Appendix A

Sequences and series

This course has for prerequisite a course (or two) of calculus. The purpose of this appendix is to review basic definitions and facts concerning sequences and series, which are needed for this course.

A.1 Sequences

An (infinite) sequence (סדרה) is an infinite ordered set of real numbers. We denote sequences by a letter which labels the sequence, and a subscript which labels the position in the sequence, for example,

$$a_1, a_2, a_3, \dots .$$

The essential fact is that to each natural number corresponds a real number. Having said that, we can define:

Definition A.1 A sequence is a function $\mathbb{N} \to \mathbb{R}$.

Definition A.2 A sequence (a_n) converges *(מתכנסת) to* $\ell \in \mathbb{R}$*, denoted*

$$\lim_{n\to\infty} a_n = \ell,$$

if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) \; : \; (\forall n > N)(|a_n - \ell| < \epsilon).$$

Definition A.3 *A sequence is called* convergent *(מתכנסת) if it converges to some (real) number; otherwise it is called* divergent *(מתבדרת) (there is no such thing as convergence to infinity; we rather say that a sequence tends (שואפת) to infinity).*

Theorem A.1 *Let a and b be convergent sequences. Then the sequences $(a+b)_n = a_n + b_n$ and $(a \cdot b)_n = a_n b_n$ are also convergent, and*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

If furthermore $\lim_{n\to\infty} b_n \neq 0$, then there exists an $N \in \mathbb{N}$ such that $b_n \neq 0$ for all $n > N$, the sequence (a_n/b_n) converges and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Comment: Since the limit of a sequence is only determined by the "tail" of the sequence, we do not care if a *finite* number of elements is not defined.

Theorem A.2 *Let f be defined on a (punctured) neighborhood of a point c. Suppose that*

$$\lim_{x\to c} f(x) = \ell.$$

If a is a sequence whose entries belong to the domain of f, such that $a_n \neq c$, and

$$\lim_{n\to\infty} a = c,$$

then

$$\lim_{n\to\infty} f(a_n) = \ell,$$

Conversely, if $\lim_{n\to\infty} f(a_n) = \ell$ for every *sequence a_n that converges to c, then $\lim_{x\to c} f(x) = \ell$.*

Comment: This theorem provides a characterization of the limit of a function at a point in terms of sequences. This is known as *Heine's characterization* of the limit.

Theorem A.3 *A convergent sequence is bounded.*

Theorem A.4 *Let* (a_n) *be a bounded sequence and let* (b_n) *be a sequence that converges to zero. Then*

$$\lim_{n\to\infty}(a_n b_n) = 0.$$

Theorem A.5 *Let* (a_n) *be a sequence with non-zero elements, such that*

$$\lim_{n\to\infty} a_n = 0.$$

Then

$$\lim_{n\to\infty} \frac{1}{|a_n|} = \infty.$$

Theorem A.6 *Let* (a_n) *be a sequence such that*

$$\lim_{n\to\infty} |a_n| = \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Theorem A.7 *Suppose that* (a_n) *and* (b_n) *are convergent sequences,*

$$\lim_{n\to\infty} a_n = \alpha \qquad \text{and} \qquad \lim_{n\to\infty} b_n = \beta,$$

and $\beta > \alpha$*. Then there exists an* $N \in \mathbb{N}$*, such that*

$$b_n > a_n \quad \forall n > N,$$

i.e., the sequence (b_n) *is eventually greater (term-by-term) than the sequence* (a_n)*.*

Corollary A.1 Let (a_n) be a sequence and $\alpha,\beta \in \mathbb{R}$. If the sequence (a_n) converges to α and $\alpha > \beta$ then eventually $a_n > \beta$.

Theorem A.8 Suppose that (a_n) and (b_n) are convergent sequences,

$$\lim_{n\to\infty} a_n = \alpha \qquad \text{and} \qquad \lim_{n\to\infty} b_n = \beta,$$

and there exists an $N \in \mathbb{N}$, such that $a_n \le b_n$ for all $n > N$. Then $\alpha \le \beta$.

Theorem A.9 (Sandwich) Suppose that (a_n) and (b_n) are sequences that converge to the same limit ℓ. Let (c_n) be a sequence for which there exists an $N \in \mathbb{N}$ such that

$$a_n \le c_n \le b_n \quad \forall n > N.$$

Then

$$\lim_{n\to\infty} c_n = \ell.$$

Definition A.4 A sequence (a_n) is called increasing (עולה) if $a_{n+1} > a_n$ for all n. It is called non-decreasing (לא יורדת) if $a_{n+1} \ge a_n$ for all n. We define similarly a decreasing and a non-increasing sequence.

Theorem A.10 (Bounded + monotonic = convergent) Let (a_n) be a non-decreasing sequence bounded from above. Then it is convergent.

Definition A.5 Let (a_n) be a sequence. A subsequence (תת סדרה) of (a_n) is any sequence

$$a_{n_1}, a_{n_2}, \dots,$$

such that

$$n_1 < n_2 < \cdots.$$

More formally, (b_n) is a subsequence of (a_n) if there exists a monotonically increasing sequence of natural numbers (n_k), such that $b_k = a_{n_k}$ for all $k \in \mathbb{N}$.

Definition A.6 *Let* (a_n) *be a sequence. A number* ℓ *is called a* partial limit (a_n) *(גבול חלקי) if* ℓ *is the limit of a subsequence of* (a_n).

Lemma A.1 *Any sequence contains a subsequence which is either non-decreasing or non-increasing.*

Corollary A.2 (Bolzano-Weierstraß) *Every bounded sequence has a convergent subsequence. That is, it has at least one partial limit.*

Theorem A.11 *If a bounded sequence has a unique partial limit, then the sequence is convergent and this partial limit is in fact its limit.*

In many cases, we would like to know whether a sequence is convergent even if we do not know what the limit is. We will now provide such a convergence criterion.

Definition A.7 *A sequence* (a_n) *is called a* Cauchy sequence *if*

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) \; : \; (\forall m, n > N)(|a_n - a_m| < \epsilon).$$

Theorem A.12 *A sequence converges if and only if it is a Cauchy sequence.*

Theorem A.13 *Let* (a_n) *be a bounded sequence. Then, the sequence*

$$b_n = \sup\{a_k \; : \; k \geq n\}$$

is monotonically decreasing, hence has a limit which is called the superior limit *(גבול עליון) of the sequence* (a_n),

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \; : \; k \geq n\}.$$

Furthermore, the superior limit is the largest partial limit of (a_n).

Likewise, the sequence

$$c_n = \inf\{a_k \,:\, k \geq n\}$$

is monotonically increasing, hence has a limit which is called the inferior limit *(גבול תחתון) of the sequence* (a_n)*,*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \,:\, k \geq n\}.$$

Furthermore, the inferior limit is the smallest partial limit of (a_n)*.*

The following is obvious:

Proposition A.1 *For every sequence* (a_n)*,*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

A.2 Series

Let (a_n) be a sequence. We define the sequence of *partial sums* (סכומים חלקיים) of (a_n) by

$$S_n^a = \sum_{k=1}^n a_k.$$

Note that (S_n^a) does not depend on the order of summation (a finite summation). If the sequence (S_n^a) converges, we will interpret its limit as the infinite sum of the sequence.

Definition A.8 *A sequence* (a_n) *is called* summable *(סכימה) if the sequence of partial sums* (S_n^a) *converges. In this case we write*

$$\sum_{k=1}^\infty a_k = \lim_{n\to\infty} S_n^a.$$

The sequence of partial sum is called a series *(טור). Often, the term series refers also to the limit of the sequence of partial sums.*

Comment: Note the terminology:

sequence is summable = series is convergent.

Theorem A.14 *Let (a_n) and (b_n) be summable sequences and $\gamma \in \mathbb{R}$. Then the sequences $(a_n + b_n)$ and (γa_n) are summable and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$

and

$$\sum_{n=1}^{\infty}(\gamma a_n) = \gamma \sum_{n=1}^{\infty} a_n.$$

An important question is how to identify summable sequences, even in cases where the sum is not known. This brings us to discuss *convergence criteria*.

Theorem A.15 (Cauchy's criterion) *The sequence a is summable if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$, such that*

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon \quad \forall m > n > N.$$

Corollary A.3 *If a is summable then*

$$\lim_{n\to\infty} a_n = 0.$$

The vanishing of the sequence is only a *necessary* condition for its summability. It is not sufficient as proves the *harmonic series*, $a_n = 1/n$,

$$S_n^a = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \cdots.$$

This grouping of terms shows that the sequence of partial sums is unbounded.

Theorem A.16 *A non-negative sequence is summable if and only if the series (i.e., the sequence of partial sums) is bounded.*

Theorem A.17 (Comparison test) *If $0 \le a_n \le b_n$ for every n and the sequence (b_n) is summable, then the sequence (a_n) is summable.*

Theorem A.18 (Limit comparison) *Let (a_n) and (b_n) be positive sequences, such that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \gamma \neq 0.$$

Then (a_n) is summable if and only if (b_n) is summable.

Theorem A.19 (Ratio test (מבחן המנה)) *Let $a_n > 0$ and suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

(i) If $r < 1$ then (a_n) is summable. (ii) If $r > 1$ then (a_n) is not summable.

Definition A.9 *Let f be integrable on any segment $[a,b]$ for some a and $b > a$. Then*

$$\int_a^\infty f(t)\,dt = \lim_{x\to\infty} \int_a^x f(t)\,dt.$$

Theorem A.20 (Integral test) *Let f be a positive decreasing function on $[1,\infty)$ and set $a_n = f(n)$. Then $\sum_{n=1}^\infty a_n$ converges if and only if $\int_1^\infty f(t)\,dt$ exists.*

We have dealt with series of non-negative sequences. The case of non-positive sequences is the same as we can define $b_n = (-a_n)$. Sequences of non-fixed sign are a different story.

Definition A.10 *A series* $\sum_{n=1}^{\infty} a_n$ *is said to be* absolutely convergent *(מתכנסת בהחלט) if* $\sum_{n=1}^{\infty} |a_n|$ *converges. A series that converges but does not converge absolutely is called* conditionally convergent *(מתכנסת בתנאי).*

Theorem A.21 *Every absolutely convergent series is convergent. Also, a series is absolutely convergent, if and only if the two series formed by its positive elements and its negative elements both converge.*

Theorem A.22 (Leibniz) *Suppose that* a_n *is a non-increasing sequence of non-negative numbers, i.e.,*

$$a_1 \geq a_2 \geq \cdots \geq 0,$$

and

$$\lim_{n\to\infty} a_n = 0.$$

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Definition A.11 *A* rearrangement *(סידור מחדש) of a sequence* a_n *is a sequence*

$$b_n = a_{f(n)},$$

where $f : \mathbb{N} \to \mathbb{N}$ *is one-to-one and onto.*

Theorem A.23 (Riemann) *If* $\sum_{n=1}^{\infty} a_n$ *is conditionally convergent then for every* $\alpha \in \mathbb{R}$ *there exists a rearrangement of the series that converges to* α.

Theorem A.24 *If* $\sum_{n=1}^{\infty} a_n$ *converges absolutely then any rearrangement* $\sum_{n=1}^{\infty} b_n$ *converges absolutely, and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

In other word, the order of summation does not matter in an absolutely convergent series.

Theorem A.25 *Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, and that c_n is any sequence that contains all terms $a_n b_k$. Then*

$$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{k=1}^{\infty} b_k\right).$$