K. For every m,n define $f_{n,m} \in C(K)$:

$$f_{n,m}(x) = \begin{cases} 1 & d(x,x_n) \le 1/m \\ 0 & d(x,x_n) > 2/m \end{cases},$$

with some continuous gluing between the two domains. This sequence of functions separates between points. By the Stone-Weierstraß theorem, polynomials of $f_{n,m}$ with rational coefficients form a countable dense set.

Conversely, suppose that $C(K)$ is separable. let f_n be a dense sequence in the unit ball of $C(K)$. Define the map $\varphi : K \to [-1,1]^{\aleph_0}$ (the **Hilbert cube**),

$$\varphi(x) = (f_1(x), f_2(x), \dots).$$

The norm on the Hilbert cube is

$$\|a\| = \left(\sum_{n=1}^{\infty} \frac{a_n^2}{n^2} \right)^{1/2}.$$

φ is continuous, and because the f_n separate between points it is one-to-one[1]. Since K is compact, φ is a homeomorphism, and since the Hilbert cube is metrizable, so is K. ■

Schauder basis

Every vector space has a **Hamel basis**, which is a purely algebraic construct. Every Hilbert space has an **orthonormal basis**, which builds upon the inner product. A normed space $(\mathscr{X}, (\cdot,\cdot))$ is said to have a **Schauder basis** (e_n) if for every $x \in \mathscr{X}$ there is a unique sequence of scalars (α_n) such that

$$x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

It can be shown that if a normed space has a Schauder basis then the space is separable. The harder question is whether a separable Banach space necessarily has a Schauder basis. This question was raised originally by Banach. For a long time all known examples of Banach spaces were found to have such a basis. In 1973, Enflo constructed a separable Banach space that does not have a Schauder basis.

[1] If they did not separate between two points they could not approximate uniformly a function taking distinct values at these two points.

Exercise 2.2 Show that if a normed space has a Schauder basis then it is separable. ■

Linear operators

Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. We have already proven that a linear transformation $T : \mathscr{X} \to \mathscr{Y}$ is continuous if and only if it is bounded (we proved it in Chapter 1, but the theorem was for general normed space). We denote the space of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$ by $B(\mathscr{X};\mathscr{Y})$. It is made into a vector space over $\mathbb{C}$ by pointwise operations:

$$(\alpha S + \beta T)(x) = \alpha S(x) + \beta T(x).$$

As usual we define:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

Note that $\|x\|$ is a norm in $\mathscr{X}$ and $\|Tx\|$ is a norm in $\mathscr{Y}$.

Proposition 2.10 $\|\cdot\|$ is a norm on $B(\mathscr{X};\mathscr{Y})$.

Proof. It is evident that $\|\cdot\|$ is homogeneous and positive. If $\|T\| = 0$ then $Tx = 0$ for all $x \in \mathscr{X}$, i.e., T is the zero transformation. It remains to prove the triangle inequality. It follows from the triangle inequality in $\mathscr{X}$ that

$$\|S+T\| = \sup_{\|x\|=1} \|(S+T)x\| \le \sup_{\|x\|=1} (\|Sx\| + \|Tx\|) \le \sup_{\|x\|=1} \|Sx\| + \sup_{\|x\|=1} \|Tx\| = \|S\| + \|T\|.$$

■

The following proposition shows that for $B(\mathscr{X};\mathscr{Y})$ to be complete it is sufficient that the target space $\mathscr{Y}$ be complete:

Proposition 2.11 Let $\mathscr{X}$ be a normed space and let $\mathscr{Y}$ be a Banach space. Then $B(\mathscr{X};\mathscr{Y})$ is a Banach space.

Proof. Let $T_n \in B(\mathscr{X};\mathscr{Y})$ be a Cauchy sequence.

Step 1: define the limit T: For every $x \in \mathscr{X}$,

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \le \|T_n - T_m\| \|x\|,$$

which implies that $(T_n x)$ is a Cauchy sequence in $\mathscr{Y}$, hence converges to a limit. We denote

$$Tx = \lim_{n\to\infty} T_n x.$$

Step 2: T is linear: T is a linear transformation from $\mathscr{X}$ to $\mathscr{Y}$, because

$$T(\alpha x+\beta y)=\lim_{n\to\infty}T_n(\alpha x+\beta y)=\lim_{n\to\infty}(\alpha T_n x+\beta T_n y)=\alpha Tx+\beta Ty.$$

Step 3: T is bounded: By the continuity of the norm, for every $x\in\mathscr{X}$

$$\|Tx\|=\lim_{n\to\infty}\|T_n x\|\le(\limsup\|T_n\|)\,\|x\|.$$

Since the Cauchy sequence T_n is bounded, it follows that $T\in B(\mathscr{X};\mathscr{Y})$.

Step 4: T is the limit of T_n: Let $\varepsilon>0$ be given. There exists an N such that for all $m,n>N$:

$$\|T_n-T_m\|<\varepsilon.$$

By the continuity of the norm, for every $x\in\mathscr{X}$ and $n>N$,

$$\|T_n x-Tx\|=\lim_{m\to\infty}\|T_n x-T_m x\|\le\left(\limsup_{m\to\infty}\|T_n-T_m\|\right)\|x\|<\varepsilon\|x\|,$$

i.e., $\|T_n-T\|<\varepsilon$, which proves that $T_n\to T$. ∎

Definition 2.3 Let $\mathscr{X}$ be a normed space. The space of all bounded linear functionals on $\mathscr{X}$ is called its **dual space** and it is denoted by $\mathscr{X}^*$.

Corollary 2.12 Let $\mathscr{X}$ be a normed space. Then, $\mathscr{X}^*$ is a Banach space.

Proof. It follows from Proposition 2.11 and the completeness of the field of scalars. ∎

Notation 2.1 Let $\mathscr{X}$ be a normed space. Let $x\in\mathscr{X}$ and $f\in\mathscr{X}^*$. We will denote the action of f on x by

$$\langle f,x\rangle.$$

This notation makes it plain that the function $(f,x)\mapsto f(x)$ is bilinear.

TA material 2.1 The converse of the last proposition is also true:

Proposition 2.13 let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. If $B(\mathscr{X},\mathscr{Y})$ is a Banach space then $\mathscr{Y}$ is a Banach space.

Proof. Let $f\in\mathscr{X}^*$ be a functional of positive norm (i.e., there is an $x\in\mathscr{X}$ such that $\langle f,x\rangle\ne 0$). Define $T:\mathscr{Y}\to B(\mathscr{X},\mathscr{Y})$:

$$y\mapsto\langle f,\cdot\rangle y.$$

T is clearly linear; it is also bounded as

$$\|Ty\|_{B(\mathscr{X},\mathscr{Y})} = \sup_{\|x\|=1} \|\langle Ty,x\rangle\|_{\mathscr{Y}} = \sup_{\|x\|=1} \|\langle f,x\rangle y\|_{\mathscr{Y}} \le \|f\|_{\mathscr{X}^*}\|y\|_{\mathscr{Y}}$$

Let (y_n) be a Cauchy sequence in $\mathscr{Y}$. Then

$$\|Ty_n - Ty_m\|_{B(\mathscr{X},\mathscr{Y})} = \|f\|_{\mathscr{X}^*}\|y_n - y_m\|_{\mathscr{Y}}.$$

Thus, (Ty_n) is a Cauchy sequence in $B(\mathscr{X},\mathscr{Y})$. By assumption, there exists an $S \in B(\mathscr{X},\mathscr{Y})$ such that $Ty_n \to S$. Let $x_0 \in \mathscr{X}$ such that $\langle f,x_0\rangle = 1$ and set $y = S(x_0)$. Then,

$$\|y_n - y\|_{\mathscr{Y}} = \|S(x_0) - \langle f,x_0\rangle y_n\|_{\mathscr{Y}} = \|S(x_0) - Ty_n(x_0)\|_{\mathscr{Y}} \le \|S - Ty_n\|_{B(\mathscr{X},\mathscr{Y})}\|x_0\|_{\mathscr{X}} \to 0,$$

hence $y_n \to y$ and $\mathscr{Y}$ is complete. ■

Proposition 2.14 Let $p \ge 1$ and set $q = p/(p-1)$ (if $p = 1$ then $q = \infty$). Then, ℓ_p^* is isometric to ℓ_q.

Comment 2.2 For $p = 2$ the proposition states that $\ell_2^* \cong \ell_2$, which we already know from the Riesz representation theorem.

Proof. Consider the case $p > 1$; the proof for $p = 1$ follows the same lines.

Step 1: construct a linear map $T : \ell_q \to \ell_p^*$: Let $y \in \ell_q$. We associate with y a linear functional on ℓ_p:

$$T(y)x = \sum_{n=1}^{\infty} y_n x_n.$$

The right hand side is well defined because by Hölder's inequality

$$\sum_{n=1}^{N} |y_n x_n| \le \left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{N} |y_n|^q\right)^{1/q} \le \|y\|_q \|x\|_p.$$

Note that not only $T(y)$ is a linear functional; the mapping $T : y \mapsto T(y)$ is also linear.

Step 2: $T : \ell_q \to \ell_p^*$ is bounded: By Hölder's inequality:

$$|T(y)x| \le \|y\|_q \|x\|_p,$$

which implies that $\|T(y)\|_{\ell_p^*} \le \|y\|_q$. Thus, T is indeed a mapping from ℓ_q to ℓ_p^*, and moreover, $T \in B(\ell_q;\ell_p^*)$. Having shown that $T(y) \in \ell_p^*$, we denote its action by $T(y)x = \langle T(y), x\rangle$.

Step 3: T is onto: Let $F \in \ell_p^*$ and define

$$y_n = \langle F, e_n \rangle,$$

where e_n is the n-th unit vector. We have thus mapped an element in $F \in \ell_p^*$ into a sequence y. We are going to show that the sequence y is in ℓ_q.

Define the sequence of sequences:

$$x_k^{(n)} = \begin{cases} |y_k|^{q-1} e^{-\imath \arg y_k} & k \le n \\ 0 & k > n. \end{cases}$$

Then,

$$\|x^{(n)}\|_p = \left(\sum_{k=1}^n |y_k|^{p(q-1)} \right)^{1/p} = \left(\sum_{k=1}^n |y_k|^q \right)^{1/p},$$

and

$$\langle F, x^{(n)} \rangle = \sum_{k=1}^n x_k^{(n)} \langle F, e_k \rangle = \sum_{k=1}^n x_k^{(n)} y_k = \sum_{k=1}^n |y_k|^q.$$

It follows that

$$\sum_{k=1}^n |y_k|^q = \langle F, x^{(n)} \rangle \le \|F\|_{\ell_p^*} \|x^{(n)}\|_p = \|F\|_{\ell_p^*} \left(\sum_{k=1}^n |y_k|^q \right)^{1/p},$$

i.e.,

$$\left(\sum_{k=1}^n |y_k|^q \right)^{1/q} \le \|F\|_{\ell_p^*}.$$

Since this inequality holds uniformly in n, it follows that $y \in \ell_q$ and $\|y\|_q \le \|F\|_{\ell_p^*}$.

Since F is continuous:

$$(\forall x \in \ell_p) \quad \langle F, x \rangle = \langle F, \lim_n \sum_{k=1}^n x_k e_k \rangle = \sum_{n=1}^\infty x_n \langle F, e_k \rangle = \sum_{n=1}^\infty x_n y_n = T(y)x.$$

Thus, F belong to the image of T, i.e., T is surjective.

Step 4: T is an isometry: We have shown both that $\|T(y)\|_{\ell_p^*} \le \|y\|_q$ and $\|T(y)\|_{\ell_p^*} \ge \|y\|_q$, which proves that T is an isometry. ■

Corollary 2.15 The spaces ℓ_p, $p \ge 1$, are Banach spaces.

Proof. For $p \ge 1$,

$$\ell_p \cong \ell_q^*,$$

and the dual of any normed space is a Banach space. ■

Comment 2.3 Another immediate corollary is that $(\ell_p^*)^* \cong \ell_p$ for $1 < p < \infty$. On the other hand, $\ell_\infty^* \neq \ell_1$.

Proposition 2.16 $c_0^* \cong \ell_1$.

Proof. Left as an exercise. ∎

2.2 The uniform boundedness principle

This section is concerned with the Banach-Steinhaus theorem and its ramifications. This theorem is one of the four "cornerstones" of Banach spaces, together with the **Hahn-Banach theorem** (which in fact applies to general vector spaces), the **open mapping theorem**, and the **closed graph theorem**.

2.2.1 The Banach-Steinhaus theorem

Theorem 2.17 — Banach-Steinhaus, 1927 (חסימות במידה שווה). Let $\mathscr{B}$ be a Banach space and let $\{\mathscr{Y}_\alpha \mid \alpha \in A\}$ be normed spaces. Let $T_\alpha \in B(\mathscr{B};\mathscr{Y}_\alpha)$ be a family of bounded linear operators. If

$$(\forall x \in \mathscr{B}) \qquad \sup_{\alpha\in A} \|T_\alpha x\| < \infty,$$

then

$$\sup_{\alpha\in A} \|T_\alpha\| < \infty.$$

Proof. Consider the sequence of sets

$$B_n = \{x \in \mathscr{B} \mid \sup_{\alpha\in A} \|T_\alpha x\| \le n\}.$$

These sets are closed and cover $\mathscr{B}$. Since $\mathscr{B}$ is complete, it follows from Baire's category theorem that there exists an m such that B_m contains an open ball. That is, there exist an $x_0 \in \mathscr{B}$ and a $\rho > 0$, such that for all $x \in B(x_0,\rho)$:

$$\sup_{\alpha\in A} \|T_\alpha x\| \le m.$$

For all $\|x\| = 1$ and $\alpha \in A$,

$$\|T_\alpha x\| = \frac{2}{\rho}\left\|T_\alpha\left(\frac{\rho}{2}x\right)\right\| \le \frac{2}{\rho}\left\|T_\alpha\left(x_0 + \frac{\rho}{2}x\right) - T_\alpha(x_0)\right\| \le \frac{4m}{\rho}\|x\|,$$

which implies that $\|T_\alpha\| \le 4m/\rho$ for all $\alpha \in A$. ∎

Comment 2.4 Note how similar is this proof to the proof that a weakly converging sequence in a Hilbert space is bounded. This is not a coincidence. The Banach-Steinhaus theorem is a generalization of the former. Indeed, suppose that a sequence y_n weakly converges to y. Defining

$$T_n = (\cdot, y_n),$$

this sequence of bounded linear functionals satisfies

$$|T_n x| = |(x, y_n)| \to |(x, y)|,$$

hence the sequence $(T_n x)$ is bounded for every $x \in \mathscr{H}$. It follows from the Banach-Steinhaus theorem that $\|T_n\| = \|y_n\|$ is bounded.

Proposition 2.18 Let T_n be a sequence in $B(\mathscr{B}; \mathscr{B}')$, where both $\mathscr{B}$ and $\mathscr{B}'$ are Banach spaces. Then, $T_n x$ converges for all $x \in \mathscr{B}$ if and only if:

① $T_n x$ converges for every x in some dense subset A of $\mathscr{B}$.
② $\sup_n \|T_n\| < \infty$.

Proof. **The "only if" direction**: If $T_n x$ converges for all $x \in \mathscr{B}$ then it certainly converges for every x in some dense subset of $\mathscr{B}$. The boundedness of $\|T_n\|$ follows from the Banach-Steinhaus theorem.

The "if" direction: Suppose that the two conditions hold; denote

$$\sup_n \|T_n\| = M.$$

Let $\varepsilon > 0$ be given. For every $x \in \mathscr{B}$ corresponds a $y \in A$ such that $\|x - y\| < \varepsilon$. Since $T_n y$ converges, there exists an N, such that for every $n, m > N$,

$$\|T_n y - T_m y\| < \varepsilon.$$

Then,

$$\|T_n x - T_m x\| \le \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| \le (2M + 1)\varepsilon.$$

It follows that $T_n x$ is a Cauchy sequence hence converges. ■

Exercise 2.3 Let $\mathscr{X}$, $\{\mathscr{Y}_\alpha\}$ be normed spaces and let $T_\alpha : \mathscr{X} \to \mathscr{Y}_\alpha$ be a family of bounded linear operators. Prove that the set

$$E = \{x \in \mathscr{X} \mid \sup \|T_\alpha x\| = \infty\}$$

is either empty or a dense G_δ set (a countable intersection of open sets). Hint: if E is not dense, then its complement contains a ball. Any vector can be translated by translation and dilation into any ball. ■

Exercise 2.4 Let $\mathscr{X}$ be a Banach space, $\mathscr{Y}$ a normed space, and $T_n \in B(\mathscr{X}, \mathscr{Y})$ a sequence satisfying

$$\lim_{n\to\infty} T_n x \equiv T x \qquad \text{exists for all } x \in \mathscr{X}.$$

① Show that T is a bounded linear operator.
② Show that $\lim_{x\to 0} T_n x = 0$ uniformly in n.
③ Show that T is not necessarily bounded if $\mathscr{X}$ is not complete. Hint: consider $f_n(x) = \sum_{k=1}^n x_k$ in the space c_c of sequences the are zero from some point, with an appropriate norm.

■

2.2.2 Applications

We will now see a number of applications of the Banach-Steinhaus theorem:

Proposition 2.19 Let $\mathscr{H}$ be a Hilbert space and let $B(\cdot,\cdot)$ be a bilinear form. If $B(\cdot, y)$ is continuous for every y and $B(x,\cdot)$ is continuous for every x, then B is bounded.

Proof. Fix $x \in \mathscr{H}$ and consider the linear functional $\overline{B(x,\cdot)}$. By assumption it is continuous, hence bounded; denote its bound by K_x:

$$|B(x,y)| \le K_x \|y\|.$$

Next, for given $0 \ne y \in \mathscr{H}$ consider the bounded linear functional

$$F_y = \frac{B(\cdot, y)}{\|y\|}.$$

Then,

$$\sup_y |\langle F_y, x\rangle| = \sup_y \frac{|B(x,y)|}{\|y\|} \le K_x,$$

i.e., the functionals F_y are pointwise bounded. It follows from the Banach-Steinhaus theorem that there exists an $M > 0$, such that

$$\sup_y \|F_y\| \le M.$$

Thus, for every $x, y \in \mathscr{H}$:

$$|B(x,y)| = |\langle F_y, x\rangle| \|y\| \le M \|x\| \|y\|,$$

which proves that B is bounded. ∎

Comment 2.5 Where did we use the continuity of $B(\cdot, y)$? It is a necessary condition for the Banach-Steinhaus theorem.

TA material 2.2 Note that we nowhere used the inner-product, which means that the proposition holds for bilinear forms in Banach spaces. On the other hand, completeness is essential. As an example for why completeness is needed, let $\mathscr{X} \subset L_1([0,1])$ be the space of real-valued polynomials; it is not complete. Define the bilinear form,

$$B(p,q) = \int_0^1 p(x)q(x)\,dx.$$

B is continuous with respect to each argument as:

$$|B(p,q)| \le \left(\max_x |p(x)|\right) \|q\|_1.$$

However B is not bounded for let $p_n(x) = (n+1)x^n$, then

$$\|p_n\| = \int_0^1 (n+1)x^n\,dx = 1,$$

whereas

$$|B(p_n, p_n)| = \int_0^1 (n+1)^2 x^{2n}\,dx \frac{(n+1)^2}{2n+1} \to \infty.$$

Pointwise convergence of Fourier series

Consider the space $C[0,2\pi]$ of functions f satisfying $f(0) = f(2\pi)$ (continuous functions on a circle). Recall that

$$S_n f(x) = \frac{1}{2\pi} \int_0^{2\pi} D_n(x-t) f(t)\,dt,$$

where

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$$

is the Dirichlet kernel.

Proposition 2.20 For every $x_0 \in [0,2\pi]$ there exists a function $f \in C[0,2\pi]$ such that the Fourier series of f does not converge at x_0.

Proof. Define the following sequence of linear functionals on $C[0,2\pi]$:

$$F_n(f) = S_n f(x_0).$$

These functionals are bounded, for

$$|F_n(f)| \le \frac{1}{2\pi}\int_0^{2\pi} |D_n(x_0-t)f(t)|\,dt \le K_n \|f\|_\infty.$$

We need to show that there exists an f for which $\langle F_n, f\rangle$ does not converge. By the Banach-Steinhaus theorem,

$$\sup_n \|F_n\| = \infty \qquad \text{implies} \qquad \exists f \in C[0,2\pi], \; \sup_n |\langle F_n, f\rangle| = \infty.$$

The proposition will be proved if we show that $\|F_n\| \to \infty$. ∎

Proposition 2.21 For F_n defined as above:

$$\lim_{n\to\infty} \|F_n\| = \infty.$$

Proof. Suppose we took

$$f(t) = \begin{cases} 1 & D_n(x_0-t) > 0 \\ -1 & D_n(x_0-t) < 0 \\ 0 & D_n(x_0-t) = 0 \end{cases},$$

then

$$\langle F_n, f\rangle = S_n f(x_0) = \frac{1}{2\pi}\int_0^{2\pi} |D_n(x_0-t)|\,dt = \frac{1}{2\pi}\int_0^{2\pi} |D_n(t)|\,dt.$$

Of course, f is not continuous, however, for every ε we can find a function $f_\varepsilon \in C[0,2\pi]$ (of norm one as well) that differs from f over a small set, such that

$$\langle F_n, f_\varepsilon\rangle \ge \frac{1}{2\pi}\int_0^{2\pi} |D_n(t)|\,dt - \varepsilon.$$

It follows that

$$\|F_n\| \ge \frac{1}{2\pi}\int_0^{2\pi} |D_n(t)|\,dt.$$

It is a simple exercise to show that the right hand side diverges as $n \to \infty$ (it grows like $\log n$). ∎

Exercise 2.5 Prove that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |D_n(t)|\,dt = \infty.$$

■

Polynomial interpolation

Proposition 2.22 Given $n+1$ points $\{(x_i, y_i)\}_{i=0}^n$ in the plane with all the x_i distinct, there exists a unique polynomial $p_n \in \Pi_n$, such that

$$p(x_i) = y_i \qquad \text{for all } i = 0, 1, \dots, n.$$

Proof. Uniqueness is immediate: if p_n and q_n satisfy the required conditions, then their difference is a polynomial of degree up to n that vanishes at $n+1$ points.

The existence of p_n is proved by providing a formula:

$$p_n(x) = \sum_{i=0}^n y_i \prod_{j\neq i} \left(\frac{x - x_j}{x_i - x_j} \right).$$

This representation of the polynomial is called the **Lagrange representation**. ■

Definition 2.4 Let $f \in C[-1,1]$ be real-valued. Let $Q_n = (x_0, \dots, x_n)$ be distinct points on $[-1,1]$; Q_n is called a **partition** (חלוקה) of size n. The **interpolation polynomial** of f through the points x_i is the unique polynomial $p_n \in \Pi_n$ satisfying

$$p(x_i) = f(x_i) \qquad \text{for all } i = 0, 1, \dots, n.$$

Question: is p_n a good approximation of f? In particular, does $p_n \to f$ as $n \to \infty$? It can be shown that there exists for every $f \in C[-1,1]$ a sequence of partitions (Q_n) such that $p_n \to f$ (the convergence is in $C[-1,1]$, i.e., it is uniform). The following theorem shows, somewhat surprisingly, that the opposite is not true: there exists for every sequence (Q_n) of partitions a function $f \in C[-1,1]$, such that p_n does not converge to f.

Theorem 2.23 — Faber, 1914. For every sequence of partitions $Q_n = (x_0^{(n)}, \dots, x_n^{(n)})$ there exists a function $f \in C[-1,1]$ for which

$$\limsup_{n\to\infty} \|f - p_n\|_\infty = \infty,$$

where p_n is the interpolation polynomial of f with respect to the partition Q_n.

Proof. We will prove the theorem for uniform partitions, $x_k^{(n)} = -1 + 2k/n$. Denote by $L_n : C[-1,1] \to \Pi_n[-1,1]$ the mapping of a function f to its polynomial interpolation with respect to Q_n:

$$(L_n f)(x) = \sum_{i=0}^{n} f(x_i) \prod_{j \neq i} \left(\frac{x - x_j}{x_i - x_j} \right).$$

L_n is a linear operator; it is bounded because

$$\|L_n f\| \le \|f\| \max_x \sum_{i=0}^{n} \prod_{j \neq i} \left| \frac{x - x_j}{x_i - x_j} \right|.$$

We will show that

$$\lim_{n\to\infty} \|L_n\| = \infty,$$

and it will follow from the Banach-Steinhaus theorem that there exists an $f \in C[-1,1]$ for which

$$\lim_{n\to\infty} \|L_n f\| = \infty.$$

Let $\hat{x} = \frac{1}{2}(x_0 + x_1)$. There exists a unit vector f satisfying

$$f(x_i) = \operatorname{sgn} \prod_{j \neq i} \left(\frac{\hat{x} - x_j}{x_i - x_j} \right).$$

Then,

$$(L_n f)(\hat{x}) = \sum_{i=0}^{n} \prod_{j \neq i} \left| \frac{\hat{x} - x_j}{x_i - x_j} \right| = \sum_{i=0}^{n} \prod_{j \neq i} \left| \frac{(-1 + 1/n) - (-1 + 2j/n)}{(-1 + 2i/n) - (-1 + 2j/n)} \right| = \sum_{i=0}^{n} \prod_{j \neq i} \left| \frac{1/2 - j}{i - j} \right|.$$

It follows that

$$\|L_n f\| \ge \sum_{i=0}^{n} \prod_{j \neq i} \left| \frac{1/2 - j}{i - j} \right|.$$

One can finally show that the right hand side is $O(n)$. ■

■ **Example 2.7 — Runge.** The polynomial interpolations of

$$f(x) = \frac{1}{1 + 25x^2}$$

on the interval $[-1,1]$ using uniform partitions diverge. This is surprising as f is smooth. ■

2.3 Isomorphisms

Definition 2.5 Two normed spaces $\mathscr{X}$ and $\mathscr{Y}$ are called **isomorphic** if there is a linear bijection $T : \mathscr{X} \leftrightarrows \mathscr{Y}$ such that both T and T^{-1} are continuous. T is called an **isomorphism** (i.e., an isomorphism between normed spaces is a homeomorphism that preserves the linear structure).

Proposition 2.24 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed space. Let $T : \mathscr{X} \twoheadrightarrow \mathscr{Y}$ be a surjective, linear norm-preserving map (i.e., an isometry). Then T is an isomorphism.

Proof. It is given that T is linear, surjective, and

$$\|Tx\|_{\mathscr{Y}} = \|x\|_{\mathscr{X}}$$

for all $x \in \mathscr{X}$. Hence, T is bounded, and therefore continuous. Since $\ker T = \{0\}$, T is also injective hence T^{-1} is well-defined. Finally, for every $y \in \mathscr{Y}$,

$$\|y\|_{\mathscr{Y}} = \|T^{-1}y\|_{\mathscr{X}}$$

so that T^{-1} is also bounded and hence continuous. ∎

Comment 2.6 The converse is not true: an isomorphism is not necessarily an isometry. An isomorphism is an equivalence relation in the category of **topological vector spaces**.

Proposition 2.25 — Characterization of isomorphisms. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. They are isomorphic if and only if there is a linear mapping $T : \mathscr{X} \to \mathscr{Y}$ onto $\mathscr{Y}$ and constants $c, C > 0$, such that

$$(\forall x \in \mathscr{X}) \quad c\|x\|_{\mathscr{X}} \le \|Tx\|_{\mathscr{Y}} \le C\|x\|_{\mathscr{X}}.$$

Proof. Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic. Then, there is a linear bijection $T : \mathscr{X} \leftrightarrows \mathscr{Y}$ and constants c, C, such that $\|T\| = C$ and $\|T^{-1}\| = 1/c$. It follows that

$$\|Tx\|_{\mathscr{Y}} \le C\|x\|_{\mathscr{X}} \quad \text{and} \quad \|x\|_{\mathscr{X}} = \|T^{-1}(Tx)\|_{\mathscr{X}} \le \frac{1}{c}\|Tx\|_{\mathscr{Y}}.$$

Conversely, suppose that the assumptions hold. It follows that T is bounded, $\|T\| \le C$, $\ker T = \{0\}$, hence T is injective and T^{-1} is well-defined. Finally, T^{-1} is continuous as

$$\|T^{-1}y\|_{\mathscr{X}} \le \frac{1}{c}\|T(T^{-1}y)\|_{\mathscr{Y}} = \frac{1}{c}\|y\|_{\mathscr{Y}},$$

i.e., $\|T^{-1}\| \le 1/c$. ∎

Corollary 2.26 Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on a vector space $\mathscr{X}$ if and only if there exist constant c, C such that $c\|x\|_1 \le \|x\|_2 \le C\|x\|_1$. (Two such norms are said to be **equivalent**).

Proof. Apply the previous characterization with T the identity. ■

Please note that two norms on the same space are equivalent if every ball $B(0,r)$ with respect to one norm is contained in some ball $B(0,\rho)$ with respect to the other norm.

We next construct an example that shows that isomorphic normed spaces are not necessarily isometric.

Proposition 2.27 The spaces c and c_0 are isomorphic.

Proof. Recall that c is the subspace of ℓ_∞ consisting of converging sequences and c_0 is the subspace of c consisting of sequences converging to zero. For $x \in c$ with limit $\bar{x}$, define

$$Tx = (\bar{x}, x_1 - \bar{x}, x_2 - \bar{x}, \dots) \in c_0.$$

(The limit of x is encoded in the first element of Tx.) T is linear; it is injective because $\ker T = \{0\}$. It is onto because for $y \in c_0$:

$$y = T(y_1 + y_2, y_1 + y_3, \dots).$$

Finally, T and T^{-1} are both continuous because

$$\|Tx\| \le \max\{|\bar{x}|, \|x\| + |\bar{x}|\} \le 2\|x\|,$$

and

$$\|T^{-1}y\| \le 2\|y\|.$$

This completes the proof. ■

But are c and c_0 isometric? The operator T defined above is not an isometry: take the sequence $x = (5,4,4,\dots)$, then $\|x\| = 5$ and $\|Tx\| = 4$. Yet, isomorphisms are not unique, hence it might well be that an isometric exists. We will show that this is not the case.

Definition 2.6 Let C be a convex set in a vector space $\mathscr{V}$. A point $x \in C$ is called an **extreme point** (נקודה קיצון) if x is not an interior point of any open segment contained in C.

Comment 2.7 An extreme point is a boundary point. The converse is not true. A closed half plane, for example, has boundary points but no extreme points.

Proposition 2.28 The closed unit ball in c_0 has no extreme points.

Proof. Let $x \in c_0$, $\|x\| \le 1$. Since x converges to zero, there is a k for which $|x_k| < 1/2$. The open segment

$$\left(x - \frac{1}{2}e_k, x + \frac{1}{2}e_k\right),$$

where e_k is the k-th unit vector, is contained in the unit ball, hence x is not an extreme point. ∎

Proposition 2.29 The closed unit ball in c has an extreme point.

Proof. Consider the point $x = (1, 1, \dots)$. Suppose it were not an extreme point. This would imply the existence of a vector $0 \ne y \in c$ and an $\varepsilon > 0$ such that

$$(x - \varepsilon y, x + \varepsilon y) \subset B(0,1).$$

Since y has a non-zero entry, say y_k, either $x_k - \varepsilon y_k$ or $x_k + \varepsilon y_k$ is greater than one, i.e., either $x - \varepsilon y$ or $x + \varepsilon y$ is not in the unit ball of c. ∎

The fact that c and c_0 are not isometric follows from the following proposition:

Proposition 2.30 Let $T : \mathscr{X} \to \mathscr{Y}$ be a linear isometry. If $x \in \mathscr{X}$ is an extreme point in the unit ball of $\mathscr{X}$, then Tx is an extreme point in the unit ball of $\mathscr{Y}$

Proof. Since isometries preserve all metric properties and being an extreme point is a metric property the assertion holds trivially. ∎

We now return to the mere concept of isomorphism:

Proposition 2.31 Separability is an isomorphism invariant.

Proof. Let $T : \mathscr{X} \to \mathscr{Y}$ be an isomorphism. Suppose that $\mathscr{X}$ is separable, and let $A \subset \mathscr{X}$ be a countable, dense set. Let $y \in \mathscr{Y}$; there exists a sequence $(x_n) \in A$ such that

$$x_n \to T^{-1}y.$$

By the continuity of T:

$$Tx_n \to T(T^{-1}y) = y.$$

This proves that the countable set $T(A)$ is dense in $\mathscr{Y}$. ∎

Recall that in Hilbert spaces all separable spaces are isomorphic. Is it also the case for Banach spaces?

Proposition 2.32 Let $\mathscr{X}$ and $\mathscr{Y}$ be isomorphic normed spaces, then $\mathscr{X}^*$ and $\mathscr{Y}^*$ are isomorphic.

Comment 2.8 "Isomorphic" can be replaced by "isometric". We will prove the latter after we learn the Hahn-Banach theorem.

Proof. Let $T : \mathscr{X} \to \mathscr{Y}$ be an isomorphism. T induces a natural map, its **adjoint**, $T^* : \mathscr{Y}^* \to \mathscr{X}^*$, defined by[2]

$$[T^*(y^*)](x) = \langle y^*, Tx \rangle,$$

(or $T^*(y^*) = y^* \circ T$). T^* is linear as the map $y^* \mapsto y^* \circ T$ is linear. T^* is injective as $y_1^* \circ T = y_2^* \circ T$ implies that $y_1^* = y_2^*$ (here we use the fact that T is invertible). T^* is also surjective as for any $x^* \in \mathscr{X}^*$,

$$x^* = (x^* \circ T^{-1}) \circ T = T^*(x^* \circ T^{-1}).$$

Thus, T^* is invertible, and

$$(T^*)^{-1}(x^*) = x^* \circ T^{-1}.$$

It remains to show that T^* and $(T^*)^{-1}$ are both continuous. Note that $(T^*)^{-1}$ relates to T^{-1} in the same way as T^* relates to T, therefore, it is sufficient to verify the boundedness of one of them:

$$\begin{aligned} \|T^*\| &= \sup_{\|y^*\|=1} \|T^*(y^*)\| = \sup_{\|y^*\|=1} \sup_{\|x\|=1} \|T^*(y^*)(x)\| = \sup_{\|y^*\|=1} \sup_{\|x\|=1} \|y^*(Tx)\| \\ &\le \sup_{\|y^*\|=1} \sup_{\|x\|=1} \|y^*\| \|T\| \|x\| = \|T\|, \end{aligned}$$

i.e., T^* is bounded. ∎

The following corollary shows that unlike Hilbert spaces, two separable Banach spaces can be non-isomorphic.

[2] We will study the adjoint operator in more detail in a later section.

Corollary 2.33 ℓ_1 and c_0 are not isomorphic.

Proof. If ℓ_1 and c_0 were isomorphic then it would follow from the last proposition that

$$\ell_1^* = \ell_\infty \qquad \text{and} \qquad c_0^* = \ell_1$$

are isomorphic. This is not true as ℓ_1 is separable and ℓ_∞ is not, and separability is an isomorphic invariant. ∎

2.3.1 Finite-dimensional normed spaces

In this section we will study in depth finite-dimensional normed spaces. We will denote by ℓ_1^n the space of n-tuples $\mathscr{F}^n$ with the norm

$$\|x\|_1 = \sum_{k=1}^n |x_k|.$$

Proposition 2.34 The space ℓ_1^n is a Banach spaces, and its unit sphere is relatively compact (i.e., its closure is compact).

Proof. Consider a Cauchy sequence $x^{(n)}$ in ℓ_1^n. Then, every component $x_k^{(n)}$ is a Cauchy sequence and hence converges to a limit x_k. It follows that $x^{(n)} \to x$. Since the unit sphere is closed and bounded it is relatively compact (true for any finite-dimensional normed space). ∎

Theorem 2.35 Let $\mathscr{X}$ be an n-dimensional normed spaces. Then $\mathscr{X}$ is isomorphic to ℓ_1^n.

Proof. Let $(u_1, \dots, u_n)$ be a basis for $\mathscr{X}$. Every $x \in \mathscr{X}$ has a unique representation as

$$x = \sum_{k=1}^n \alpha_k(x) u_k,$$

i.e., $\alpha_k : \mathscr{X} \to \mathscr{F}$ returns the n-th component of a vector with respect to the (arbitrarily) chosen basis. Define the operator $T : \mathscr{X} \to \ell_1^n$:

$$Tx = (\alpha_1(x), \dots, \alpha_n(x)).$$

T is linear, injective, and surjective, with

$$T^{-1}(a_1,\ldots,a_n) = \sum_{k=1}^{n} a_k u_k.$$

It remains to prove that T and T^{-1} are continuous. Let $a \in \ell_1^n$. By the triangle inequality:

$$\left\|T^{-1}a\right\|_{\mathscr{X}} \le \sum_{k=1}^{n} |a_k|\|u_k\|_{\mathscr{X}} \le \left(\max_k \|u_k\|_{\mathscr{X}}\right)\|a\|_1,$$

which proves that T^{-1} is continuous; note how essential the fact that the dimension is finite is. Since T^{-1} is continuous and the unit ball of ℓ_1^n is relatively compact it follows that

$$\min_{\|a\|_1=1} \|T^{-1}a\|_{\mathscr{X}} = M$$

exists, and moreover, $M > 0$. Now, for all $x \in \mathscr{X}$:

$$\|x\|_{\mathscr{X}} = \left\|T^{-1}Tx\right\|_{\mathscr{X}} = \left\|T^{-1}\left(\frac{Tx}{\|Tx\|_1}\right)\right\|_{\mathscr{X}} \|Tx\|_1 \ge M\|Tx\|_1,$$

which completes the proof. ∎

Corollary 2.36 Let $\mathscr{X}$ and $\mathscr{Y}$ be finite-dimensional normed spaces of the same dimension n and over the same field $\mathscr{F}$. Then $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic.

Proof. Isomorphism is an equivalence relation and every n-dimensional normed space is isomorphic to ℓ_1^n. ∎

Lemma 2.37 Completeness is an isomorphic invariant.

Proof. Let $T : \mathscr{X} \to \mathscr{Y}$ be an isomorphism. Suppose that $\mathscr{X}$ is complete. Let (y_n) be a cauchy sequence in $\mathscr{Y}$. Then,

$$\|T^{-1}y_n - T^{-1}y_m\| \le \|T^{-1}\|\|y_n - y_m\|,$$

i.e., $(T^{-1}y_n)$ is a Cauchy sequence in $\mathscr{X}$, and hence converges, so say $x \in \mathscr{X}$. By the continuity of T,

$$y_n = T(T^{-1}y_n) \to Tx,$$

i.e., $\mathscr{Y}$ is complete. ∎

Corollary 2.38 Every finite-dimensional normed space is complete (i.e., a Banach space).

Proof. Completeness is invariant under isomorphisms, and every n-dimensional normed space is isomorphic to ℓ_1^n, which is complete. ■

Corollary 2.39 Every linear subspace of a finite-dimensional normed space is closed.

Proof. Closedness is invariant under isomorphisms (it is a topological property), and any linear subspace of ℓ_1^n is closed. ■

Corollary 2.40 All the norms on $\mathbb{C}^n$ are equivalent.

Proof. We have seen that (i) all norms on $\mathbb{C}^n$ are isomorphic and (ii) an isomorphism on the same space implies equivalent norms. ■

Corollary 2.41 Every linear transformation from a finite-dimensional normed space into a normed space $\mathscr{X}$ is bounded.

Proof. Let $\mathscr{Y}$ be a finite dimensional normed space and let $S:\mathscr{Y}\to\mathscr{X}$ be a linear transformation. Since $\mathscr{Y}$ is isomorphic to ℓ_1^n there is an isomorphism

$$\varphi:\ell_1^n\to\mathscr{Y}.$$

Then,

$$T=\varphi\circ S:\ell_1^n\to\mathscr{X}$$

is a linear transformation, which is bounded if an only if S is bounded.

For $a=(a_1,\dots,a_n)\in\ell_1^n$,

$$\|Ta\|_{\mathscr{X}}=\left\|\sum_{k=1}^n a_k T(e_k)\right\|_{\mathscr{X}}\le\left(\max_k\|T(e_k)\|_{\mathscr{X}}\right)\|a\|_1,$$

which completes the proof. ■

Corollary 2.42 In a finite-dimensional normed space every bounded, closed set is compact.

Proof. Follows again from the isomorphism to ℓ_1^n. ■

We have shown that every n-dimensional normed space is isomorphic to ℓ_1^n. We have seen that by choosing a basis $\{u_1, \dots, u_n\}$ in $\mathscr{X}$, and defining $T : \ell_1^n \to \mathscr{X}$ as

$$T(a) = \sum_{k=1}^{n} a_k u_k,$$

we have,

$$\|Ta\|_{\mathscr{X}} \le \left(\max_k \|u_k\|_{\mathscr{X}}\right) \|a\|_1,$$

and

$$\|T^{-1}x\|_1 \le \left(\max_{\|y\|=1} \|T^{-1}y\|_1\right) \|x\|_{\mathscr{X}}.$$

We can always have $\|T\| = 1$ be choosing a basis (u_k) of unit vectors. The question is whether we can simultaneously control $\|T^{-1}\|$. This is an important question if we are looking for isomorphisms that do not distort distances by too much. A bound on $\|T^{-1}\|$ can be obtained using the following theorem:

Theorem 2.43 — Auerbach. Let $\mathscr{X}$ be an n-dimensional normed space. Then, there is a basis (u_k) and there are n functionals $(f_k) \in \mathscr{X}^*$, such that $\|u_k\| = \|f_k\| = 1$ and

$$f_j(u_i) = \delta_{ij}$$

(The functionals f_k are **the dual basis** of u_k).

Proof. Choose a basis $(e_1, \dots, e_n)$ in $\mathscr{X}$; every vector $x \in \mathscr{X}$ has a unique representation

$$x = \sum_{k=1}^{n} x_k e_k.$$

Next, define the function, $D : \mathscr{X}^n \to \mathscr{F}$:

$$D(x_1, \dots, x_n) = \det((x_i)_j).$$

(D looks like a volume form, but of course, there is no such this in a Banach space). By the compactness of the unit sphere, there exist n vectors $\{u_k\}_{k=1}^n$, on the unit sphere of $\mathscr{X}$, such that

$$D(u_1, \dots, u_k) = \max_{\|x_1\| = \dots = \|x_n\| = 1} |D(x_1, \dots, x_n)|.$$

The $\{u_k\}$ are independent, otherwise the determinant would vanish.

For every $i = 1, 2, \dots, n$ define the functional $f_i : \mathscr{X} \to \mathscr{F}$:

$$f_i(x) = \frac{D(u_1, \dots, u_{i-1}, x, u_{i+1}, u_n)}{D(u_1, \dots, u_k)}.$$

The f_i are linear functionals, they have norm 1 (maximized for $x = u_i$), and $f_i(u_j) = \delta_{ij}$. ∎

Comment 2.9 The basis $\{u_k\}$ is not unique, but it uniquely determined the dual basis. In Hilbert spaces the basis maximizing the volume form is orthonormal.

Corollary 2.44 Let $\mathscr{X}$ be an n-dimensional normed space. There exists an isomorphism $T : \ell_1^n \to \mathscr{X}$ such that $\|T\| = 1$ and $\|T^{-1}\| \le n$.

Proof. Choose a basis of unit vectors $\{u_k\}$ in $\mathscr{X}$ as above along with its dual basis $\{f_j\}$. Define $T : \ell_1^n \to \mathscr{X}$ by

$$T(a) = \sum_{k=1}^n a_k u_k.$$

Clearly,

$$\|T(a)\|_{\mathscr{X}} \le \sum_{k=1}^n |a_k| \|u_k\| = \|a\|_1,$$

namely, $\|T\| \le 1$. The norm is in fact equal to one as for $a = e_k$, $T(a) = u_k$, hence $\|T(a)\| = 1 = \|a\|_1$.

We turn to evaluate the norm of T^{-1}. For every $j = 1, 2, \dots, n$,

$$f_j\left(\sum_{k=1}^n a_k u_k\right) = \sum_{k=1}^n a_k T_j(u_k) = a_j,$$

i.e., f_j returns the j-th component with respect to the basis $\{u_k\}$. Thus,

$$T^{-1}(x) = (f_1(x), \dots, f_n(x)).$$

It follows that,

$$\|T^{-1}(x)\|_1 = \sum_{k=1}^n |f_k(x)| \le \sum_{k=1}^n \|f_k\|_{\mathscr{X}^*} \|x\|_{\mathscr{X}} \le n \|x\|_{\mathscr{X}},$$

where we used the property $\|f_k\|_{\mathscr{X}^*}$ of the dual basis. ∎

Lemma 2.45 — Riesz. Let $\mathscr{X}$ be a normed space and let $\mathscr{X}_0$ be a closed subspace (in a strict sense). Then, for every $\varepsilon > 0$ there exists a unit vector $x \in \mathscr{X}$, $\|x\| = 1$, such that

$$d(x, \mathscr{X}_0) \geq 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Choose an arbitrary $y \in \mathscr{X} \setminus \mathscr{X}_0$ and denote

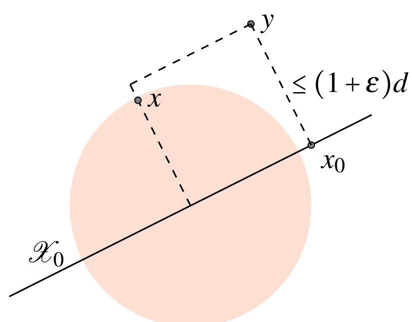
$$d = d(y, \mathscr{X}_0).$$

$d > 0$ because $\mathscr{X}_0$ is closed. By the definition of the distance of a point from a set as an infimum, there is an $x_0 \in \mathscr{X}_0$ such that[3]

$$\|y - x_0\| \leq (1 + \varepsilon) d.$$

Set

$$x = \frac{y - x_0}{\|y - x_0\|}.$$



For all $z \in \mathscr{X}_0$,

$$\|x - z\| = \left\| \frac{y - x_0}{\|y - x_0\|} - z \right\| = \frac{\|y - z\overbrace{\|y - x_0\| - x_0}^{\in \mathscr{X}_0}\|}{\|y - x_0\|} \geq \frac{d}{d(1 + \varepsilon)} = \frac{1 - \varepsilon}{1 - \varepsilon^2} \geq 1 - \varepsilon.$$

■

Riesz's lemma allows us to characterize the finite-dimensional normed spaces:

Theorem 2.46 A normed space $\mathscr{X}$ has finite dimension if and only if its unit ball $\mathfrak{B}_{\mathscr{X}}$ is **totally bounded** (חסום כליל). That is, for every $\varepsilon > 0$ there exists a finite cover of $\mathfrak{B}_{\mathscr{X}}$ by balls of radius ε.

[3] Unlike in Hilbert space, there is no guarantee that a minimizer exists.

Comment 2.10 A compact set is totally bounded (obvious: express the set as a union of open balls of radius ε around each point; by compactness there exists a finite sub-cover). The converse is not true. In a complete metric space, however, a closed, totally bounded set is compact. Why? Take a sequence x_n. For every $\varepsilon > 0$ the space can be covered by a finite number of ε-balls, and at least of one of them contains an infinite subsequence. We may proceed inductively and, using a diagonalization argument, construct a subsequence that satisfies Cauchy's criterion. Since the space is complete, this subsequence converges. In a metric space compactness coincides with sequential compactness.

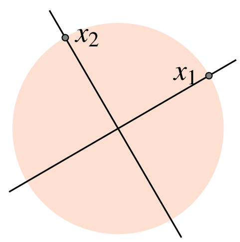
Proof. We have already seen (Corollary 2.42) that in a finite-dimensional normed space every closed and bounded set is compact, hence $\mathfrak{B}_{\mathscr{X}}$ is totally bounded.

Conversely, let $\mathfrak{B}_{\mathscr{X}}$ be totally bounded. Suppose, by contradiction, that $\mathscr{X}$ has infinite dimension. Take $x_1 \in \partial\mathfrak{B}_{\mathscr{X}} = S$, and set

$$\mathscr{X}_1 = \operatorname{Span}\{x_1\}.$$

Since $\mathscr{X}$ is infinite-dimensional, $\mathscr{X}_1$ is a closed proper subspace of $\mathscr{X}$. By Riesz's lemma there exists an $x_2 \in S$ such that

$$d(x_2, \mathscr{X}_1) > \tfrac{1}{2}.$$



Set

$$\mathscr{X}_2 = \operatorname{Span}\{x_1, x_2\}.$$

$\mathscr{X}_2$ is a closed proper subspace of $\mathscr{X}$. By Riesz's lemma there exists an $x_3 \in S$ such that

$$d(x_3, \mathscr{X}_2) > \tfrac{1}{2}.$$

Proceed inductively to construct an infinite sequence $(x_n) \subset S$, such that the distance between any pair is larger that $\frac{1}{2}$. It follows that $S \subset \mathfrak{B}_{\mathscr{X}}$ cannot be covered by a finite number of balls of radius $\frac{1}{4}$, i.e., it is not totally bounded.

∎

Corollary 2.47 In an infinite-dimensional normed space the closed unit ball is not compact.

Proof. Compactness implies total boundedness, and we have just seen that if the closed unit ball is totally bounded then the space is finite-dimensional. ■

TA material 2.3

Definition 2.7 Let $\mathscr{X}$ and $\mathscr{Y}$ be finite dimensional spaces of the same dimension. Their **Banach-Mazur distance** is

$$d(\mathscr{X},\mathscr{Y}) = \inf\{\|T\|\|T^{-1}\| \mid T:\mathscr{X}\to\mathscr{Y} \text{ is an isomorphism}\}.$$

Note that we can always choose $\|T\|$ or $\|T^{-1}\|$ to be 1 by multiplying T by a scalar. Note also that if $c, C > 0$ are the constants in the definition of the isomorphism, then C can be chosen to be $\|T\|$ or larger, and c can be chosen to be $\|T^{-1}\|$ or smaller. Hence

$$d(\mathscr{X},\mathscr{Y}) = \inf\left\{\frac{C}{c} \mid c, C > 0 \text{ are isomorphism constants}\right\}.$$

Proposition 2.48 Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathcal{Z}$ be finite dimensional spaces of the same dimension. Then,

① $d(\mathscr{X},\mathscr{Y}) = d(\mathscr{Y},\mathscr{X})$.
② $d(\mathscr{X},\mathscr{Y}) \le d(\mathscr{X},\mathcal{Z})\,d(\mathcal{Z},\mathscr{Y})$.
③ $d(\mathscr{X},\mathscr{Y}) \ge 1$ with equality if and only if $\mathscr{X}$ and $\mathscr{Y}$ are isometric.

Corollary 2.49 The logarithm of the Banach-Mazur distance is a metric on the collection of n-dimensional normed spaces modulo isometries.

■ **Example 2.8** We proved that for every n-dimensional normed space $\mathscr{X}$,

$$d(\mathscr{X},\ell_1^n) \le n.$$

■

■ **Example 2.9** Taking the identity map, it follows from the Cauchy-Schwarz identity that

$$d(\ell_1^n,\ell_2^n) \le \sqrt{n}.$$

It can be shown, using the generalized parallelogram identity, that this is in fact an equality. ■

Exercise 2.6 Let $\mathscr{X}$ be a normed space. Every subspace of $\mathscr{X}$ is closed. Prove that $\mathscr{X}$ is finite-dimensional. ▪

Exercise 2.7

① Show that ℓ_1^n and ℓ_∞^n (R) are isometric if and only if $n \le 2$. Hint: characterize the unit balls for $n = 3$.

② Let $p \in [1,2)$. Prove that

$$d(\ell_p^n, \ell_2^n) = n^{1/p-1/2}.$$

Conclude that there are no $p, q \in [1,2)$ such that ℓ_p^n and ℓ_q^n are isometric.

▪

2.3.2 Compact sets in infinite-dimensional spaces

In finite-dimensional normed spaces, the compact sets coincide with the closed and bounded sets. This is not so in infinite-dimensional spaces. The question then is what else is needed to guarantee the compactness of closed bounded sets. This question is of practical value in many applications, as compactness is a major tool, e.g., in existence proofs. In this subsection we will see a two particular examples.

Proposition 2.50 Let $\mathscr{X} = \ell_p$, $1 \le p < \infty$ (the same applies to $\mathscr{X} = c_0$). Let P_n be the projections:

$$P_n(x_1, x_2, \dots) = (x_1, \dots, x_n, 0, 0, \dots)$$

(P_n projects ℓ_p onto ℓ_p^n). A closed bounded set $A \subset \mathscr{X}$ is compact if and only if

$$\lim_{n\to\infty} \|P_n x - x\| = 0$$

uniformly on A. That is, for every $\varepsilon > 0$ there exists an N such that

$$\sup_{\substack{x\in A\\ n>N}} \|P_n x - x\| < \varepsilon.$$

Proof. **Part 1: Suppose A is compact**: It follows that it is totally bounded. Given $\varepsilon > 0$, there exists a finite set $C \subset A$ such that

$$A \subseteq \bigcup_{y\in C} B(y, \varepsilon).$$

(such a set C is called an **ε-net** (אפסילון רשת) of A). In other words:

$$\sup_{x\in A}\min_{y\in C}\|x-y\|<\varepsilon.$$

Since $P_n y\to y$ for every $y\in C$ (a property of the space ℓ_p) and since C is a finite set, then there is an N:

$$\sup_{\substack{y\in C\\ n>N}}\|P_n y-y\|<\varepsilon.$$

Thus,

$$\begin{aligned}\sup_{\substack{x\in A\\ n>N}}\|P_n x-x\| &= \sup_{\substack{x\in A\\ n>N}}\min_{y\in C}\|P_n x-x\|\\ &\le \sup_{\substack{x\in A\\ n>N}}\min_{y\in C}\left(\|P_n x-P_n y\|+\|P_n y-y\|+\|y-x\|\right)\\ &\le \sup_{\substack{x\in A\\ n>N}}\min_{y\in C}\left(\|P_n x-P_n y\|+\|y-x\|\right)+\sup_{n>N}\max_{y\in C}\|P_n y-y\|\\ &<3\varepsilon,\end{aligned}$$

where in the last step we used the fact that $\|P_n\|=1$. This proves that $P_n x\to x$ uniformly on A.

Part 2: Suppose $P_n x\to x$ **uniformly in** A: Let $\varepsilon>0$ be given. There is an $n\in\mathbb{N}$ such that:

$$\sup_{x\in A}\|P_n x-x\|<\varepsilon.$$

Consider the set

$$B_n=\{P_n y \mid y\in A\},$$

i.e., the set of elements of A truncated at the n-th entry. B_n is isomorphic to a closed and bounded set in ℓ_p^n, hence it is compact (and totally bounded). It follows that B_n can be covered by an finite ε-net: there is a finite set of vectors $C\subset A$, such that

$$B_n\subset\bigcup_{y\in C}B(P_n y,\varepsilon).$$

In other words,

$$\sup_{x\in A}\min_{y\in C}\|P_n x-P_n y\|<\varepsilon.$$

Hence,

$$\begin{aligned}\sup_{x\in A}\min_{y\in C}\|x-y\| &\le \sup_{x\in A}\min_{y\in C}\left(\|x-P_n x\|+\|P_n x-P_n y\|+\|P_n y-y\|\right)\\ &\le \sup_{x\in A}\|x-P_n x\|+\sup_{x\in A}\min_{y\in C}\|P_n x-P_n y\|+\max_{y\in C}\|P_n y-y\|\\ &<3\varepsilon.\end{aligned}$$

This proves that $A\subset\bigcup_{y\in C}B(y,3\varepsilon)$, i.e., A is totally bounded. Since the space is complete and A is both totally bounded and closed, A is compact. ■

The next theorem is classical (used, for example, to prove the existence of solutions to initial value problems). The proof may differ somewhat from the proof you are acquainted with.

Theorem 2.51 — Arzela-Ascoli, 1883. Let K is a compact metric space and let $C(K)$ be the space of continuous functions on K endowed with the supremum norm (uniform convergence topology). Let A be a closed, bounded set in $C(K)$. Then A is compact if it is **equi-continuous** (רציף במידה אחידה), that is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{f\in A} \sup_{d(x,y)<\delta} |f(x) - f(y)| < \varepsilon.$$

Proof. **Part 1: Suppose that A is closed, bounded and equi-continuous**: Let $\varepsilon > 0$ be given. Let $\delta > 0$ be given by the definition of equi-continuity. Since K is compact, then it is totally bounded and there exists a finite set of points

$$K' = (x_1, x_2, \dots, x_n)$$

that forms a δ-net, namely

$$\sup_{x\in K} \min_{y\in K'} d(x,y) < \delta.$$

It follows that

$$\sup_{f\in A} \sup_{x\in K} \min_{y\in K'} |f(x) - f(y)| < \varepsilon.$$

Consider the set

$$\{(f(x_1), f(x_2), \dots, f(x_n)) \mid f \in A\} \subset \mathbb{R}^n.$$

It is bounded and therefore totally bounded (it is not necessarily compact, because it is not necessarily closed). It follows that there exists a finite set of functions $A' \subset A$ such that

$$\sup_{f\in A} \min_{g\in A'} \max_{y\in K'} |f(y) - g(y)| < \varepsilon.$$

Then,

$$\begin{aligned}
\sup_{f\in A} \min_{g\in A'} \sup_{x\in K} |f(x) - g(x)| &= \sup_{f\in A} \min_{g\in A'} \sup_{x\in K} \min_{y\in K'} |f(x) - g(x)| \\
&\le \sup_{f\in A} \min_{g\in A'} \sup_{x\in K} \min_{y\in K'} (|f(x) - f(y)| + |f(y) - g(y)| + |g(y) - g(x)|) \\
&\le \sup_{f\in A} \sup_{g\in A'} \sup_{x\in K} \min_{y\in K'} (|f(x) - f(y)| + |g(y) - g(x)|) \\
&\quad + \sup_{f\in A} \min_{g\in A'} \max_{y\in K'} |f(y) - g(y)| < 3\varepsilon.
\end{aligned}$$

It follows that $A \subset \bigcup_{g \in A'} B(g, 3\varepsilon)$, i.e., A is totally bounded, and since $C(K)$ is a complete metric space, A is also compact.

Part 2: Suppose that A is compact: Let $\varepsilon > 0$ be given. Since A is totally bounded there exists a finite set $A' \subset A$ such that

$$\sup_{f \in A} \min_{g \in A'} \| f - g \| < \varepsilon.$$

Since each of the $g \in A'$ is (uniformly) continuous, there is a $\delta > 0$ such that

$$\max_{g \in A'} \sup_{d(x,y) < \delta} |g(x) - g(y)| < \varepsilon.$$

Now,

$$\begin{aligned}
\sup_{f \in A} \sup_{d(x,y) < \delta} |f(x) - f(y)| &= \sup_{f \in A} \sup_{d(x,y) < \delta} \min_{g \in A'} |f(x) - f(y)| \\
&\le \sup_{f \in A} \sup_{d(x,y) < \delta} \min_{g \in A'} \left(|f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \right) \\
&< 3\varepsilon.
\end{aligned}$$

This proves that the set A is equi-continuous. ■

2.4 The Hahn-Banach theorem

The Hahn-Banach theorem (proved independently by Hans Hahn and Stefan Banach in the late 1920s) is a central result in functional analysis. Its context is the following: in many cases we study Banach spaces $\mathscr{B}$ through their duals $\mathscr{B}^*$. The question is what is the guarantee that the dual is not trivial (contains only the zero functional). The Hahn-Banach theorem allows the extension of bounded linear functionals defined on a subspace of $\mathscr{B}$ to the whole space, and in particular shows that there are "enough" continuous linear functionals to make the study of $\mathscr{B}^*$ interesting. Specifically, a consequence of the Hahn-Banach theorem is that if $\langle f, x \rangle$ for all f in a dense set in $\mathscr{X}^*$, then $x = 0$.

Note that in a finite dimensional vector space we are used to study vectors through their coordinates. What is the coordinate of a vector? The value returned by a bounded functional. In particular, it takes n independent linear functional in order to "know everything" about a vector. The situation becomes more complicated in infinite-dimensional spaces. It is in this context that the Hahn-Banach theorem tells us that there the dual space is sufficiently rich in order to study a vector through the values returned by bounded linear functionals.

Theorem 2.52 — Hahn-Banach. Let $\mathscr{V}$ be a real vector space and let p a functional over $\mathscr{V}$ satisfying:

① Sub-linearity: $p(x+y) \le p(x) + p(y)$.

② Homogeneity: for every $\alpha \geq 0$, $p(\alpha x) = \alpha p(x)$.

Let $\mathscr{Y} \subset \mathscr{V}$ be a linear subspace and let $f : \mathscr{Y} \to \mathbb{R}$ be a linear functional satisfying:

$$f \leq p|_{\mathscr{Y}}.$$

Then, there is a linear functional F on $\mathscr{V}$ satisfying $F|_{\mathscr{Y}} = f$ and $F \leq p$.

Comments 2.1

① Important: despite the fact that this theorem is associated with Banach, $\mathscr{V}$ doesn't need to be a Banach space; in fact it doesn't even need to be normed.

② Note the condition that the vector space be over $\mathbb{R}$.

③ There is no requirement that the linear functional f be bounded (there is no norm).

④ An important functional p that satisfies the required conditions in the case of a normed space is the norm.

Proof. The Hahn-Banach theorem follows from the axiom of choice. Consider the collection of all **linear extensions** $(F, \mathscr{V}_F)$ of $(f, \mathscr{Y})$, where $\mathscr{Y} \subseteq \mathscr{V}_F \subseteq \mathscr{V}$, $F|_{\mathscr{Y}} = f$, satisfying $F \leq p|_{\mathscr{V}_F}$. We define a partial order among linear extensions according to their domains: $F_1 \geq F_2$ if $\mathscr{V}_{F_1} \supseteq \mathscr{V}_{F_2}$ and $F_1|_{\mathscr{V}_{F_2}} = F_2$. Take any chain $\{(F_\alpha, \mathscr{V}_\alpha) \mid \alpha \in A\}$ and set

$$D = \bigcup_\alpha \mathscr{V}_\alpha.$$

For every $x \in D$ there exists an α such that $x \in \mathscr{V}_\alpha$, hence we can unambiguously define a functional F in D by $\langle F, x\rangle = \langle F_\alpha, x\rangle$. The functional F is a linear extension of f that satisfies $F \leq p|_D$; it is an upper bound of the chain. It follows from Zorn's lemma that there exists a maximal element $(F, \mathscr{V}_F)$ to all linear extension of f.

It remain to show that $\mathscr{V}_F = \mathscr{V}$. For that, we show that if $\mathscr{V}_F \subsetneq \mathscr{V}$, then we can extend F, contradicting its maximality.

So generally, let's show that if $\mathscr{Y} \subsetneq \mathscr{V}$, then we can extend f. Take $x_0 \in \mathscr{V} \setminus \mathscr{Y}$, and set

$$\mathscr{V}_1 = \operatorname{Span}\{\mathscr{Y}, x_0\}.$$

Every vector $x \in \mathscr{V}_1$ has a unique representation as $x = y + \alpha x_0$, where $y \in \mathscr{Y}$. Define a linear functional $f_1 : \mathscr{V}_1 \to \mathbb{R}$, by setting $\langle f_1, x_0\rangle = \beta$, i.e.,

$$\langle f_1, y + \alpha x_0\rangle = \langle f, y\rangle + \alpha\beta,$$

and β will be chosen later. Note that $f_1|_{\mathscr{Y}} = f$, i.e., f_1 is a **linear extension** of f on $\mathscr{V}_1$. We want $f_1 \leq p|_{\mathscr{V}_1}$, i.e., we want for every $\alpha \in \mathbb{R}$,

$$\langle f, y\rangle + \alpha\beta \leq p(y + \alpha x_0).$$

In particular, we want for $\alpha = \pm 1$, and any y, y'

$$\langle f, y\rangle + \beta \le p(y + x_0) \quad \text{and} \quad \langle f, y'\rangle - \beta \le p(y' - x_0),$$

which means that

$$\langle f, y'\rangle - p(y' - x_0) \le \beta \le p(y + x_0) - \langle f, y\rangle.$$

Such a β exists because for every y, y':

$$\begin{aligned}\langle f, y'\rangle + \langle f, y\rangle &= \langle f, y' + y\rangle = \langle f, y + x_0 + y' - x_0\rangle \\ &\overset{f\le p}{\le} p(y + x_0 + y' - x_0) \overset{p \text{ is sub-lin.}}{\le} p(y + x_0) + p(y' - x_0),\end{aligned}$$

which implies that

$$\sup_{y'} \left(\langle f, y'\rangle - p(y' - x_0)\right) \le \inf_{y} \left(p(y + x_0) - \langle f, y\rangle\right).$$

Having chosen β this way we have for every $\alpha \ge 0$:

$$\langle f_1, y + \alpha x_0\rangle = \alpha \left\langle f_1, \frac{y}{\alpha} + x_0\right\rangle \le \alpha p\left(\frac{y}{\alpha} + x_0\right) = p(y + \alpha x_0),$$

whereas for $\alpha < 0$,

$$\langle f_1, y + \alpha x_0\rangle = |\alpha| \left\langle f_1, \frac{y}{|\alpha|} - x_0\right\rangle \le |\alpha| p\left(\frac{y}{|\alpha|} - x_0\right) = p(y + \alpha x_0).$$

It follows that $f_1|_{\mathscr{Y}} = f$ and $f_1 \le p|_{\mathscr{Y}_1}$. ■

Exercise 2.8 Prove the complex version of Hahn-Banach theorem: let $\mathscr{X}$ be a complex vector space and p a semi-norm on $\mathscr{X}$ (note that this is slightly different from the real case). If $\mathscr{Y} \subset \mathscr{X}$ is a linear subspace and $f : \mathscr{Y} \to \mathbb{C}$ is a linear functional satisfying $|f| \le p|_{\mathscr{Y}}$, then there is a linear functional $F : \mathscr{X} \to \mathbb{C}$ extending f and satisfying $|F| \le p$. Hint: consider $\mathscr{X}$ as a real vector space, and extend $f_1 = \operatorname{Re} f$ to a linear functional g on $\mathscr{X}$; finally define $F(x) = g(x) - ig(ix)$. ■

The following is a first application of the Hahn-Banach theorem:

Theorem 2.53 — Extension theorem. Let $\mathscr{X}$ be a normed space and $\mathscr{Y}$ a linear subspace. Let $f \in \mathscr{Y}^*$. Then there is a linear extension $F \in \mathscr{X}^*$ of f (namely $F|_{\mathscr{Y}} = f$) such that $\|F\| = \|f\|$.

Comment 2.11 Sometimes people refer to this theorem as the Hahn-Banach theorem.

Comment 2.12 We proved this theorem for Hilbert spaces. There, however, we used an orthogonal projection from $\mathscr{X}$ to $\mathscr{Y}$, something that has no analog in Banach spaces.

Proof. The theorem holds for complex normed spaces. The proof starts by considering real spaces:

Case 1: $\mathscr{X}$ is a real vector space: For $x \in \mathscr{X}$ define $p(x) = \|f\|\|x\|$. Then p satisfies

$$p(x+y) = \|f\|\|x+y\| \le \|f\|\left(\|x\| + \|y\|\right) = p(x+y),$$

and for $\alpha > 0$,

$$p(\alpha x) = \|f\|\,\|\alpha x\| = \alpha\,\|f\|\|x\| = \alpha\, p(x).$$

Moreover, for $y \in \mathscr{Y}$,

$$\langle f, y\rangle \le |\langle f, y\rangle| \le \|f\|\|y\| = p(y).$$

It follows from the Hahn-Banach theorem there is a linear functional F on $\mathscr{X}$ such that $F|_{\mathscr{Y}} = f$ and

$$F(x) \le p(x) = \|f\|\|x\|.$$

That is, $F \in \mathscr{X}^*$ with $\|F\| \le \|f\|$; since F is an extension of f, it follows that $\|F\| = \|f\|$. (Please note that the sign does not constitute a problem since we can always replace x by $-x$.)

Case 2: $\mathscr{X}$ is a complex vector space: We denote by $\mathscr{X}_{\mathbb{R}}$ the real vector space obtained by restricting the scalars to $\mathbb{R}$ (the sets $\mathscr{X}$ and $\mathscr{X}_{\mathbb{R}}$ are the same sets, but x and $\imath x$ are not co-linear in $\mathscr{X}_{\mathbb{R}}$). A functional can be linear over $\mathscr{X}_{\mathbb{R}}$ but not over $\mathscr{X}$, because we only require that

$$\langle f, \alpha x + \beta y\rangle = \alpha\langle f, x\rangle + \beta\langle f, y\rangle$$

for $\alpha, \beta \in \mathbb{R}$.

Every linear functional f on $\mathscr{Y}$ can be decomposed as

$$f = u + \imath v,$$

where u and v are the real and imaginal parts of f. u and v are linear functionals on $\mathscr{Y}_{\mathbb{R}}$ (but no necessarily on $\mathscr{Y}$!), since for $\alpha \in \mathbb{R}$,

$$\langle f, \alpha x\rangle = u(\alpha x) + \imath v(\alpha x) = \alpha\langle f, x\rangle = \alpha\left(u(x) + \imath v(x)\right),$$

namely,

$$u(\alpha x) = \alpha\, u(x) \qquad \text{and} \qquad v(\alpha x) = \alpha\, v(x).$$

We will thus write $u(x) = \langle u, x\rangle_{\mathbb{R}}$ and $v(x) = \langle v, x\rangle_{\mathbb{R}}$.

f is in fact determined by u. Since f is linear over $\mathbb{C}$:

$$\langle f, \imath x\rangle = \langle u, \imath x\rangle_{\mathbb{R}} + \imath\langle v, \imath x\rangle_{\mathbb{R}} = \imath\langle f, x\rangle = \imath\langle u, x\rangle_{\mathbb{R}} - \langle v, x\rangle_{\mathbb{R}},$$

from which follows that

$$\langle v, x\rangle_{\mathbb{R}} = -\langle u, \imath x\rangle_{\mathbb{R}},$$

hence

$$\langle f, x\rangle = \langle u, x\rangle_{\mathbb{R}} - \imath\langle u, \imath x\rangle_{\mathbb{R}}.$$

Consider now the functional u. For $y \in \mathscr{Y}$:

$$|\langle u, y\rangle_{\mathbb{R}}| = |u(y)| \le \|u(y) + \imath v(y)\| = |\langle f, y\rangle| \le \|f\|\|y\|,$$

hence $u \in \mathscr{Y}_{\mathbb{R}}^*$ with $\|u\| \le \|f\|$. By the first part of this proof u can be extended into a linear functional U on $\mathscr{X}_{\mathbb{R}}$, such that $\|U\| \le \|f\|$.

Define $F \in \mathscr{X}_{\mathbb{R}}^*$ by

$$\langle F, x\rangle_{\mathbb{R}} = \langle U, x\rangle_{\mathbb{R}} - \imath\langle U, \imath x\rangle_{\mathbb{R}}.$$

F is a linear functional on $\mathscr{X}_{\mathbb{R}}$ satisfying:

$$\langle F, \imath x\rangle_{\mathbb{R}} = \langle U, \imath x\rangle_{\mathbb{R}} - \imath\langle U, -x\rangle_{\mathbb{R}} = \langle U, \imath x\rangle_{\mathbb{R}} + \imath\langle U, x\rangle_{\mathbb{R}} = \imath\langle F, x\rangle_{\mathbb{R}},$$

i.e., F is linear over $\mathbb{C}$, $F \in \mathscr{X}^*$. F is an extension of f since for $y \in \mathscr{Y}$:

$$\langle F, y\rangle = \langle U, y\rangle_{\mathbb{R}} - \imath\langle U, \imath y\rangle_{\mathbb{R}} = \langle u, y\rangle_{\mathbb{R}} - \imath\langle u, \imath y\rangle_{\mathbb{R}} = \langle f, y\rangle.$$

It remains to bound the norm of F. The Hahn-Banach theorem only guarantees a bound on the norm of U. For $x \in \mathscr{X}$

$$|\langle F, x\rangle| = e^{-\imath \arg\langle F, x\rangle}\langle F, x\rangle = \langle F, e^{-\imath \arg\langle F, x\rangle} x\rangle = \langle U, e^{-\imath \arg\langle F, x\rangle} x\rangle_{\mathbb{R}},$$

where the last equality follows because the imaginary part must be zero. Hence,

$$|\langle F, x\rangle| \le \langle U, e^{-\imath \arg\langle F, x\rangle} x\rangle_{\mathbb{R}} \le \|f\|\|x\|,$$

from which follows that $\|F\| \le \|f\|$, and since F is an extension of f, $\|F\| = \|f\|$. ■

The Hahn-Banach theorem and the resulting extension theorem have many applications, some of which we will see now.

Proposition 2.54 Let $\mathscr{X}$ be a normed space and let $\mathscr{Y} \subset \mathscr{X}$ be a linear subspace. Denote by $\mathscr{Y}^{\perp} \subset \mathscr{X}^*$:

$$\mathscr{Y}^{\perp} = \{f \in \mathscr{X}^* \mid f|_{\mathscr{Y}} = 0\}.$$

Then for all $x \in \mathscr{X} \setminus \mathscr{Y}$:

$$d(x, \mathscr{Y}) = \max_{f \in \mathfrak{B}_{\mathscr{Y}^{\perp}}} |\langle f, x\rangle|.$$

(This holds trivially for $x \in \mathscr{Y}$.)

Comment 2.13 Note that there is no minimizer of $d(x,\mathscr{Y})$ in $\mathscr{Y}$ but there is a maximizer of $|\langle f,x\rangle|$ in $\mathfrak{B}_{\mathscr{Y}^\perp}$. This hints to a type of compactness of the unit ball in $\mathscr{X}^*$.

Comment 2.14 Does it remind you something? What happens in a Hilbert space? One can write an explicit expression for $d(x,\mathscr{Y})$ in terms of a Gram matrix.

Proof. Denote $d = d(x,\mathscr{Y})$. For $f \in \mathfrak{B}_{\mathscr{Y}^\perp}$, we have for all $y \in \mathscr{Y}$:

$$\sup_{f\in\mathfrak{B}_{\mathscr{Y}^\perp}} |\langle f,x\rangle| = \sup_{f\in\mathfrak{B}_{\mathscr{Y}^\perp}} \inf_{y\in\mathscr{Y}} |\langle f,x-y\rangle| \le \left(\sup_{f\in\mathfrak{B}_{\mathscr{Y}^\perp}} \|f\|\right) \inf_{y\in\mathscr{Y}} \|x-y\| = d(x,\mathscr{Y}).$$

i.e.,

$$d(x,\mathscr{Y}) \ge \sup_{f\in\mathfrak{B}_{\mathscr{Y}^\perp}} |\langle f,x\rangle|.$$

If d is zero (which can happen if $\mathscr{Y}$ is a dense linear subset), then

$$\max_{f\in\mathfrak{B}_{\mathscr{Y}^\perp}} |\langle f,x\rangle| = 0,$$

and the proposition holds.

Suppose then that $d > 0$. We need to show that there is an $f \in \mathfrak{B}_{\mathscr{Y}^\perp}$ such that $\langle f,x\rangle = d$. Let $\mathscr{Z} = \operatorname{Span}\{\mathscr{Y},x\}$; every $z \in \mathscr{Z}$ has a unique representation:

$$z = y + \alpha x.$$

Define a linear functional f_0 on $\mathscr{Z}$:

$$\langle f_0, y+\alpha x\rangle = \alpha d,$$

i.e., $f_0 \in \mathscr{Y}^\perp$ (with $\mathscr{Y}$ as a subset of $\mathscr{Z}$), and $\langle f_0,x\rangle = d$.

For every $y \in \mathscr{Y}$ and $\alpha \in \mathbb{R}$:

$$|\langle f_0, y+\alpha x\rangle| = |\alpha| d \le |\alpha| \underbrace{\|y/\alpha + x\|}_{\text{dist of } -y/\alpha \text{ from } x} = \|y+\alpha x\|,$$

which implies that $\|f_0\| \le 1$. Take $y_n \in \mathscr{Y}$ such that $d(x,y_n) \to d$, then

$$d = |\langle f_0, x-y_n\rangle| \le \|f_0\|\|x-y_n\| \to \|f_0\| d,$$

which implies that $\|f_0\| \ge 1$, i.e., $\|f_0\| = 1$. We have thus found an $f_0 \in \mathfrak{B}_{\mathscr{Y}^\perp}$ linear over $\mathscr{Z}$, such that $\langle f_0,x\rangle = d$. The only problem is that f_0 is a linear functional on $\mathscr{Z}$ rather than on $\mathscr{X}$. By the extension theorem we can extend f_0 to all of $\mathscr{X}$ without changing its norm, which completes the proof. ∎

Corollary 2.55 Let $\mathscr{X}$ be a normed space, then

$$\|x\| = \max_{f \in \mathfrak{B}_{\mathscr{X}^*}} |\langle f, x\rangle|.$$

That is, there is an $f \in \mathscr{X}^*$, $\|f\| = 1$, such that

$$\langle f, x\rangle = \|x\|.$$

Proof. Take $\mathscr{Y} = \{0\}$ in Proposition 2.54. Then $d(x, \mathscr{Y}) = \|x\|$ and $\mathscr{Y}^\perp = \mathscr{X}^*$. ∎

Comment 2.15 We have an explicit expression of the norm in terms of $\mathscr{X}^*$.

The following corollary, though by now immediate, it of utter importance. It asserts that in order for a vector to be zero it is sufficient to “test" it with a dense set of bounded linear functionals.

Corollary 2.56 Let $\mathscr{X}$ be a normed space and let A be a dense set in $\mathscr{X}^*$. If $\langle f, x\rangle = 0$ for all $f \in A$ then $x = 0$.

Proof. Let $f \in B_{\mathscr{X}^*}$. There exists a sequence $f_n \in A$ such that $f_n \to f$. Then,

$$0 = \langle f_n, x\rangle \to \langle f, x\rangle,$$

which by Corollary 2.55 implies that

$$\|x\| = \max_{f \in \mathfrak{B}_{\mathscr{X}^*}} |\langle f, x\rangle| = 0.$$

∎

Corollary 2.57 Let $\mathscr{X}$ be a normed space and let $\mathscr{Y} \subset \mathscr{X}$ be a linear subspace. Then $x \in \overline{\mathscr{Y}}$ if and only if every bounded linear functional that vanishes on $\mathscr{Y}$ vanishes also on x.

In particular, $\mathscr{Y}$ in dense in $\mathscr{X}$ if and only the only functional f that vanishes on $\mathscr{Y}$ is the zero functional.

Proof. Suppose that $x \in \overline{\mathscr{Y}}$ and let x_n be a sequence in $\mathscr{Y}$ that converges to x. For all $f \in \mathscr{Y}^\perp$,

$$\langle f, x\rangle = \lim_{n\to\infty} \langle f, x_n\rangle = 0.$$

Conversely, suppose that $x \in \mathscr{X}$ satisfies that $\langle f, x\rangle = 0$ for all $f \in \mathscr{Y}^\perp$. Then,

$$d(x, \mathscr{Y}) = \max_{f \in \mathfrak{B}_{\mathscr{Y}^\perp}} |\langle f, x\rangle| = 0,$$

i.e., $x \in \overline{\mathscr{Y}}$.

In particular, if $\mathscr{Y}$ is dense in $\mathscr{X}$, then $x \in \overline{\mathscr{Y}} = \mathscr{X}$ implies that every bounded linear functional that vanishes on $\mathscr{Y}$ vanishes also on x, but the condition holds trivially. ■

The following theorem, which is also a consequence of the Hahn-Banach theorem, shows that the separability of the dual implies the separability of the space.

Theorem 2.58 Let $\mathscr{X}$ be a normed space. If $\mathscr{X}^*$ is a separable then so is $\mathscr{X}$.

Proof. Let (f_n) be a dense sequence in $\mathscr{X}^*$. Choose (x_n) such that $\|x_n\| \le 1$ and $|\langle f_n, x_n\rangle| \ge \frac{1}{2}\|f_n\|$; this is always possible by the definition of the norm.

Let $\mathscr{Y} = \operatorname{Span}\{x_n\}$; it is separable because we can take all combinations of $\{x_n\}$ with rational coefficients. Let $f \in \mathscr{Y}^\perp$. By the density of (f_n), there exists a subsequence $f_{n_k} \to f$. Then

$$\frac{1}{2}\|f_{n_k}\| \le |\langle f_{n_k}, x_{n_k}\rangle| = |\langle f - f_{n_k}, x_{n_k}\rangle| \le \|f - f_{n_k}\| \to 0,$$

which proves that $f_{n_k} \to 0$, i.e., $f = 0$.

We have just proved that every functional that vanishes on $\mathscr{Y}$ is the zero functional. It follows by Corollary 2.57 that $\mathscr{Y}$ is dense in $\mathscr{X}$, i.e., $\mathscr{X}$ is separable. ■

Proposition 2.59 Let $\{x_\alpha \mid \alpha \in A\}$ be a collection of vectors in a normed space $\mathscr{X}$. If

$$\forall f \in \mathscr{X}^* \qquad \sup_\alpha |\langle f, x_\alpha\rangle| < \infty,$$

then

$$\sup_\alpha \|x_\alpha\| < \infty.$$

Comment 2.16 See how similar it is to the Banach-Steinhaus theorem. In the former, the sequence was a sequence of functionals, and

$$\forall x \in \mathscr{X} \qquad \sup_\alpha |\langle f_\alpha, x\rangle| < \infty,$$

implies

$$\sup_\alpha \|f_\alpha\| < \infty.$$

In a Hilbert space $\mathscr{X}$ and $\mathscr{X}^*$ are isomorphic, so the two coincide. This is not the case in a Banach space.

Proof. Define $F_\alpha \in (\mathscr{X}^*)^*$ as follows:

$$\langle F_\alpha, f \rangle = \langle f, x_\alpha \rangle$$

(later on, we will call F_α the natural inclusion of x_α in $\mathscr{X}^{**}$). It is given that for every $f \in \mathscr{X}^*$:

$$\sup_{\alpha \in A} |\langle F_\alpha, f \rangle| = \sup_{\alpha \in A} |\langle f, x_\alpha \rangle| < \infty.$$

It follows from the Banach-Steinhaus theorem that

$$\sup_{\alpha \in A} \|F_\alpha\| < \infty.$$

We will be done if we show that $\|F_\alpha\| = \|x_\alpha\|$ (the natural inclusion is norm-preserving). Since

$$|\langle F_\alpha, f \rangle| = |\langle f, x_\alpha \rangle| \le \|f\| \|x_\alpha\|,$$

it follows that $\|F_\alpha\| \le \|x_\alpha\|$. By Corollary 2.55,

$$\forall \alpha \in A \quad \exists g_\alpha \in B_{\mathscr{X}^*} \quad \text{such that} \quad \|x_\alpha\| = \langle g_\alpha, x_\alpha \rangle.$$

Hence

$$\|F_\alpha\| = \sup_{f \in \mathfrak{B}_{\mathscr{X}^*}} |\langle F_\alpha, f \rangle| \ge |\langle F_\alpha, g_\alpha \rangle| = |\langle g_\alpha, x_\alpha \rangle| = \|x_\alpha\|,$$

i.e., $\|F_\alpha\| = \|x_\alpha\|$. ∎

Comment 2.17 We have proved along the way that $\mathscr{X}$ is isometric to a subspace of $(\mathscr{X}^*)^*$. This point will be elaborated when we deal more in depth with duality.

Exercise 2.9

① Let $\mathscr{X} = \mathbb{R}^3$ with the Euclidean norm and let

$$\mathscr{Y} = \{(x,y,z) \in \mathscr{X} \mid x + 2y = 0\}.$$

Define on $\mathscr{Y}$ the functional $\langle f, (x,y,z) \rangle = x - y$. Find $\|f\|$ and extend f to $\mathscr{X}$ without changing the norm.

② Let $\mathscr{X} = \mathbb{R}^3$ with the norm $\|\cdot\|_1$ and let

$$\mathscr{Y} = \{(x,y,z) \in \mathscr{X} \mid x + 2y = z = 0\}.$$

Define on $\mathscr{Y}$ the functional $\langle f, (x,y,z)\rangle = x$. Find $\|f\|$ and find *two* norm-preserving extensions of f to $\mathscr{X}$.

■

2.4.1 Analytic Banach-space-valued functions

Consider a complex Banach space $\mathscr{B}$, and let $x : \mathcal{D} \to \mathscr{B}$ be defined on a domain $\mathcal{D} \subseteq \mathbb{C}$. x is called **analytic** if it is differentiable for all $z \in \mathcal{D}$, namely if

$$x'(z) = \lim_{h\to 0} \frac{x(z+h) - x(z)}{h} \qquad \text{exists,}$$

i.e., if there exists a function $x' : \mathcal{D} \to \mathscr{B}$, such that

$$\lim_{h\to 0} \left\| \frac{x(z+h) - x(z) - hx'(z)}{h} \right\| = 0.$$

x is called **weakly analytic** if for all $f \in \mathscr{B}^*$,

$$\langle f, x(\cdot)\rangle : \mathcal{D} \to \mathbb{C}$$

is analytic.

Interestingly, analyticity and weak-analyticity coincide. The proof relies on the Hahn-Banach theorem.

Theorem 2.60 x is analytic if and only if it is weakly analytic.

Proof. **Part 1: suppose that x is analytic**: For every $f \in \mathscr{B}^*$:

$$\frac{\langle f, x(z+h)\rangle - \langle f, x(z)\rangle}{h} - \langle f, x'(z)\rangle = \left\langle f, \frac{x(z+h) - x(z) - hx'(z)}{h} \right\rangle.$$

The right hand side tends to zero as $h \to 0$, hence x is weakly analytic.

Part 2: suppose that x is weakly analytic: Fix z. We will show that

$$\frac{x(z+h) - x(z)}{h} - \frac{x(z+k) - x(z)}{k}$$

tends to zero when $h, k \to 0$ (independently). This will prove that x is analytic as for any sequence $h_n \to 0$, there exists, given $\varepsilon > 0$, an index N, such that for every $n, m > N$:

$$\left\| \frac{x(z+h_n) - x(z)}{h_n} - \frac{x(z+h_m) - x(z)}{h_m} \right\| < \varepsilon,$$

which implies that

$$\frac{x(z+h_n)-x(z)}{h_n}$$

is a Cauchy sequence in $\mathscr{B}$ and hence converges (the limit is independent of the sequence).

Let $f \in \mathscr{B}^*$. Let $B(z,r)$ be a closed ball contained in $\mathcal{D}$. Let $|h| < r/2$ and $|k| < r/2$. By the weak analyticity of x it follows from Cauchy's formula that

$$\langle f, x(z+\eta)\rangle = \frac{1}{2\pi\iota}\oint_{\partial B(z,r)} \frac{\langle f, x(\zeta)\rangle}{\zeta-(z+\eta)}\,d\zeta,$$

hence

$$\begin{aligned}\frac{\langle f,x(z+h)\rangle - \langle f,x(z)\rangle}{h} &= \frac{1}{2\pi\iota}\oint_{\partial B(z,r)} \langle f,x(\zeta)\rangle \frac{1}{h}\left(\frac{1}{\zeta-z-h}-\frac{1}{\zeta-z}\right)d\zeta \\ &= \frac{1}{2\pi\iota}\oint_{\partial B(z,r)} \langle f,x(\zeta)\rangle \frac{1}{(\zeta-z-h)(\zeta-z)}\,d\zeta,\end{aligned}$$

and

$$\begin{aligned}&\frac{\langle f,x(z+h)\rangle - \langle f,x(z)\rangle}{h} - \frac{\langle f,x(z+k)\rangle - \langle f,x(z)\rangle}{k} = \\ &\quad = \frac{1}{2\pi\iota}\oint_{\partial B(z,r)} \langle f,x(\zeta)\rangle \frac{h-k}{(\zeta-z-h)(\zeta-z-k)(\zeta-z)}\,d\zeta.\end{aligned}$$

Since $\langle f, x(\cdot)\rangle$ is bounded in $B(z,r)$, we can bound $|\langle f, x(\zeta)\rangle|$ by a constant M_f (it depends on f). We can then evaluate the right hand side as follows:

$$\frac{1}{|h-k|}\left|\frac{\langle f,x(z+h)\rangle - \langle f,x(z)\rangle}{h} - \frac{\langle f,x(z+k)\rangle - \langle f,x(z)\rangle}{k}\right| \le \frac{1}{2\pi}\frac{2\pi r M_f}{r(r/2)^2} = \frac{4M_f}{r^2}.$$

We can rewrite this as follows:

$$\frac{1}{|h-k|}\left|\left\langle f, \underbrace{\frac{x(z+h)-x(z)}{h} - \frac{x(z+k)-x(z)}{k}}_{\text{a collection } x_\alpha}\right\rangle\right| \le \frac{4M_f}{r^2}.$$

It follows from Proposition 2.59 that there exists a constant M such that

$$\sup_{|h|,|k|<r/2} \frac{1}{|h-k|}\left\|\frac{x(z+h)-x(z)}{h} - \frac{x(z+k)-x(z)}{k}\right\| \le M.$$

This completes the proof. ∎

For a continuous functions $\mathcal{D} \to \mathscr{B}$ one can define path integrals in exactly the same way as we define the Riemann integral of a continuous real-valued function.

Proposition 2.61 Let $x : \mathcal{D} \to \mathscr{B}$ be analytic. Then for every closed curve of finite-length γ:

$$\int_\gamma x(z)\,dz = 0.$$

Proof. For every $f \in \mathscr{B}^*$ it follows from its continuity and linearity that

$$\left\langle f, \int_\gamma x(z)\,dz \right\rangle = \int_\gamma \langle f, x(z)\rangle\,dz = 0,$$

where the second equality holds because $\langle f, x(\cdot)\rangle$ is analytic. Since the application of a functional f of $\int_\gamma x(z)\,dz$ yields zero for every $f \in \mathscr{B}^*$ is follows that it is zero. ■

Exercise 2.10 Prove Cauchy's formula for analytic Banach-space-valued integrals: Let $D \subset \mathbb{C}$ be an open set and $\mathscr{X}$ a complex Banach space. Let $x : D \to \mathscr{X}$ be an analytic function and let $\gamma \subset D$ be a closed, simple, smooth curve of finite length encircling a point $z \in D$. Prove that

$$x(z) = \frac{1}{2\pi\imath} \oint_\gamma \frac{x(\zeta)}{\zeta - z}\,d\zeta.$$

■

Exercise 2.11 Let $\mathscr{X}$ be a Banach space. A function $f : [a,b] \to \mathscr{X}$ is said to be of **bounded variation** if

$$\sup \sum_i \|f(b_i) - f(a_i)\| < \infty,$$

where the supremum is over all choices of finitely many disjoint intervals (a_i, b_i) in $[a,b]$. Prove that if for any $\phi \in \mathscr{X}^*$ the function $\phi \circ f : [a,b] \to \mathscr{F}$ is of bounded variation, then f is of bounded variation (that is, the weak and strong notions of bounded variation are equivalent). ■

Exercise 2.12 Let $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be an infinite-dimensional normed space.

① Construct an unbounded bijective linear operator $T : \mathscr{X} \leftrightarrows \mathscr{X}$. Hint: recall the construction of an unbounded linear functional.

② Use T to construct a new norm $\|\cdot\|_T$ on $\mathscr{X}$ such that $T : (\mathscr{X}, \|\cdot\|_{\mathscr{X}}) \to$

$(\mathscr{X}, \|\cdot\|_T)$ is an isometry.

③ Conclude that the norms $, \|\cdot\|_{\mathscr{X}}$ and $, \|\cdot\|_T$ are isomorphic but not equivalent.

TA material 2.4 The space $c \subset \ell_\infty$ comprises convergent sequences. We want to extend the notion of the limit of a sequence to ℓ_∞.

We may use the extension theorem resulting from the Hahn-Banach theorem to conclude:

Corollary 2.62 Let lim be the limit functional on $c \subset \ell_\infty$. Since $\|\lim\| = 1$ then there is an extension of lim to a continuous linear functional in ℓ_∞^*

The problem of this extension is that although we get the wanted linearity property of the limit, we do not necessarily get other properties, such as dependence only on the tail of the sequence. The following proposition shows that we can do better:

Proposition 2.63 There exists $f \in \ell_\infty^*$ such that for any $x \in \ell_\infty$:

①
$$\liminf x_n \le \langle f, x\rangle \le \limsup x_n,$$
In particular, $\|f\| = 1$ and if $x \in c$ then $\langle f, x\rangle = \lim x_n$.

② For every $k \in \mathbb{N}$,
$$\langle f, \{x_n\}\rangle = \langle f, \{x_{n+k}\}\rangle,$$
i.e. f depends only on the tail of the sequence.

The functional f is called a **Banach limit** of sequences. Note that it is not unique.

2.4.2 Geometric version of the Hahn-Banach theorem

Definition 2.8 Let $\mathscr{V}$ be a vector space. A linear subspace $\mathscr{Y} \subset \mathscr{V}$ is called **maximal** (or of **co-dimensional one**) if $\mathscr{Y}$ is not contained in any other (strict) subspace of $\mathscr{V}$.

Suppose that $\mathscr{Y} \subset \mathscr{V}$ is a maximal linear subspace. Let $x_0 \in \mathscr{V} \setminus \mathscr{Y}$. Then

$$\operatorname{Span}\{\mathscr{Y}, x_0\} = \mathscr{V},$$

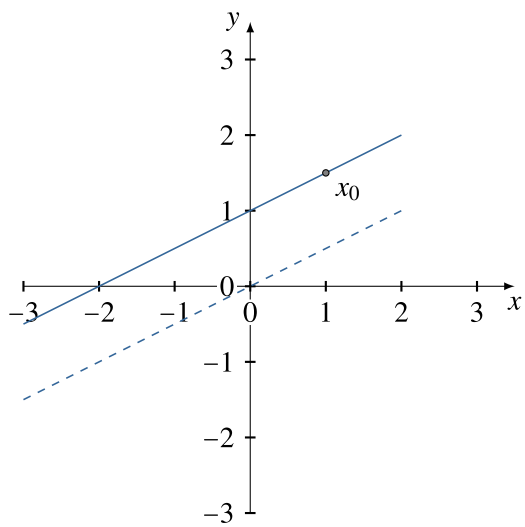
i.e., every $x \in \mathscr{V}$ has a unique representation

$$x = y + \alpha x_0.$$

Definition 2.9 Let $\mathscr{V}$ be a vector space, let $\mathscr{Y}$ be a linear subspace, and let $x_0 \in \mathscr{V}$. The set

$$x_0 + \mathscr{Y}$$

is called a **plane**. If $\mathscr{Y}$ is maximal, then we call this plane a **hyperplane** (**על מישור**). (A hyperplane is a translation of a maximal subspace.)



Proposition 2.64 Let $\mathscr{V}$ be a vector space. Let f be a non-zero linear functional. The set

$$\ker f$$

is a maximal subspace (and in particular a hyperplane).

Proof. Since $f \neq 0$ there exists a vector x_0 such that $\langle f, x_0 \rangle \neq 0$. For every $x \in \mathscr{V}$:

$$\left\langle f, x - \frac{\langle f, x \rangle}{\langle f, x_0 \rangle} x_0 \right\rangle = 0,$$

i.e.,

$$x = \underbrace{\frac{\langle f, x \rangle}{\langle f, x_0 \rangle} x_0}_{\in \operatorname{Span}\{x_0\}} + \underbrace{x - \frac{\langle f, x \rangle}{\langle f, x_0 \rangle} x_0}_{\in \ker f},$$

which proves that $\operatorname{Span}\{\ker f, x_0\} = \mathscr{V}$. ■

Definition 2.10 Let $\mathscr{V}$ be a vector space and $S \subset \mathscr{V}$. A point $x \in S$ is called an **internal point** (נקודה תוך) if

$$\forall y \in \mathscr{V} \quad \exists \varepsilon > 0 \quad \text{such that} \quad x + (-\varepsilon, \varepsilon) y \subset S.$$

That is, the intersection of every line $x + ty$ with S contains a segment with center x.

Comment 2.18 An internal point replaces the notion of an interior point when there is no topology. Since lines are well-defined in vector spaces and they have a natural topology, we define a notion of "interiority" through lines. In a normed space, an interior point is also an internal point. The converse is not true. Take for example $\mathbb{R}^2$ (with, say, the Euclidean norm) and the set

$$S = \{(x,y) \mid y \le 0 \text{ or } y \ge x^2\}.$$

The origin is an internal point but not an interior point.

Exercise 2.13 Let C be a convex subset of a finite-dimensional vector space $\mathscr{X}$. Prove that $x \in C$ is an internal point if and only if it is an interior point (with respect to any norm on $\mathscr{X}$). ▪

Definition 2.11 Let $\mathscr{V}$ be a vector space. A set $S \subset \mathscr{V}$ for which the origin of $\mathscr{V}$ is an internal point is called **absorbing** (**בולעת**). This is because

$$\forall y \in \mathscr{V} \quad \exists n \in \mathbb{N} \quad \text{such that} \quad y/n \in S,$$

i.e.,

$$\bigcup_{n=1}^{\infty} nS = \mathscr{V}.$$

(The entire vector space is contained in blowups of S.)

Let $\mathscr{V}$ be a vector space. Let K be an absorbing set. Define the **Minkowski functional** P_K on $\mathscr{V}$:

$$P_K(x) = \inf\{t > 0 \mid x/t \in K\}.$$

(It is finite because K is absorbing.)

Comment 2.19 If for some $\alpha > 0$,

$$\frac{x}{\alpha} \in K \qquad \text{then} \qquad P_K(x) \le \alpha.$$

Conversely, if

$$P_K(x) < \alpha,$$

then by definition

$$(\exists \beta < \alpha) \qquad \frac{x}{\beta} \in K.$$

This still doesn't guarantee that $x/\alpha \in K$. If K is convex, since the origin is in K, it follows that

$$\frac{x}{\alpha} \in K.$$

Proposition 2.65 Let $\mathscr{V}$ be a vector space. Let $K \subset \mathscr{V}$ be convex and absorbing. Then,

① P_K is non-negative.
② For every $\alpha > 0$: $P_K(\alpha x) = \alpha P_K(x)$.
③ For every $x, y \in \mathscr{V}$: $P_K(x+y) \le P_K(x) + P_K(y)$.
④ For every $x \in K$: $P_K(x) \le 1$.
⑤ x is an internal point of K if and only if $P_K(x) < 1$.

Proof. ① is obvious. ② is also clear as

$$P_K(\alpha x) = \inf\{t > 0 \mid \alpha x/t \in K\} = \inf\{\alpha t > 0 \mid x/t \in K\} = \alpha P_K(x).$$

To prove ③, suppose that

$$P_K(x) < \beta \qquad \text{and} \qquad P_K(y) < \gamma.$$

Since K is convex, $x/\beta \in K$, $y/\gamma \in K$, and

$$\frac{\beta}{\beta+\gamma}\frac{x}{\beta} + \frac{\gamma}{\beta+\gamma}\frac{y}{\gamma} = \frac{x+y}{\beta+\gamma} \in K.$$

This means that

$$P_K(x+y) \le \beta + \gamma.$$

Because this holds for every upper bounds β of γ of $P_K(x)$ and $P_K(y)$, ③ follows.

④ is obvious: for $x \in K$, $x/1 \in K$, hence $P_K(x) \le 1$.

It remains to prove ⑤. Suppose that x is an internal point of K. Then there is an $\varepsilon > 0$ such that $x + \varepsilon x \in K$, i.e., $x/(1+\varepsilon)^{-1} \in K$, which proves that $P_K(x) < 1$.

Conversely, suppose that $P_K(x) < 1$. Then there is an $\varepsilon > 0$ such that $P_K(x) < 1 - \varepsilon$, i.e.,

$$\frac{x}{1-\varepsilon} \in K.$$

Let $y \in \mathscr{V}$ be arbitrary. Because K is absorbing there exists a $\delta > 0$, such that $\delta y I \subset K$ (here I is the unit interval). Because K is convex,

$$x + \varepsilon\delta y I = (1-\varepsilon)\frac{x}{1-\varepsilon} + \varepsilon\delta y I \subset K,$$

which proves that x is an internal point. ■

Exercise 2.14 Let K be a convex, absorbing set in a real linear space $\mathscr{X}$. Assume K is symmetric, i.e. if $x \in K$ then $(-x) \in K$.

① Show that P_K is a semi-norm on $\mathscr{X}$.
② Show that if, in addition, K doesn't contain an entire line passing through the origin, then P_K is a norm on $\mathscr{X}$.
③ Find the open unit ball B of that norm and prove that

$$B \subset K \subset \overline{B},$$

where the inclusion on either side may be strict.

Definition 2.12 Let $\mathscr{V}$ be a vector space and let $M, N \subset \mathscr{V}$. A linear functional f on $\mathscr{V}$ is said to **separate** (**מפריד**) M and N if

$$\sup \operatorname{Re}\langle f, N\rangle \le \inf \operatorname{Re}\langle f, M\rangle,$$

where

$$\langle f, N\rangle = \{\langle f, x\rangle \mid x \in N\}.$$

Proposition 2.66 f separates between M and N if and only if it separates between $M - N$ and $\{0\}$.

Proof. For every $x \in M$ and $y \in N$:

$$\operatorname{Re}\langle f, x\rangle \le \operatorname{Re}\langle f, y\rangle$$

if and only if for every $x \in M$ and $y \in N$:

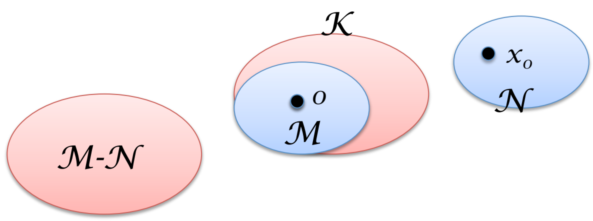
$$\operatorname{Re}\langle f, x - y\rangle \le 0 = \operatorname{Re}\langle f, 0\rangle.$$

■

Theorem 2.67 — Separation theorem. Let M and N be disjoint convex sets in a vector space $\mathscr{V}$. If at least one of them, say M, has an internal point, then there exists a non-zero linear functional that separates between M and N.

Proof. Consider the case where $\mathscr{V}$ is a real vector space. Since for every point p, f separates between M and N if and only if it separates between $M-p$ and $N-p$, then we can assume without loss of generality that 0 is an internal point of M.

Let $x_0 \in N$. Then $(-x_0)$ is an internal point of the convex set $M-N$, and 0 is an internal point of the convex set $K = x_0 + M - N$. Since M and N are disjoint then $x_0 \notin K$. If we prove the existence of a linear functional that separates K from x_0, then the same functional separates between $M-N$ and 0, and also between M and N. Thus, we may assume that $M = K$ is a convex set containing the internal point 0 (it is absorbing) and that $N = \{x_0\}$.



Consider the Minkowski functional P_K. Since $x_0 \notin K$ then $P_K(x_0) \geq 1$. On the linear subspace $\text{Span}\{x_0\}$ define a linear functional $\langle f, \alpha x_0 \rangle = \alpha P_K(x_0)$. For $\alpha > 0$, $\langle f, \alpha x_0 \rangle \leq P_K(\alpha x_0)$ whereas for $\alpha < 0$,

$$\langle f, \alpha x_0 \rangle = \alpha P_K(x_0) \leq 0 \leq P_K(\alpha x_0).$$

By the Hahn-Banach theorem there exists a linear extension F of f on $\mathscr{V}$ such that $\langle F, x \rangle \leq P_K(x)$. For $x \in K$:

$$\langle F, x \rangle \leq P_K(x) \leq 1,$$

whereas

$$\langle F, x_0 \rangle = \langle f, x_0 \rangle = P_K(x_0) \geq 1,$$

i.e. F separates K from $\{x_0\}$.

If the space is complex it is possible to treat separately real and imaginary parts as we did before. ■

TA material 2.5 We will show that the separation theorem does not hold in the absence of an internal point. Let $\mathscr{X}$ be the space of real-valued sequences that are zero from some point onwards. Let K comprise the subset for which the last non-zero coordinate is positive. A direct calculation shows that K is convex but It has no internal points.

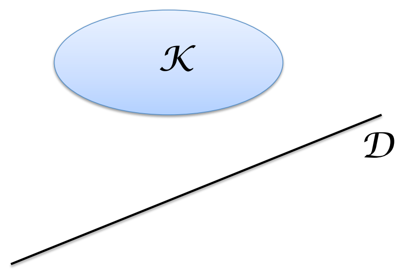
We now show that there is no non-trivial linear functional separating K from $\{0\}$. Let $f \neq 0$ be a linear functional on $\mathscr{X}$. W.l.o.g. we can assume that $f(K) \geq 0$ (otherwise take $-f$). Since $e_k \in K$ for every k it follows that $f(e_k) \geq 0$. Since $\{e_k\}$

spans $\mathscr{X}$ and $f \neq 0$, there is a j such that $\langle f, e_j \rangle > 0$. Now, if $\langle f, e_{j+1} \rangle = 0$, then $-e_j + e_{j+1} \in K$ but $\langle f, -e_j + e_{j+1} \rangle = -\langle f, e_j \rangle < 0$. Therefore $\langle f, e_{j+1} \rangle > 0$, but then

$$\langle f, -ej + \frac{\langle f, e_j \rangle}{2\langle f, e_{j+1} \rangle} e_{j+1} \rangle = -\frac{1}{2} \langle f, e_j \rangle < 0,$$

which contradicts the fact that the argument is in K.

Theorem 2.68 — Geometric Hahn-Banach. Let $\mathscr{V}$ be a vector space and let $K \subset \mathscr{V}$ be a convex set whose points are all internal points. Let D be a plane disjoint of K. Then there is a hyperplane $\mathscr{Y}$ that contains D and is disjoint of K.



Proof. Without loss of generality we may assume that D is a linear subspace (i.e., it contains the origin). From the separation theorem follows the existence of a non-zero linear functional F and a real number γ such that

$$\sup \operatorname{Re}\langle F, K \rangle \leq \gamma \leq \inf \operatorname{Re}\langle F, D \rangle.$$

Define $\langle f, x \rangle = \operatorname{Re}\langle F, x \rangle$, i.e.,

$$\langle F, x \rangle = \langle f, x \rangle - \imath \langle f, \imath x \rangle.$$

f is a linear functional over $\mathscr{V}_{\mathbb{R}}$. We have

$$\sup \langle f, K \rangle \leq \gamma \leq \inf \langle f, D \rangle,$$

and since $0 \in D$, $\gamma \leq 0$. Since D is a linear subspace, if there exists an $x \in D$ for which $\langle f, x \rangle \neq 0$, then either $\langle f, x \rangle$ or $\langle f, -x \rangle$ is negative, which is a contradiction. Hence, $f|_D = 0$. It follows that $F|_D = 0$. The set

$$\ker F$$

is a hyperplane that contains D. It remains to show that $\ker F$ and K are disjoint. Suppose that $x_0 \in K \cap \ker F$. Let $y \in \mathscr{V}$ such that $\langle f, y \rangle > 0$ (there necessarily exists such point y). Since x_0 is an internal point of K there exists a $\delta > 0$ such that $x_0 + \delta y \in K$. Then

$$\langle f, x_0 + \delta y \rangle = \langle f, x_0 \rangle + \delta \langle f, y \rangle > 0,$$

which contradicts the fact that $\sup \langle f, K \rangle \leq 0$. ■

Comment 2.20 We proved that we can assume the separating functional f to vanish on the hyperplane. We will use this fact later.

Comment 2.21 Even though we proved the separation theorem and the geometric Hahn-Banach theorem using the (non-geometric) Hahn-Banach theorem, each of these theorems is equivalent to the two other.

Proposition 2.69 Let $\mathscr{X}$ be a normed space and let $f \neq 0$ be a linear functional over $\mathscr{X}$. f is continuous if and only if $\ker f$ is closed. Moreover, if f is not continuous then $\ker f$ is dense in $\mathscr{X}$.

Proof. The easy part: if f is continuous then $\ker f = f^{-1}(0)$ is closed as it is the pre-image of a closed set.

The harder part: if $\ker f$ is closed, define $L = f^{-1}(1)$[4]. Take $x_0 \in L$. Then

$$L = x_0 + \ker f,$$

which is closed. Note that $0 \notin L$, hence there is an open ball $B(0,r)$ that does not intersect L.

For every $x \in B(0,r)$, $|\langle f,x\rangle| < 1$. Why? Suppose there were an $x \in B(0,r)$ for which $|\langle f,x\rangle| \geq 1$, then

$$\left\langle f, \frac{x}{\langle f,x\rangle}\right\rangle = 1,$$

i.e., $\frac{x}{\langle f,x\rangle} \in B(0,r) \cap L$, which is a contradiction. It follows that for every $x \in \mathscr{X}$:

$$|\langle f,x\rangle| = \frac{2\|x\|}{r}\left|\left\langle f, \frac{rx}{2\|x\|}\right\rangle\right| \leq \frac{2\|x\|}{r},$$

i.e., f is bounded.

For the last part of the proposition, suppose that f is not continuous. We have already shown that $\ker f$ is not closed. Since $\ker f$ is a maximal linear subspace of $\mathscr{X}$, its closure, which is a linear subspace of $\mathscr{X}$ that contains $\ker f$ must be equal to $\mathscr{X}$ (by Proposition 2.64). ∎

Corollary 2.70 A non-continuous linear functional $f \neq 0$ maps every open set onto $\mathscr{F}$.

[4] L is not empty because since $f \neq 0$ there exists an x_0 such that $\langle f,x_0\rangle = 0$. Then $x_0/\langle f,x_0\rangle \in L$.

Proof. Let f be a non-continuous linear functional and let $U \subset \mathscr{X}$ be an open set. Let $a \in \mathscr{F}$. We need to show that there is an $x \in U$ such that $\langle f, x \rangle = a$.

Let x_0 satisfy $\langle f, x_0 \rangle = \alpha \neq 0$. Then

$$\left\langle f, \frac{a x_0}{\alpha} \right\rangle = a.$$

By the previous proposition, the set

$$L = \frac{a x_0}{\alpha} + \ker f$$

is dense, hence intersects the open set U. That is, there exists an $x \in U$ of the form

$$x = \frac{a x_0}{\alpha} + y,$$

where $y \in \ker f$, which implies that $\langle f, x \rangle = a$. ■

Corollary 2.71 A linear functional $f \neq 0$ that separates two sets A, B, with at least one of them having an interior (not internal) point (נקודת פנים) is continuous.

Proof. If A has an interior point, then it contains, in particular, an open set U. If f were non-continuous it would map U to the entire scalar field, hence would not separate between the sets. ■

Corollary 2.72 Let $\mathscr{X}$ be a normed space and let K_1, K_2 be disjoint convex sets with at least one of them having an interior point (again, not internal). Then there is a bounded linear functional $f \neq 0$ that separates K_1 and K_2.

Proof. It follows from the Separation Theorem that there is an $f \neq 0$ that separates K_1 and K_2. It follows from the previous corollary that it is continuous/bounded. ■

Exercise 2.15 Let $\mathscr{X}$ be a normed space and $A, B \subset \mathscr{X}$ disjoint convex sets such that B is closed and A is compact. Prove that there exists an $f \in \mathscr{X}^*$ such that

$$\sup \operatorname{Re}\langle f, A \rangle < \inf \operatorname{Re}\langle f, B \rangle,$$

i.e., the separation is strict. ■

2.5 Duality

2.5.1 Reflexivity

Definition 2.13 Let $\mathscr{X}$ be a normed space. The mapping $\iota : \mathscr{X} \to \mathscr{X}^{**}$ defined by

$$\langle \iota(x), \cdot \rangle = \langle \cdot, x \rangle$$

is called the **natural inclusion** (השיכון הטבעי) of $\mathscr{X}$ into $\mathscr{X}^{**}$.

Proposition 2.73 ι is a linear, norm-preserving map.

Proof. We proved it already in the course of proving Proposition 2.59. Let's recall how: one the one hand, for every $f \in \mathscr{X}^*$:

$$|\langle \iota(x), f \rangle| = |\langle f, x \rangle| \le \|f\| \|x\|,$$

which proves that $\|\iota(x)\| \le \|x\|$. On the other hand, by (a corollary of) the Hahn-Banach theorem,

$$\|x\| = \max_{\|f\|=1} |\langle f, x \rangle| = \max_{\|f\|=1} |\langle \iota(x), f \rangle| \le \max_{\|f\|=1} \|\iota(x)\| \|f\| = \|\iota(x)\|,$$

which proves that $\|\iota(x)\| = \|x\|$. ∎

Comment 2.22 ι is not necessarily an isometry; an isometry has to be reversible, i.e., surjective (a norm-preserving linear map is of course injective).

Definition 2.14 A Banach space $\mathscr{X}$ is called **reflexive** (רפלקסיבי) if ι is an isometry.

Comment 2.23 Reflexivity was first introduced by Hahn in 1927. The term was coined in 1939 by Lorch.

Comment 2.24 For $\mathscr{X}$ to be reflexive it is not sufficient that $\mathscr{X}$ and $\mathscr{X}^{**}$ be isometric. The natural inclusion ι has to be an isometry.

Why are reflexive spaces important? We will get an answer toward the end of this section in the context of weak convergence.

Proposition 2.74 Every finite-dimensional Banach space is reflexive.

Proof. Let $\dim \mathscr{X} = n$. Since $\dim \mathscr{X}^* = n$ it follows that $\dim \mathscr{X}^{**} = n$. Since ι is norm preserving, it is injective and hence also surjective. ∎

■ **Example 2.10** Every Hilbert space is reflexive. This follows from the Riesz representation theorem. Every $f \in \mathscr{H}^*$ has a representation $y_f \in \mathscr{H}$, such that

$$\langle f, x\rangle = (x, y_f).$$

The mapping $f \mapsto y_f$ is an *anti-isomorphism* between the inner-product spaces $\mathscr{H}^*$ and $\mathscr{H}$ with

$$(f,g) = (y_g, y_f).$$

By the same representation theorem, every $F \in \mathscr{H}^{**}$ has a representation,

$$\langle F, f\rangle = (f, f_F) = (y_{f_F}, y_f).$$

The mapping $F \mapsto y_{f_F}$ is an isomorphism between the inner-product spaces $\mathscr{H}^{**}$ and $\mathscr{H}$. It remains to show that it is the natural inclusion. Indeed,

$$\langle \iota(y_{f_F}), f\rangle = \langle f, y_{f_F}\rangle = (y_{f_F}, y_f),$$

namely,

$$F = \iota(y_{f_F}).$$

■

■ **Example 2.11** The spaces ℓ_p, $1 < p < \infty$, are reflexive. This is because $\mathscr{X}^{**} \cong \mathscr{X}$ and the isometry is precisely the natural inclusion.

Recall that the isometry $T : \ell_p^* \to \ell_q$ is defined via

$$[T(f)]_n = \langle f, e_n\rangle,$$

for $f \in L_p^*$. Conversely, for $y \in L_q$,

$$\langle T^{-1}(y), x\rangle = \sum_{n=1}^{\infty} y_n x_n.$$

TO CONTINUE ■

2.5.2 Quotient spaces

Definition 2.15 Let $\mathscr{V}$ be a vector space and let M be a linear subspace. We say that x, y are **equivalent mod-M** if $x - y \in M$. The set of equivalence classes is called the **quotient space** (מרחב מנה) $\mathscr{V}/M$. We denote the equivalence class of $x \in \mathscr{V}$ by $[x] \in \mathscr{V}/M$:

$$[x] = \{y \in \mathscr{V}, y - x \in M\} = x + M.$$

As usual, we endow $\mathscr{V}/M$ with a vector space structure. For $\alpha, \beta \in \mathscr{V}/M$:

$$\alpha + \beta = [x + y],$$

where $x \in \alpha$ and $y \in \beta$. This definition is independent of representatives as if $[x] = [x']$ and $[y] = [y']$ then

$$[x+y] = [x'+y'].$$

Similarly, for $\alpha \in \mathscr{V}/M$ and $a \in \mathbb{F}$:

$$a\alpha = [ax],$$

where $x \in \alpha$. The mapping $\pi : x \mapsto [x]$ is called the **natural homomorphism** of $\mathscr{V}$ onto $\mathscr{V}/M$[5].

Let $\mathscr{X}$ be a normed space and let M be a closed linear subspace. We endow $\mathscr{X}/M$ with a norm:

$$\|\alpha\|_{\mathscr{X}/M} = \inf_{y\in\alpha} \|y\|_{\mathscr{X}}. \tag{2.1}$$

For $x \in \mathscr{X}$:

$$\|[x]\|_{\mathscr{X}/M} = \inf_{m\in M} \|x-m\|_{\mathscr{X}} = d(x,M).$$

Proposition 2.75 (2.1) is a norm on $\mathscr{X}/M$.

Proof. **Positivity**: Suppose that $\alpha \in \mathscr{X}/M$ satisfies

$$\|\alpha\|_{\mathscr{X}/M} = 0.$$

Then, for $x \in \alpha$ there exists a sequence $m_n \in N$ such that

$$\lim_{n\to\infty} \|x-m_n\| = 0.$$

Since M is closed it follows that $x \in M$, i.e., $\alpha = [x] = 0$.

Homogeneity: Let $\alpha \in \mathscr{X}/M$ and $x \in \alpha$, then

$$\|a\alpha\|_{\mathscr{X}/M} = \|[ax]\|_{\mathscr{X}/M} = \inf_{m\in M} \|ax-m\|_{\mathscr{X}} = |a| \inf_{m\in M} \|x-m/a\|_{\mathscr{X}} = |a|\|\alpha\|_{\mathscr{X}/M}.$$

Triangle inequality: The triangle inequality follows from:

$$\begin{aligned}
\|\alpha+\beta\|_{\mathscr{X}/M} &= \inf_{y\in\alpha+\beta} \|y\|_{\mathscr{X}} \\
&= \inf_{x\in\alpha, z\in\beta} \|x+z\|_{\mathscr{X}} \\
&\le \inf_{x\in\alpha, z\in\beta} \left(\|x\|_{\mathscr{X}} + \|z\|_{\mathscr{X}}\right) \\
&= \|\alpha\|_{\mathscr{X}/M} + \|\beta\|_{\mathscr{X}/M}.
\end{aligned}$$

■

[5] A homomorphism between vector spaces preserves the linear structure but it is not necessarily injective.

Proposition 2.76 If $\mathscr{B}$ is a Banach space and M is a closed linear subspace then $\mathscr{B}/M$ is a Banach space.

Proof. Let α_n be a Cauchy sequence in $\mathscr{B}/M$. It is sufficient to show that α_n has a converging subsequence (any Cauchy sequence that has a converging subsequence is convergent).

We can construct a subsequence $\beta_k = \alpha_{n_k}$. such that

$$\|\beta_{k+1} - \beta_k\|_{\mathscr{B}/M} < \frac{1}{2^k}.$$

Let $x_k \in \beta$: for every k there is an element $m_k \in M$, such that

$$\|x_{k+1} - x_k - m_k\|_{\mathscr{B}} < \frac{1}{2^k}.$$

Define

$$w_n = \sum_{k=1}^{n-1}(x_{k+1} - x_k - m_k) = x_n - x_1 - \sum_{k=1}^{n-1} m_n.$$

The Weierstraß M-test Iimplies that w_n converges; denote the limit by w; denote $\beta = [w + x_1]$. Then,

$$\begin{aligned}
\|\beta_n - \beta\|_{\mathscr{B}/M} &= \|[x_n] - [w + x_1]\|_{\mathscr{B}/M} \\
&= \|[x_n - x_1] - [w]\|_{\mathscr{B}/M} \\
&= \|[w_n] - [w]\|_{\mathscr{B}/M} \\
&= d(w_n - w, M) \\
&\le \|w_n - w\|_{\mathscr{X}}.
\end{aligned}$$

Since the right hand side converges to zero, $\beta_n \to \beta$. ■

The following proposition is obvious from the definition of the norm in the quotient space:

Proposition 2.77 Let $\mathscr{X}$ be a normed space and let $M \subset \mathscr{X}$ be a linear subspace. The natural homomorphism $\pi : \mathscr{X} \to \mathscr{X}/M$ maps the open unit ball of $\mathscr{X}$ onto the open unit ball of $\mathscr{X}/M$.

Proof. Let $x \in \mathscr{X}$. By definition

$$\|\pi(x)\|_{\mathscr{X}/M} = \inf_{y \in [x]} \|y\|_{\mathscr{X}} \le \|x\|_{\mathscr{X}},$$

which proves that
$$\pi : \mathfrak{B}_{\mathscr{X}} \to \mathfrak{B}_{\mathscr{X}/M}.$$
It remains to show that π is surjective. Let $\alpha \in \mathfrak{B}_{\mathscr{X}/M}$. Then,
$$1 > \|\alpha\|_{\mathscr{X}/M} = \inf_{y\in\alpha} \|y\|_{\mathscr{X}},$$
which implies that there exists a $y \in \alpha$ such that $\|y\|_{\mathscr{X}} < 1$, i.e., $\alpha \in \mathfrak{B}_{\mathscr{X}/M}$ is the image under π of an element $y \in \mathfrak{B}_{\mathscr{X}}$. ■

Proposition 2.78 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. Let $T : \mathscr{X} \to \mathscr{Y}$ be a linear transformation onto $\mathscr{Y}$, such that $\mathfrak{B}_{\mathscr{X}}$ is mapped onto $\mathfrak{B}_{\mathscr{Y}}$. Then $\mathscr{Y}$ is isometric to $\mathscr{X}/\ker T$.

Proof. Let π be the natural homomorphism of $\mathscr{X}$ onto $\mathscr{X}/\ker T$. Define $\tau : \mathscr{X}/\ker T \to \mathscr{Y}$ as follows:
$$\tau : \alpha \mapsto Tx,$$
where $x \in \alpha$; this definition does not depend on the representative x.

τ is linear and surjective (because T is surjective). τ is injective because $\tau[\alpha] = 0$ implies $Tx = 0$ for $x \in \alpha$, i.e., $x \in \ker T = [0]$. It follows that τ is a bijective map mapping the open unit ball onto the open unit ball; hence τ^{-1} satisfies the same property.

We conclude that $\|\tau\| \le 1$ and $\|\tau^{-1}\| \le 1$, which implies that
$$\|\tau\| = \|\tau^{-1}\| = 1.$$
Finally, for every $\alpha \in \mathscr{X}/\ker T$:
$$\|\alpha\|_{\mathscr{X}/\ker T} = \|\tau^{-1} \circ \tau(\alpha)\|_{\mathscr{X}/\ker T} \le \|\tau(\alpha)\|_{\mathscr{Y}} \le \|\alpha\|_{\mathscr{X}/\ker T},$$
from which follows that τ is an isometry. ■

Lemma 2.79 Let $\mathscr{B}$ be a Banach space and let A be a dense set in $B(0,\rho)$. Every $x \in B(0,\rho)$ can be represented as:
$$x = \sum_{j=1}^{\infty} \lambda_j x_j,$$
where $x_j \in A$ and $\sum_{j=1}^{\infty} |\lambda_j| < 1$ (i.e., $\lambda = (\lambda_j)$ is in the unit ball of ℓ_1).

Proof. We can assume without loss of generality that $\rho = 1$ (just scale otherwise). Given $x \in \mathfrak{B}_{\mathscr{B}}$, take $\delta > 0$ such that

$$\|(1+\delta)x\| < 1.$$

Fix $\varepsilon > 0$ and take $x_1 \in A$ such that

$$\|(1+\delta)x - x_1\| < \varepsilon.$$

Since $[(1+\delta)x - x_1]/\varepsilon \in \mathfrak{B}_{\mathscr{B}}$, there exists an $x_2 \in A$ such that

$$\left\| \frac{(1+\delta)x - x_1}{\varepsilon} - x_2 \right\| < \varepsilon.$$

Take now $x_3 \in A$ such that

$$\left\| \frac{(1+\delta)x - x_1 - \varepsilon x_2}{\varepsilon^2} - x_3 \right\| < \varepsilon,$$

i.e.,

$$\|(1+\delta)x - x_1 - \varepsilon x_2 - \varepsilon^2 x_3\| < \varepsilon^3.$$

Continuing inductively, we get that

$$x = \frac{1}{1+\delta} \sum_{j=1}^{\infty} \varepsilon^{j-1} x_j,$$

and the limit exists. Since $\lambda_j = \varepsilon^{j-1}/(1+\delta)$ we have

$$\sum_{j=1}^{\infty} \lambda_j = \frac{1}{(1+\delta)(1-\varepsilon)},$$

which we can make less than 1 by taking ε sufficiently small. ■

The following proposition is interesting. Recall that every separable Hilbert space is isometric to ℓ_2. This is not true for Banach spaces. Instead every separable Banach space is isomorphic to a quotient space of ℓ_1.

Proposition 2.80 Every separable Banach space $\mathscr{B}$ is isometric to a quotient space of ℓ_1.

Proof. Let (x_n) be a dense sequence in $\mathfrak{B}_{\mathscr{B}}$. For $\lambda \in \ell_1$ define the map $T : \ell_1 \to \mathscr{B}$,

$$T(\lambda) = \sum_{j=1}^{\infty} \lambda_j x_j.$$

It is a linear map with

$$\|T(\lambda)\|_{\mathscr{B}} \le \sum_{j=1}^{\infty} |\lambda_j| = \|\lambda\|_1,$$

from which follows that $\|T\| \le 1$. Lemma 2.79 states that T maps $\mathfrak{B}_{\ell_1}$ *onto* $\mathfrak{B}_{\mathscr{B}}$. It follows from Proposition 2.78 that $\mathscr{B}$ is isometric to $\ell_1/\ker T$. ■

Comment 2.25 $\ker T$ depends on the choice of the dense sequence (x_n), so this construction is far from being unique.

2.5.3 The dual of a linear transformation

Notation 2.2 We usually denote elements of a normed space $\mathscr{X}$ by $x, y, \dots$ and elements of its dual space $\mathscr{X}^*$ by $f, g, \dots$. In parts of this section we will deal with duals of duals, duals of duals of duals, and even duals of duals of duals of duals.... in such cases it will be more convenient to use $x \in \mathscr{X}$, $x^* \in \mathscr{X}^*$, $x^{**} \in \mathscr{X}^{**}$, etc.

Definition 2.16 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces and let $T \in B(\mathscr{X}, \mathscr{Y})$. We define $T^* : \mathscr{Y}^* \to \mathscr{X}^*$ which we call the **dual** of T (**אופרטור צמוד**):

$$\langle T^*(y^*), \cdot \rangle = \langle y^*, T(\cdot) \rangle,$$

namely, $T^*(y^*) = y^* \circ T$.

Proposition 2.81 If $T \in B(\mathscr{X}, \mathscr{Y})$ then $T^* \in B(\mathscr{Y}^*, \mathscr{X}^*)$ and $\|T^*\| = \|T\|$.

Proof. Linearity of T^* results from the linearity of the map $y^* \mapsto y^* \circ T$. To calculate the norm of T^* we first note that

$$\|T^*(y^*)\| = \sup_{x \in \mathfrak{B}_{\mathscr{X}}} |\langle T^*(y^*), x \rangle| = \sup_{x \in \mathfrak{B}_{\mathscr{X}}} |\langle y^*, T(x) \rangle| \le \|y^*\| \|T\|,$$

from which we conclude that $\|T^*\| \le \|T\|$. On the other hand,

$$\begin{aligned} \|T(x)\| &= \max_{y^* \in \mathfrak{B}_{\mathscr{Y}^*}} |\langle y^*, T(x) \rangle| \\ &= \max_{y^* \in \mathfrak{B}_{\mathscr{Y}^*}} |\langle T^*(y^*), x \rangle| \\ &\le \max_{y^* \in \mathfrak{B}_{\mathscr{Y}^*}} \|T^*\| \|y^*\| \|x\| = \|T^*\| \|x\|, \end{aligned}$$

from which we deduce that $\|T\| \le \|T^*\|$, hence $\|T\| = \|T^*\|$. ■

Proposition 2.82 If $T \in B(\mathscr{X}, \mathscr{Y})$ is an isometry then so is $T^* \in B(\mathscr{Y}^*, \mathscr{X}^*)$.

Proof. We first show that T^* is injective: let $y^* \in \ker T^*$. Then, for every $y \in \mathscr{Y}$,

$$\langle y^*, y\rangle = \langle y^*, T(T^{-1}(y))\rangle = \langle T^*(y^*), T^{-1}(y)\rangle = 0,$$

i.e., $y^* = 0$, namely $\ker T^* = \{0\}$.

We then show that T^* is surjective: for every $x^* \in \mathscr{X}^*$:

$$\langle x^*, \cdot\rangle = \langle x^*, T^{-1}T(\cdot)\rangle = \langle (T^{-1})^*(x^*), T(\cdot)\rangle = \langle T^*(T^{-1})^*(x^*), \cdot\rangle,$$

namely $x^* = T^*(T^{-1})^*(x^*) \in \text{Image}(T^*)$.

Finally, T^* is an isometry because for every $y^* \in \mathscr{Y}^*$:

$$\begin{aligned}
\|T^*(y^*)\| &= \sup_{x \in \mathfrak{B}_{\mathscr{X}}} |\langle T^*(y^*), x\rangle| \\
&= \sup_{x \in \mathfrak{B}_{\mathscr{X}}} |\langle y^*, T(x)\rangle| \\
&= \sup_{y \in \mathfrak{B}_{\mathscr{Y}}} |\langle y^*, y\rangle| \\
&= \|y^*\|,
\end{aligned}$$

where we have used the fact that T is an isometry. ■

Comment 2.26 The fact that $\mathscr{X}$ isometric to $\mathscr{Y}$ implies that $\mathscr{X}^*$ is isometric to $\mathscr{Y}^*$ is not surprising (if two spaces are "the same" then so must be their duals). The interesting fact is that if T is an isometry between normed spaces then $(T^*)^{-1}$ is an isometry between their duals.

2.5.4 The dual space of a Banach space

Our next goal is to derive tools for calculating the dual space of a given Banach space. For this, we need some new constructions:

Definition 2.17 Let $\mathscr{B}$ be a Banach space. Let $M \subset \mathscr{B}$ and $N \subset \mathscr{B}^*$ be linear subspaces (not necessarily closed). We denote:

$$M^\perp = \{f \in \mathscr{B}^* \mid \langle f, M\rangle = \{0\}\}$$
$$N_\perp = \{x \in \mathscr{B} \mid \langle N, x\rangle = \{0\}\}.$$

Proposition 2.83 $M^\perp$ and $N_\perp$ are closed linear subspaces.

Proof. Let $f \in \overline{M^\perp}$. Then there exists a sequence $f_n \to f$ such that

$$\forall x \in M \qquad \langle f_n, x\rangle = 0,$$

from which follows that[6]

$$\forall x \in M \qquad \langle f, x\rangle = 0,$$

i.e., $f \in M^\perp$, hence $M^\perp = \overline{M^\perp}$. A similar argument holds for $N_\perp$. ■

Proposition 2.84

$$(M^\perp)_\perp = \overline{M}.$$

Proof. The easy part: let $x \in M$. Then,

$$\forall f \in M^\perp \qquad \langle f, x\rangle = 0,$$

which follows that $x \in (M^\perp)_\perp$, i.e.,

$$M \subset (M^\perp)_\perp,$$

but since the latter is closed, it follows that

$$\overline{M} \subset (M^\perp)_\perp.$$

The harder part: suppose that $x \notin \overline{M}$. Since $\overline{M}$ is closed there is an open ball $B(x,r)$ that does not intersect $\overline{M}$. It follows from the geometric Hahn-Banach theorem (well, one of its sequels) that there exists a *bounded* linear functional f that vanishes on M and not on x:

$$f \in M^\perp \qquad \text{and} \qquad \langle f, x\rangle \neq 0,$$

i.e., $x \notin (M^\perp)_\perp$. ■

Proposition 2.85 Let M be a closed subspace of a Banach space $\mathscr{B}$. Then:

① M^* is isometric to $\mathscr{B}^*/M^\perp$.
② $(\mathscr{B}/M)^*$ is isometric to $M^\perp$.

[6] Because

$$|\langle f_n, x\rangle - \langle f, x\rangle| = |\langle f_n - f, x\rangle| \le \|f_n - f\|\|x\|.$$

Proof. **Part 1**: Let $f \in M^*$. It follows from the Hahn-Banach theorem that f can be extended into a functional $F \in \mathscr{B}^*$. We define a map

$$\sigma : M^* \to \mathscr{B}^*/M^\perp$$

as follows:

$$\sigma(f) = [F].$$

The mapping σ is well-defined (i.e., independent of the extension) because if F,G are both extensions of f, then $F|_M = G|_M$, i.e., $F - G \in M^\perp$. In other words, we identify the Hahn-Banach extension as an operator $M^* \to \mathscr{B}^*/M^\perp$.

σ is linear. It is also onto because let $[F] \in \mathscr{B}^*/M^\perp$, then

$$[F] = \sigma(F|_M).$$

It remains to show that σ is norm-preserving; injectivity follows at once.

It is easy to see that

$$\|\sigma(f)\| = \|[F]\| = \inf_{G \in [F]} \|G\| = \|f\|,$$

where we used the fact that there exists an extension G that has the same norm as f.

Part 2: Define

$$\tau : (\mathscr{B}/M)^* \to \mathscr{B}^*$$

as follows:

$$\langle \tau(f), x \rangle = \langle f, [x] \rangle.$$

Note that if $x \in M$ then $[x] = [0]$, so that $\langle \tau(f), x \rangle = \langle f, [0] \rangle = 0$, which shows that in fact, $\tau : (\mathscr{B}/M)^* \to M^\perp$.

τ is linear. It is onto because $g \in M^\perp$ uniquely defines a functional $f \in (\mathscr{B}/M)^*$ satisfying,

$$\langle f, [x] \rangle = \langle g, x \rangle$$

i.e.,

$$g = \tau(f).$$

As above it remains to show that τ is norm-preserving. Let $f \in (\mathscr{B}/M)^*$ and let $[x]$ be on the unit sphere of $\mathscr{B}/M$. For every $\varepsilon > 0$ there exists a $y \in [x]$ such that $\|y\| \le 1 + \varepsilon$. On the one hand,

$$\langle f, [x] \rangle = \langle \tau(f), y \rangle \le \|\tau(f)\|(1 + \varepsilon).$$

Since this holds for every $[x]$ and every $\varepsilon > 0$:

$$\|f\| \le \|\tau(f)\|.$$

On the other hand,

$$\langle \tau(f), x \rangle = \langle f, [x] \rangle \le \|f\| \|[x]\| \le \|f\| \|x\|,$$

which implies that $\|\tau(f)\| \le \|f\|$. This completes the proof. ■

Proposition 2.86 If $\mathscr{B}$ is reflexive then so is $\mathscr{B}^{**}$.

Proof. It is given that $\iota : \mathscr{B} \to \mathscr{B}^{**}$ is an isometry. By Proposition 2.82 so are

$$\iota^* : \mathscr{B}^{***} \to \mathscr{B}^* \qquad \text{and} \qquad \iota^{**} : \mathscr{B}^{**} \to \mathscr{B}^{****}.$$

To prove that $\mathscr{B}^{**}$ is reflexive we need to show that ι^{**} is the natural inclusion. For every $x^{***} \in \mathscr{B}^{***}$:

$$\langle \iota^{**}(\cdot), x^{***} \rangle = \langle \cdot, \iota^*(x^{***}) \rangle$$

But since ι is an isomorphism and it is the natural inclusion,

$$\begin{aligned}\langle \cdot, \iota^*(x^{***}) \rangle &= \langle \iota(\iota^{-1}(\cdot)), \iota^*(x^{***}) \rangle \\ &= \langle \iota^*(x^{***}), \iota^{-1}(\cdot) \rangle \\ &= \langle x^{***}, \iota(\iota^{-1}(\cdot)) \rangle \\ &= \langle x^{***}, \cdot \rangle.\end{aligned}$$

Thus,

$$\langle \iota^{**}(\cdot), x^{***} \rangle = \langle x^{***}, \cdot \rangle,$$

which means that ι^{**} is the natural inclusion $\mathscr{X}^{**} \to \mathscr{X}^{****}$. ■

Exercise 2.16 More generally, show that if $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic and $\mathscr{X}$ is reflexive then $\mathscr{Y}$ is reflexive. ■

Proposition 2.87 Let $\mathscr{B}$ be a Banach space:

$$\mathscr{B} \text{ is reflexive} \quad \text{if and only if} \quad \mathscr{B}^* \text{ is reflexive.}$$

Proof. **Suppose that $\mathscr{B}^*$ is reflexive**. To avoid confusion we denote the natural inclusions

$$\iota : \mathscr{B} \to \mathscr{B}^{**} \qquad \text{and} \qquad \iota_1 : \mathscr{B}^* \to \mathscr{B}^{***}.$$

The latter is invertible., hence for every $x^{***} \in \mathscr{B}^{***}$

$$\langle x^{***}, \cdot \rangle = \langle \iota_1(\iota_1^{-1}(x^{***})), \cdot \rangle = \langle \cdot, \iota_1^{-1}(x^{***}) \rangle.$$

Choose

$$x^{***} \in [\text{Image}(\iota)]^{\perp}.$$

Then:

$$0 = \langle x^{***}, \iota(\cdot) \rangle = \langle \iota(\cdot), \iota_1^{-1}(x^{***}) \rangle = \langle \iota_1^{-1}(x^{***}), \cdot \rangle,$$

from which we conclude that $\iota_1^{-1}(x^{***}) = 0$, hence also $x^{***} = 0$.

Thus,

$$x^{***} \in [\text{Image}(\iota)]^{\perp} \implies x^{***} = 0.$$

This implies that Image(ι) is dense in $\mathscr{B}^{**}$. Since it is closed, Image$(\iota) = \mathscr{B}^{**}$, i.e., $\mathscr{B}$ is reflexive.

Suppose that $\mathscr{B}$ is reflexive. This implies that $\mathscr{B}^{**}$ is reflexive, and from the first part follows that $\mathscr{B}^{*}$ is reflexive. ■

Since we proved in Theorem 2.58 that $\mathscr{B}$ is separable only if $\mathscr{B}^{*}$ is separable, we conclude that:

Corollary 2.88 Let $\mathscr{B}$ be a Banach space:

$$\mathscr{B} \text{ reflexive and separable if and only if } \mathscr{B}^{*} \text{ reflexive and separable.}$$

Proof. There seems to be a missing part: suppose $\mathscr{B}$ is reflexive and separable. Since $\mathscr{B}$ is isometric to $\mathscr{B}^{**}$ the latter is separable, hence $\mathscr{B}^{*}$ is also separable. ■

Proposition 2.89 A closed linear subspace of a reflexive Banach space is reflexive.

Proof. Let M be a closed linear subspace of a Banach space $\mathscr{B}$. We need to show that the natural inclusion $\iota : M \to M^{**}$ is onto.

Let $m^{**} \in M^{**}$. We can extend m^{**} into a functional $\sigma(m^{**})$ on $\mathscr{B}^{*}$ as follows:

$$\langle \sigma(m^{**}), x^{*} \rangle = \langle m^{**}, x^{*}|_M \rangle.$$

Since $\mathscr{B}$ is reflexive, $\sigma(m^{**}) = \iota(x)$ for some $x \in \mathscr{B}$. Thus for every $m^{**} \in M^{**}$ there exists an $x \in \mathscr{B}$ such that

$$\langle m^{**}, x^{*}|_M \rangle = \langle \iota(x), x^{*} \rangle = \langle x^{*}, x \rangle.$$

We now claim that $x \in M$. Otherwise, we can find an x^{*} that $x^{*}|_M = 0$ and $\langle x^{*}, x \rangle \neq 0$ (here we use the fact that M is closed!) obtaining a contradiction.

So, for every $m^{**} \in M^{**}$ there exists an $m \in M$ such that

$$\langle \iota(m), x^{*} \rangle = \langle m^{**}, x^{*}|_M \rangle.$$

In particular, for every $m^* \in M^*$ let x^* be an extension of m^*, then

$$\langle \iota(m), x^* \rangle = \langle x^*, m \rangle = \langle m^*, m \rangle = \langle \iota(m), m^* \rangle = \langle m^{**}, m^* \rangle.$$

Thus, ι is onto, which proves that M is reflexive. ∎

Theorem 2.90 Let $\mathscr{B}$ be a Banach space and M a closed linear subspace. Then $\mathscr{B}$ is reflexive if and only if both M and $\mathscr{B}/M$ are reflexive.

Proof. ∎

2.5.5 Weak convergence in Banach spaces

We now study the notion of weak convergence in Banach spaces. As in Hilbert spaces, weak convergence is important because the unit ball of infinite-dimensional spaces is not (relatively) compact, however, it may be weakly (relatively) compact. As a result, proving that a set is bounded guarantees the existence of a weak accumulation point, which could be the object we are looking for (e.g., a minimizer or the solution to an equation).

Definition 2.18 Let $\mathscr{X}$ be a normed space. A sequence $x_n \in \mathscr{X}$ **weakly converges** to $x \in \mathscr{X}$ if

$$\lim_{n\to\infty} \langle f, x_n \rangle = \langle f, x \rangle$$

for all $f \in \mathscr{X}^*$. As in Hilbert spaces, we denote weak convergence by $x_n \rightharpoonup x$.

Proposition 2.91 Let $\mathscr{X}$ be a normed space and (x_n) a sequence. A weak limit of (x_n), if it exists, is unique.

Proof. if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then

$$\lim_{n\to\infty} \langle f, x_n \rangle = \langle f, x \rangle = \langle f, y \rangle,$$

which implies that $\langle f, x - y \rangle = 0$ for all $f \in \mathscr{X}^*$. As a consequence of the Hahn-Banach theorem $x = y$. ∎

Definition 2.19 A sequence x_n is called a **weak Cauchy sequence** if $\langle f, x_n \rangle$ is a Cauchy sequence for all $f \in \mathscr{X}^*$.

Definition 2.20 The space $\mathscr{X}$ is called **weakly sequentially complete** (**שלם סדרתית חלש**) if every weak Cauchy sequence weakly converges.

Proposition 2.92 Let $\mathscr{X}$ be a normed space. If $x_n \rightharpoonup x$ then

① (x_n) is bounded.
② $x \in \overline{\text{Span}\{x_n \mid n \in \mathbb{N}\}}$.
③ $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

Proof. ① Since

$$\forall f \in \mathscr{X} \qquad \lim_{n\to\infty} \langle f, x_n \rangle = \langle f, x \rangle,$$

it follows that

$$\forall f \in \mathscr{X} \qquad \sup_n |\langle f, x_n \rangle| < \infty,$$

and by Proposition 2.59 (the “reverse" Banach-Steinhaus therem):

$$\sup_n \|x_n\| < \infty.$$

② Let $\mathscr{V} = \text{Span}\{x_n\}$. For every $f \in \mathscr{V}^\perp$,

$$\langle f, x \rangle = \lim_{n\to\infty} \langle f, x_n \rangle = 0,$$

i.e., $x \in (\mathscr{V}^\perp)_\perp$. We have proved that $(\mathscr{V}^\perp)_\perp = \overline{\mathscr{V}}$.

③ It follows from the Hahn-Banach theorem (more precisely, from Corollary 2.55) that there is an $f \in \mathfrak{B}_{\mathscr{X}^*}$ satisfying

$$\|x\| = |\langle f, x \rangle| = \lim_{n\to\infty} |\langle f, x_n \rangle| \le \limsup_{n\to\infty} \|x_n\|.$$

■

Comment 2.27 Recall Proposition 1.60 stating that a weak Cauchy sequence in a Hilbert space is bounded. The proof was based on Baire's category theorem. The generalization to Banach spaces is based on Proposition 2.59 which relies on the Hahn-Banach theorem.

Recall also Proposition 1.58 stating that in a Hilbert space $x_n \rightharpoonup x$ implies $\|x\| \le \liminf_{n\to\infty} \|x_n\|$. The proof was based on the inner-product,

$$\|x\|^2 = \lim_{n\to\infty} (x_n, x) \le \liminf_{n\to\infty} \|x_n\| \|x\|.$$

The generalization to Banach spaces relies on Corollary 2.55, which again relies on the Hahn-Banach theorem.

We will now see a number of results that explain the importance of reflexive spaces.

Theorem 2.93 Let $\mathscr{B}$ be a reflexive Banach space. Then, $K \subset \mathscr{B}$ is bounded if and only if every sequence in K has a weakly converging subsequence (in $\mathscr{B}$). In other words, a set is bounded if and only if it is weakly sequentially relatively compact.

Proof. **Suppose that K is weakly sequentially relatively compact**. If K is not bounded then there is a sequence $x_n \in K$ satisfying $\|x_n\| > n$. This contradicts the fact that every sequence has a weakly converging (hence bounded) subsequence.

Suppose that K is bounded. Let $(x_n) \in K$ and denote $\sup_n \|x_n\| = M$ (by assumption it is finite). Let

$$\mathscr{X}_0 = \overline{\operatorname{Span}\{x_n \mid n \in \mathbb{N}\}}.$$

The space $\mathscr{X}_0$ is both separable and reflexive (since it is a closed subspace of a reflexive Banach space—Proposition 2.89). It follows from Corollary 2.88 that $\mathscr{X}_0^*$ is both separable and reflexive. Let $x_n^* \in \mathscr{X}_0^*$ be a dense sequence. Since the sequence

$$\langle x_1^*, x_n \rangle$$

is bounded, it has a convergent subsequence $\langle x_1^*, x_{1,n} \rangle$. Then because

$$\langle x_2^*, x_{1,n} \rangle$$

is bounded, it has a convergent subsequence $\langle x_2^*, x_{2,n} \rangle$. We proceed inductively.

Consider now the diagonal sequence $x_{n,n}$. For every k the sequence

$$\langle x_k^*, x_{n,n} \rangle$$

converges. Our goal is to show that there exists an $x_0 \in \mathscr{X}_0$ such that

$$\lim_{n\to\infty} \langle x_k^*, x_{n,n} \rangle = \langle x_k^*, x_0 \rangle,$$

and further infer that we can replace x_k^* by any $x^* \in \mathscr{B}^*$.

For every k,

$$\lim_{n\to\infty} \langle x_k^*, x_{n,n} \rangle = \lim_{n\to\infty} \langle \iota(x_{n,n}), x_k^* \rangle \equiv f(x_k^*).$$

f is well-defined; it is linear and satisfies for every k:

$$f(x_k^*) = \lim_{n\to\infty} \langle x_k^*, x_{n,n} \rangle \le M \|x_k^*\|.$$

Since (x_k^*) is a dense sequence in $\mathscr{X}_0^*$, for and $x^* \in \mathscr{X}_0^*$ we take a sequence $x_k^* \to x^*$, then

$$|f(x_n^*) - f(x_m^*)| \le M\|x_n^* - x_m^*\|,$$

i.e., $f(x_n^*)$ is a Cauchy sequence and hence converges (the limit is independent of the chosen sequence). Thus, f can be extended to a bounded linear operator on $\mathscr{X}_0^*$.

Since $\mathscr{X}_0$ is reflexive, there exists an $x_0 \in \mathscr{X}_0$ such that $\imath(x_0) = f$. That is, for every $x_0^* \in \mathscr{X}_0^*$:

$$\lim_{n\to\infty} \langle \imath(x_{n,n}), x_0^* \rangle = \langle \imath(x_0), x_0^* \rangle,$$

namely,

$$\lim_{n\to\infty} \langle x_0^*, x_{n,n} \rangle = \langle x_0^*, x_0 \rangle.$$

We will be done if we can replace $x_0^* \in \mathscr{X}_0^*$ by $x^* \in \mathscr{B}^*$. Let $x^* \in \mathscr{B}^*$, then

$$\begin{aligned}
\lim_{n\to\infty} \langle x^*, x_{n,n} \rangle &= \lim_{n\to\infty} \langle x^*|_{\mathscr{X}_0}, x_{n,n} \rangle \\
&= \lim_{n\to\infty} \langle \imath(x_{n,n}), x^*|_{\mathscr{X}_0} \rangle \\
&= \langle \imath(x_0), x^*|_{\mathscr{X}_0} \rangle \\
&= \langle x^*|_{\mathscr{X}_0}, x_0 \rangle \\
&= \langle x^*, x_0 \rangle,
\end{aligned}$$

from which we conclude that $x_{n,n} \rightharpoonup x_0$ in $\mathscr{B}$.

■

Corollary 2.94 Let $\mathscr{B}$ be a reflexive Banach space. Then, it is weakly sequentially complete (every weak Cauchy sequence weakly converges).

Proof. Let (x_n) be a weak Cauchy sequence in a reflexive Banach space, namely

$$\langle f, x_n \rangle$$

is a Cauchy sequence for every $f \in \mathscr{B}^*$. Hence,

$$\forall f \in \mathscr{B}^* \qquad \sup_n |\langle f, x_n \rangle| < \infty,$$

and it follows from the "reverse Banach-Steinhaus theorem" that $\sup_n \|x_n\| < \infty$. By the last theorem, (x_n) has a weakly converging subsequence. That is, there exists a subsequence $x_{n_k} \rightharpoonup x$, i.e.,

$$\forall f \in \mathscr{B}^* \qquad \lim_{k\to\infty} \langle f, x_{n_k} \rangle = \langle f, x \rangle.$$

It follows at once that the entire sequence weakly converges to x. ■

Exercise 2.17 A mode of convergence is said to satisfy the **Urysohn property** if for every sequence x_n the existence of a unique partial limit x implies that $x_n \to x$.

① Prove that convergence in a metric space satisfies the Urysohn property.
② Prove that weak convergence in a normed space satisfies the Urysohn property.

■

Exercise 2.18 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed space. A linear operator $T : \mathscr{X} \to \mathscr{Y}$ is **weakly continuous** if for every $f \in \mathscr{Y}^*$:

$$f \circ T : \mathscr{X} \to \mathscr{F}$$

is continuous. Show that T is continuous if and only if it is weakly continuous.

■

Exercise 2.19 Prove that c and c_0 are not weakly sequentially complete. ■

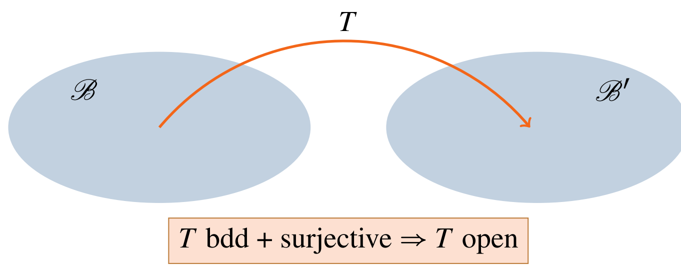
2.6 The open mapping and closed graph theorems

The main subject of this section is two theorems about linear transformations between Banach spaces.

2.6.1 The open mapping theorem

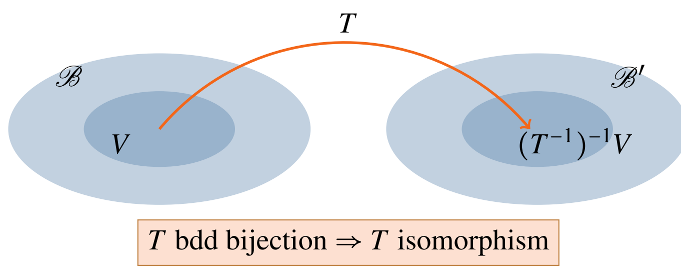
Definition 2.21 Let $\mathscr{X}$ and $\mathscr{Y}$ be topological spaces. A mapping $T : \mathscr{X} \to \mathscr{Y}$ that maps open sets into open sets is called an **open mapping**.

Theorem 2.95 — Open mapping theorem, Banach-Schauder. Let $\mathscr{B}$ and $\mathscr{B}'$ be Banach spaces. A surjective bounded linear operator $T : \mathscr{B} \twoheadrightarrow \mathscr{B}'$ is an open mapping.



Before we prove this theorem, note the following corollary:

Corollary 2.96 — Bounded bijections are isomorphisms. Let $\mathscr{B}$ and $\mathscr{B}'$ be Banach spaces and let $T : \mathscr{B} \leftrightarrows \mathscr{B}'$ be a linear bounded, bijection. Then, T is an isomorphism, namely, T^{-1} is also bounded.



Proof. We need to show that $T-1$ is continuous Let $V \subset \mathscr{B}$ be open. Then, we need to show that

$$(T^{-1})^{-1}V \qquad \text{is open in } \mathscr{B}',$$

which follows from the open mapping theorem. ∎

Proof of open mapping theorem. It is sufficient to show that T maps the open ball $\mathfrak{B}_{\mathscr{B}}$ into an open set. To show this, it is sufficient to show that there exists a $\delta > 0$ such that

$$T(\mathfrak{B}_{\mathscr{B}}) \supset \delta \mathfrak{B}_{\mathscr{B}'}.$$

Indeed, let $y \in T(\mathfrak{B}_{\mathscr{B}})$. Then there exists an $x \in \mathfrak{B}_{\mathscr{B}}$ such that $y = T(x)$. Let $\eta > 0$ be such that

$$B(x,\eta) \subset \mathfrak{B}_{\mathscr{X}},$$

then by linearity of T

$$T(\mathfrak{B}_{\mathscr{B}}) \supset T(B(x,\eta)) = T(x) + \eta\, T(\mathfrak{B}_{\mathscr{B}}) \supset y + \eta\,\delta \mathfrak{B}_{\mathscr{B}'},$$

i.e., y is an interior point of $T(\mathfrak{B}_{\mathscr{B}})$ and the latter is indeed open.

We can express $\mathscr{B}$ as follows:

$$\mathscr{B} = \bigcup_{n=1}^{n} n\mathfrak{B}_{\mathscr{B}}.$$

Since T is linear and surjective, $\mathscr{B}' = T(\mathscr{B})$, namely,

$$\mathscr{B}' = T\left(\bigcup_{n=1}^{n} n\mathfrak{B}_{\mathscr{B}}\right) = \bigcup_{n=1}^{n} n\,T(\mathfrak{B}_{\mathscr{B}}) = \bigcup_{n=1}^{n} n\overline{T(\mathfrak{B}_{\mathscr{B}})}.$$

Since $\mathscr{B}'$ is complete, Baire's category theorem asserts that one of the sets $n\overline{T(\mathfrak{B}_{\mathscr{B}})}$ has a non-empty interior, but since these sets are all blowups of the same set, it follows that $\overline{T(\mathfrak{B}_{\mathscr{B}})}$ has a non-empty interior.

Thus, there is an open ball $B(a,\delta) \subset \overline{T(\mathfrak{B}_{\mathscr{B}})}$. The set $\overline{T(\mathfrak{B}_{\mathscr{B}})}$ symmetric about zero, hence $B(-a,\delta) \subset \overline{T(\mathfrak{B}_{\mathscr{B}})}$. Since it is also convex,

$$\delta\,\mathfrak{B}_{\mathscr{B}'} \subset \overline{T(\mathfrak{B}_{\mathscr{B}})}.$$

Let $A \subset \mathfrak{B}_{\mathscr{B}}$ be a set for which $T(A)$ is dense in $T(\mathfrak{B}_{\mathscr{B}})$. Let $y \in \delta\,\mathfrak{B}_{\mathscr{B}'}$. Since $y \in \overline{T(\mathfrak{B}_{\mathscr{B}})}$, it has a representation

$$y = \sum_{n=1}^{\infty} \lambda_n T(x_n),$$

where $x_n \in A$ and $\sum_{n=1}^{\infty} |\lambda_n| < 1$ (Lemma 2.79). The sequence $\sum_{n=1}^{\infty} \lambda_n x_n$ is a Cauchy sequence because

$$\left\| \sum_{k=m}^{n} \lambda_k x_k \right\| \le \sum_{k=m}^{n} |\lambda_k|.$$

Since $\mathscr{X}$ is complete it converges to a limit $x \in \mathfrak{B}_{\mathscr{B}}$. Since T is continuous,

$$y = T(x) \in T(\mathfrak{B}_{\mathscr{B}}),$$

i.e., $\delta\,\mathfrak{B}_{\mathscr{B}'} \subset T(\mathfrak{B}_{\mathscr{B}})$, which completes the proof. ∎

▪ **Example 2.12** Consider the Banach space $c_0(\mathbb{Z})$ of two-sided infinite sequences (x_n) that tend to zero as $|n| \to \infty$. Define the linear operator $T : L^1[0,2\pi] \to c_0(\mathbb{Z})$,

$$(Tf)_n = \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\imath nx} f(x)\,dx.$$

The Riemann-Lebesgue lemma states that indeed, T maps $L^1[0,2\pi]$ into $c_0(\mathbb{Z})$, where $c_0(\mathbb{Z}) \subset \ell_\infty(\mathbb{Z})$ is endowed with the supremum norm.

T is bounded because

$$\|Tf\|_\infty \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{-\imath nx} f(x)|\,dx = \|f\|_{L^1}.$$

T is injective—the Fourier coefficients identify the function uniquely (this has to be shown). Is T surjective? Does any sequence of Fourier coefficients that decay at infinite correspond to an $L^1[0,2\pi]$ function?

No! If T were surjective, this would imply that $L^1[0,2\pi]$ and $c_0(\mathbb{Z})$ are isomorphic, but this is impossible because $c_0^* = \ell_1$ is separable whereas $L^1[0,2\pi]^* = L^\infty[0,2\pi]$ is not separable (Proposition 2.32). ■

Comment 2.28 You saw in the tutoring session that

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{\sqrt{k}}$$

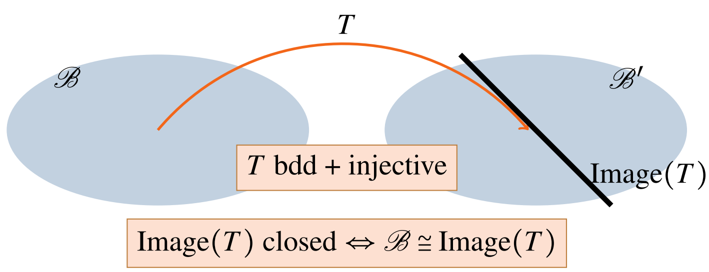
converges pointwise but is is not an $L^2[0,2\pi]$ function. In other words, the $c_0(\mathbb{Z})$ sequence $x_n = 1/\sqrt{n}$ is not the Fourier coefficients of an integrable function.

We now present a few corollaries of the open mapping theorem:

Corollary 2.97 Let $\mathscr{B}$ and $\mathscr{B}'$ be Banach spaces and let $T : \mathscr{B} \hookrightarrow \mathscr{B}'$ linear, bounded and injective. Then, Image(T) is closed if and only if there exists a constant $C > 0$ such that

$$\forall x \in \mathscr{B} \qquad \|Tx\| \geq C\|x\|.$$

Comment 2.29 This corollary states that if T is bounded and injective, then the image of T closed if and only if $T^{-1} : \text{Image}(T) \to \mathscr{B}$ is bounded.



Proof. **Suppose that** Image(T) **is closed**.Then Image(T) is a Banach space, and by Corollary 2.96 $T : \mathscr{B} \to \text{Image}(T)$ is an isomorphism.

Suppose that $\|Tx\| \geq C\|x\|$. We need to show that Image(T) is closed. Let $y \in$ Image(T). Then there exists a sequence x_n such that

$$\lim_{n\to\infty} \|Tx_n - y\| = 0.$$

The sequence Tx_n, being convergent, is a Cauchy sequence. Since

$$\|x_n - x_m\| \leq \frac{1}{C}\|T(x_n - x_m)\| = \frac{1}{C}\|Tx_n - Tx_m\|,$$

it follows that (x_n) is a Cauchy sequence and hence converges to a limit x. Finally, since T is bounded:

$$Tx = \lim_{n\to\infty} Tx_n = y,$$

i.e., $y \in \text{Image}(T)$, and therefore $\text{Image}(T)$ is closed. ∎

Corollary 2.98 Let $\mathscr{B}$ and $\mathscr{B}'$ be Banach spaces. There exists a bounded linear operator from $\mathscr{B}$ onto $\mathscr{B}'$ if and only if $\mathscr{B}'$ is isomorphic to a quotient space of $\mathscr{B}$.

Proof. **Suppose that $\mathscr{B}'$ is isomorphic to $\mathscr{B}/M$.** Denote the isomorphism $\tau : \mathscr{B}/M \to \mathscr{B}'$ and $\pi : \mathscr{B} \to \mathscr{B}/M$ be the natural homomorphism. Then $\tau \circ \pi$ is a bounded linear operator $\mathscr{B} \to \mathscr{B}'$ that is surjective.

Suppose that $T : \mathscr{B} \twoheadrightarrow \mathscr{B}'$ is linear, bounded, and surjective. Then $\ker T$ is a closed subspace of $\mathscr{B}$ and $\sigma : \mathscr{B}/\ker T \to \mathscr{B}'$ defined by

$$\sigma[x] = Tx$$

is bijective. It is also bounded as for all $y \in [x]$,

$$\|\sigma[x]\| \le \|T\|\|y\|$$

and taking the infimum over all $y \in [x]$,

$$\|\sigma[x]\| \le \|T\|\|[x]\|.$$

By Corollary 2.96 σ is an isomorphism. ∎

Corollary 2.99 Let $\mathscr{V}$ be a vector space, which is a Banach space with respect to two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose that $\|\cdot\|_1 \le C\|\cdot\|_2$. Then the two norms are equivalent, i.e., there exists a constant $c > 0$ such that $c\|\cdot\|_2 \le \|\cdot\|_1$.

Proof. Consider the mapping:

$$\text{Id} : (\mathscr{V}, \|\cdot\|_2) \to (\mathscr{V}, \|\cdot\|_1).$$

It is a linear bijection. It is also bounded as

$$\|\text{Id}(x)\|_1 = \|x\|_1 \le C\|x\|_2.$$

Hence Id is an isomorphism and its inverse is bounded. ∎

■ **Example 2.13** Let $\mathscr{B} = (C(0,1), \|\cdot\|_0)$. Let $\mathscr{X}_0 \subset \mathscr{X}$ be a closed subspace whose elements are all continuously differentiable. Then $\mathscr{X}_0$ is finite dimensional. ■

Comment 2.30 This is an interesting statement. Any infinite-dimensional closed subspace of $C(0,1)$ contains functions that are not in $C^1(0,1)$.

Proof. Since $\mathscr{X}_0$ is closed, it is a Banach space with respect to the sup-norm $\|\cdot\|_0$. Define on $\mathscr{X}_0$ the $C^1(0,1)$ norm:

$$\|f\|_1 = \|f\|_0 + \|f'\|_0.$$

Since $\|f\|_0 \le \|f\|_1$, it follows from the previous corollary that the two norms are equivalent, i.e., there exists a constant $c > 0$ such that

$$c\|f\|_1 \le \|f\|_0,$$

i.e.,

$$\|f'\|_0 \le \frac{1-c}{c}\|f\|_0.$$

It follows that the elements in

$$\mathfrak{B}_{(\mathscr{X}_0, \|\cdot\|_0)} = \{f \in \mathscr{X}_0 \mid \|f\|_0 = 1\}$$

are uniformly bounded and equi-continuous. By the Arzela-Ascoli theorem this set is compact, and in particular totally bounded.

Theorem 2.46 states that a normed space has finite dimension if and only if its unit ball is totally bounded. This concludes the proof.

■

2.6.2 The closed graph theorem

Definition 2.22 Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. Let $D(T) \subset \mathscr{X}$ be a linear subspace and $T : D(T) \to \mathscr{Y}$ a linear map. T is called **closed** if

$$D(T) \ni x_n \to x \qquad \text{and} \qquad Tx_n \to y,$$

implies that

$$x \in D(T) \qquad \text{and} \qquad y = Tx.$$

Comment 2.31 If $D(T)$ is closed, and T is bounded, then T is clearly closed, as the closure of $D(T)$ implies that $x_n \to x$ implies $x \in D(T)$. Furthermore, if T is continuous, then

$$Tx_n \to Tx.$$

It is convenient to characterize closed maps through their graph. Consider the cartesian product $\mathscr{X} \times \mathscr{Y}$, which we endow with a linear structure,

$$\alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2),$$

and a norm

$$\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

If $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces, so is their cartesian product.

Given a mapping T, its **graph** is

$$G(T) = \{(x, Tx) \mid x \in D(T)\} \subset \mathscr{X} \times \mathscr{Y}.$$

If T is linear, then $G(T)$ is a linear subspace of $\mathscr{X} \times \mathscr{Y}$.

Lemma 2.100 A linear subspace M of $\mathscr{X} \times \mathscr{Y}$ is the graph of a function if and only if it does not contain elements of the form $(0, y)$ for $y \neq 0$.

Proof. M is the graph of a function if and only if for every $x \in D(T)$:

$$(x,y) \in M \quad \text{and} \quad (x,z) \in M \quad \text{implies} \quad y = z,$$

which is turn occurs if and only if

$$(0, y - z) \in M \quad \text{implies} \quad y - z = 0.$$

∎

Proposition 2.101 — Closed operator = closed graph. A linear operator $T : D(T) \to \mathscr{Y}$ is closed if and only if its graph $G(T)$ is closed in $\mathscr{X} \times \mathscr{Y}$.

Proof. The definition of a closed map is that

$$G(T) \ni (x_n, Tx_n) \to (x,y) \qquad \text{in } \mathscr{X} \times \mathscr{Y}$$

implies that $x \in D(T)$ and $y = Tx$, namely

$$(x,y) \in G(T),$$

which is precisely the condition for $G(T)$ being closed. ∎

Corollary 2.102 Let $T : D(T) \hookrightarrow \mathscr{Y}$ be linear and injective. Then T is closed if and only if $T^{-1} : \text{Image}(T) \to D(T)$ is closed.

Proof. If T is injective then T^{-1} is well-defined on Image(T). Its graph is

$$G(T^{-1}) = \{(Tx, x) : x \in D(T)\} \subset \mathscr{Y} \times \mathscr{X}.$$

This graph is closed if and only if $G(T)$ is closed, namely

$$\{(Tx, x) : x \in D(T)\} \quad \text{is closed in} \quad \mathscr{Y} \times \mathscr{X}$$

if and only if

$$\{(x, Tx) : x \in D(T)\} \quad \text{is closed in} \quad \mathscr{X} \times \mathscr{Y}.$$

■

Definition 2.23 A linear map $T : D(T) \to \mathscr{Y}$ is called **closable** (**סגירה**) if the closure of its graph $\overline{G(T)}$ is the graph of a linear map. This linear map, if it exists, is an extension of T.

Proposition 2.103 et $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. A linear map $T : D(T) \to \mathscr{Y}$ is closable if and only if

$$D(T) \ni x_n \to 0 \quad \text{and} \quad Tx_n \to y \quad \text{implies} \quad y = 0.$$

Namely,

$$(x_n, Tx_n) \to (0, y)$$

implies $y = 0$.

Proof. The conditions

$$x_n \to 0 \quad \text{and} \quad Tx_n \to y$$

amount to the condition that $(0, y) \in \overline{G(T)}$. Thus, we need to prove that T is closable iff $(0, y) \in \overline{G(T)}$ implies that $y = 0$.

We have all the pieces: T is closable if and only if $\overline{G(T)}$ is the graph of a function, which in turn holds if and only if $(0, y) \in \overline{G(T)}$ implies that $y = 0$. ■

The following proposition shows that closed maps are not necessarily bounded.

Proposition 2.104 Let $\mathscr{B}$ and $\mathscr{B}'$ be banach spaces. Let $T : \mathscr{B} \hookrightarrow \mathscr{B}'$ be a bounded linear injective map. If Image(T) is not closed, then $T^{-1} : \text{Image}(T) \to \mathscr{B}$ is a closed linear map that is not bounded.

Proof. Since $D(T)$ is closed and T is bounded, then T is closed. It follows from Corollary 2.102 that T^{-1} is closed. If T^{-1} was bounded, it would follow from Corollary 2.97 that Image(T) is closed, which is a contradiction ■

■ **Example 2.14** Let $\mathscr{B} = \mathscr{B}' = C[0,1]$. We define the linear operator $T : \mathscr{B} \to \mathscr{B}'$,

$$(Tf)(x) = \int_0^x f.$$

T is injective. Its image is

$$\text{Image}(T) = \{f \in C^1[0,1] \mid f(0) = 0\}.$$

T is closed as

$$(f_n, \int f_n) \to (f,g) \qquad \text{in } C[0,1] \times C[0,1]$$

implies that $g = \int f$. The image of T is a linear subspace that is not closed (otherwise it would have had finite dimension). Hence, T^{-1}, which is given by

$$T^{-1} f = f'$$

is closed but not bounded. ■

■ **Example 2.15** Let $\mathscr{B} = \mathscr{B}' = L^2(0,1)$. Let $(a_0, a_1, \dots, a_N)$ functions in $C^N(0,1)$. Consider the linear operator

$$T = \sum_{k=0}^N a_k(x) \frac{d^k}{dx^k}.$$

Its domain is

$$D(T) = \{f \in L^2(0,1) \cap C^N(0,1) \mid Tf \in L^2(0,1)\}.$$

We will show that T is closable. Suppose that $f_n \to 0$ and $Tf_n \to g$ (all limits in $L^2(0,1)$). Let $\varphi \in C_0^\infty(0,1)$ (a test function). Integrating by parts we find:

$$(\varphi, Tf_n) = (T^* \varphi, f_n),$$

where

$$T^* \varphi = \sum_{k=0}^N (-1)^k \frac{d^k}{dx^k} (a_k \varphi).$$

Letting $n \to \infty$ and using the continuity of the inner-product:

$$(\varphi, g) = (T^* \varphi, 0) = 0.$$

Since the tests functions are dense in $L^2(0,1)$ it follows that $g = 0$. T is closable by Proposition 2.103. ■

We finally arrive to the closed graph theorem:

Theorem 2.105 — Closed graph. Let $\mathscr{B}$ and $\mathscr{B}'$ be banach spaces. A closed linear operator $T:\mathscr{B}\to\mathscr{B}'$ is bounded.

Comment 2.32 Closed is not equivalent to bounded in a general normed space.

Proof. It is given that

$$G(T)=\{(x,Tx):x\in\mathscr{B}\}$$

is a closed subspace of $\mathscr{B}\times\mathscr{B}'$. The map

$$\pi_2:G(T)\to\mathscr{B}'$$

defined by

$$\pi_2(x,y)=y$$

is linear. It is bounded because

$$\|\pi_2(x,Tx)\|=\|Tx\|\le\sqrt{\|x\|^2+\|Tx\|^2}=\|(x,Tx)\|.$$

Similarly, the map

$$\pi_1:G(T)\to\mathscr{B}$$

defined by

$$\pi_1(x,y)=x$$

is linear, bounded, and bijective. Hence, it follows from the open mapping theorem that π_1^{-1} is bounded. Since

$$T=\pi_2\circ\pi_1^{-1},$$

it follows that T is bounded. ■

Proposition 2.106 Let $\mathscr{B},\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces. Let

$$T_1:D(T_1)\subset\mathscr{B}\to\mathscr{B}_1 \qquad\text{and}\qquad T_2:D(T_2)\subset\mathscr{B}\to\mathscr{B}_2$$

be linear operators. Suppose that T_1 closable and T_2 is closed. If $D(T_2)\subset D(T_1)$ then there exists a constant $C>0$ such that

$$\|T_1x\|\le C\left(\|x\|+\|T_2x\|\right).$$

Proof. $G(T_2)$ is closed hence as a closed subspace of a Banach space it is a Banach space. Consider the linear operator $\pi:G(T_2)\to\mathscr{B}_1$,

$$\pi(x,T_2x)=T_1x.$$

We will show that it is closed. Suppose that the sequence $(x_n, T_2 x_n)$ converges in $G(T_2)$ and that $T_1 x_n$ converges in $\mathscr{B}_1$. Since $G(T_2)$ is closed, there exists an $x \in D(T_2)$ such that

$$(x_n, T_2 x_n) \to (x, T_2 x).$$

Since $D(T_2) \subset D(T_1)$ it follows that $x \in D(T_1)$ and since T_1 is closable, the limit of $T_1 x_n$ is $T_1 x$. Hence π is a closed operator. From the closed graph theorem it is bounded. Thus,

$$\|T_1 x\| = \|\pi(x, T_2)\| \le C\|(x, T_2 x)\| = C\sqrt{\|x\|^2 + \|T_2 x\|^2} \le C\left(\|x\| + \|T_2 x\|\right).$$

■

■ **Example 2.16** Let $\mathscr{B} = \mathscr{B}_1 = \mathscr{B}_2 = C[0,1]$. Let $T_1 : C^1[0,1] \to C[0,1]$ be defined by

$$T_1 f = f'.$$

We know that T_1 is closed (and in particular closable). Define $T_2 : C^2[0,1] \to C[0,1]$ by

$$T_2 f = f''.$$

T_2 is a closed linear operator. By the previous proposition there exists a constant C such that for all $f \in C^2[0,1]$:

$$\max_x |f'(x)| \le C\left(\max_x |f(x)| + \max_x |f''(x)|\right).$$

■