Non-Euclidean Plates

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Introduction
How do thin sheets “shape themselves”?

Nature exhibits endless systems where complex patterns emerge from a-priori structureless objects.

For example, leaves and flowers exhibit rippled shapes even though their “vein structure” seems perfectly symmetric.

Leaves and flowers are examples of thin bodies that exhibit symmetry breaking even in the absence of external forces.
The **buckling** of thin elastic materials is a well-known example of symmetry breaking.

A thin plate is compressed laterally at its ends.

For weak enough forcing the plate remains flat and compresses.

Above a critical force the plate buckles.
**Interpretation** [Euler-Bernoulli, -1750]:

Compression **changes the metric** of the plate. The “energy cost” of metric change is **linear** in the thickness of the plate.

At a critical compression it is energetically favorable to **bend**, as bending energy is only **cubic** in the plate thickness.

Once the external forces are removed the beam regains its flat conformation. This is not the case for flowers and leaves, which are buckled and wrinkled even **in the absence of external forcing**.
Torn plastic sheets

Example: another (fun) use for garbage bags

A thin sheet of polyethylene is torn by cutting a short slit and pulling either side apart.

The torn edge curls up into a wavy pattern, which remains after tearing is over.
Torn plastic sheet are reminiscent of patterns encountered in plants. The outer edge undergoes a cascade of 
spontaneous buckling transitions.

The rippled structure exhibits a fractal-like behavior. One can identify up to 6 generations of self-
similar patterns. The cutoff seems to be associated with the thickness of the sheet.
Why do torn plastic sheets behave this way?

Controlled experiments show that the ripples are not due to oscillations in tearing process. Tearing causes a plastic deformation which is symmetric (homogenous) along the tearing axis (i.e., ripples break the symmetry).

Tearing causes a change in the metric of the sheet. There is an increase in length along the tear. The magnitude of this excessive length is highest at the edge, and decays away from it.
Torn plastic sheet on which a grid of dots has been printed to measure the change of metric after tearing.

Recall Gauss’ *Theorema Egregium*: the metric determines the Gaussian curvature of the surface.

A metric of the type generated by the tearing is *hyperbolic*. The symmetry breaking results (apparently) from the difficulty of embedding a hyperbolic surface in 3D space.

Simple metric → Complex pattern
It is generally so that growth processes that generate excessive length at the growth boundary result in hyperbolic metrics, which cannot be embedded without symmetry breaking and rippling.

Leaf whose outer boundary has been treated with a growth hormone.

5 rubber bands of increasing lengths glued to each other, creating a gradually increasing “excessive length”.
Research Goals

• Derive a physical model for thin elastic sheets that are structurally symmetric, but exhibit the kind of “metric mismatch” that induces symmetry breaking.

• A model resting on “fundamental” principles.

• Predictions versus experimental data.

• Clarify the roles of geometry versus elasticity.

• New mathematical questions?
Modeling
3D and 2D elasticity

Even though we eventually want to treat thin sheets as 2D surfaces, the starting point has to be a model for 3D bodies.

Elasticity theories assume the existence of a rest configuration in which all internal forces vanish. The state of the body is described relative to this rest configuration (the deviation is called the strain). An energy functional is constructed, (usually) quadratic in the strain.

**Problem:**
We want to model thin bodies that exhibit residual stress even in the absence of forces or constraints. There is no stress-free rest configuration!
Plates and shells

Once a model for 3D bodies is formulated, one wants to take advantage of the thin aspect ratio, and derive a model for a 2D surface.

Two types of thin sheets are commonly encountered:

- **Plates**
  - Can be thought of as a continuous stack of *flat identical* surfaces glued together.
  - **Homogeneous along the thin axis.**
  - **Flat** rest configuration.
Can be thought of as a continuous stack of not necessarily flat, non-identical surfaces glued together.

• **Curved** rest configuration.

**Problem:**
Our thin sheets are neither plates nor shells.

Like **plates**: homogeneous along thin dimension.
Like **shells**: no flat rest configuration.
3D differential geometry

Domain of parametrization (local Cartesian coordinates)

Configuration

The **metric**: \[ g_{ij} = \frac{\partial r}{\partial x^i} \cdot \frac{\partial r}{\partial x^j} \] (Also known as the Cauchy-Green tensor)

Metric defines uniquely configuration [up to translation and rotation] (rigidity).

A symmetric positive-definite 2-tensor field is a metric of a 3D body if its **Riemann curvature tensor** vanishes [solvability condition].
Linear and nonlinear elasticity

The fundamental variable in elasticity theory is the deviation from the stress-free rest configuration:

\[ r(x) = x + u(x) \]

The energy density is assumed to be a function of the distance of the metric from the identity (Green-St Venant Strain):

\[ \nabla r^T \nabla r - I = \nabla u + (\nabla u)^T + (\nabla u)^T (\nabla u) \]
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Linear elasticity refers to neglecting the quadratic term (at the cost of loss of invariance under rotations!)
Linear and nonlinear elasticity

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**Linear elasticity** refers to neglecting the quadratic term (at the cost of loss of invariance under rotations!)

In **nonlinear elasticity** the quadratic term is retained. Equivalently,

\[ w(x) = \text{function}(\text{dist}(\nabla r(x), \text{SO}(3))) \]
“Incompatible” 3D elasticity

Hyper-elasticity principle [Truesdell 1952]:

The energy stored within a deformed elastic body is a volume integral of an elastic density, \( w \). This density depends only on the local metric tensor and on tensors that characterize the body, but are independent of the configuration, \( w = w(g,x) \).

- For every \( x,g \), \( w(g,x) \geq 0 \).
- For every \( x \) there exists a unique SPD tensor \( \bar{g}(x) \), such that \( w(\bar{g}(x),x) = 0 \).
- \( w \) is twice differentiable near \( \bar{g} \).

\[
\begin{align*}
  w &= \frac{1}{2} A_{ijkl} \epsilon_{ij} \epsilon_{kl} + O(\epsilon^4) \\
  \epsilon_{ij} &= \frac{1}{2} (g_{ij} - \bar{g}_{ij})
\end{align*}
\]
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- \( \omega \) is twice differentiable near \( \bar{g} \).

Truesdell’s principle was formulated in terms of displacements. We rather use the metric [Ciarlet 2005]

Very important (!!): The reference metric is not necessarily immersible in \( \mathbb{R}^3 \) (“incompatible” elasticity)

\[
\omega = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + O(\varepsilon^4) \quad \varepsilon_{ij} = \frac{1}{2} (g_{ij} - \bar{g}_{ij})
\]
Reduction to “standard” elasticity

If $\bar{g}(x)$ is an immersible metric, then there exists a rest configuration with metric $g(x) = \bar{g}(x)$.

Then, take $x$ to be a Cartesian parametrization of the rest configuration, i.e., $\bar{g}_{ij}(x) = \delta_{ij}$. Define the displacement $u(x) = r(x) - x$. Then, $\varepsilon_{ij}$ reduces to the Green-St. Venant strain tensor:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)$$

Similarly,

$$s^{ij} = \frac{dw}{d\varepsilon_{ij}} = A^{ijkl} \varepsilon_{kl}$$

generalizes the second Piola-Kirchhoff stress tensor.
What is the “elasticity tensor” $A_{ijkl}$?

Let $p$ be any point in the body. We can always define a re-parametrization $x'$ for which $\bar{g}_{ij}(x') = \delta_{ij}$ at the point $p$ (but only there!)

If the medium is isotropic then $(A')_{ijkl}$ must be isotropic:

$$(A')_{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right)$$

Lamé coefficients
It is here that the **covariant** formulation becomes valuable:

**$x$ parametrization**

$$w = \frac{1}{2} A^{ijkl} \epsilon_{ij} \epsilon_{kl}$$

$$\epsilon_{ij} = \frac{1}{2} (g_{ij} - \bar{g}_{ij})$$

**$x'$ parametrization**

$$w = \frac{1}{2} (A')^{ijkl} \epsilon'_{ij} \epsilon'_{kl}$$

$$\epsilon'_{ij} = \frac{1}{2} (g'_{ij} - \delta_{ij}) \quad \text{(at } p)$$

$$(A')^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})$$

**Jacobian of transformation**

$$\Lambda^k_i = \partial (x')^k / \partial x^i$$

$$(\Lambda^{-1})^i_k = \partial x^i / \partial (x')^k$$

**Transformation of 2-tensors**

$$\bar{g}_{ij} = \Lambda^k_i \Lambda^l_j \delta_{kl} \quad \text{(at } p)$$

**Invert**

$$(\bar{g}^{-1})^{ij} = (\Lambda^{-1})^i_k (\Lambda^{-1})^j_l \delta^{kl} \equiv \bar{g}^{ij}$$

**Transformation of 4-tensors**

$$A^{ijkl} = (\Lambda^{-1})^i_p (\Lambda^{-1})^j_q (\Lambda^{-1})^k_r (\Lambda^{-1})^l_s (A')^{pqrs}$$

**Substitute**

$$A^{ijkl} = \lambda \bar{g}^{ij} \bar{g}^{kl} + \mu (\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk})$$

**Isotropy**: elastic tensor uniquely determined by reference metric and Lamé coefficients.
Recapitulation

Elastic energy density:

\[ w = \frac{1}{2} A^{ijkl} \epsilon_{ij} \epsilon_{kl} \]  
Truncated at quadratic order in strain

Strain:

\[ \epsilon_{ij} = \frac{1}{2} (g_{ij} - \bar{g}_{ij}) \]  
Expressed in terms of metric rather than displacement
Reference metric \( \bar{g}_{ij} \) non-immersible ("incompatible")

Elastic tensor:

\[ A^{ijkl} = \lambda \bar{g}^{ij} \bar{g}^{kl} + \mu (\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk}) \]  
\( \bar{g}^{ij} \) is the reciprocal of \( \bar{g}_{ij} \)

Total elastic energy:

\[ E = \int w(x) \sqrt{|g|} \, dx^1 \, dx^2 \, dx^3 \]  
Volume element induced by reference metric
Recapitulation

Elastic energy density:

\[ w = \frac{1}{2} A^{ijkl} \epsilon_{ij} \epsilon_{kl} \]

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Static “incompatible” elasticity: the equilibrium configuration is the metric that minimizes the energy functional \( E \) over all metrics that are immersible in \( \mathbb{R}^3 \) (all flat metrics, or metrics for which the Riemann curvature tensor vanishes).

Volume element induced by reference metric
Recapitulation

**Elastic energy density:**

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Corresponding Euler-Lagrange equations: divergence of stress is zero + constraint of immersibility.
First mathematical questions

We defined the equilibrium by the **immersible** $g_{ij}$ that minimizes a weighted $L^2$ distance from non-immersible $\bar{g}_{ij}$. **Does a minimizer exist?**

The equilibrium energy is the $L^2$ distance of $\bar{g}_{ij}$ from the nearest ummersible $g_{ij}$. **Is it bounded away from zero?**

Otherwise, **can you approach** (in $L^2$) a **non-immersible metric with a sequence of immersible metrics?** [curvature depends on 2nd derivative of metric]

Many recent theorems about continuity of immersion on metric [Ciarlet, C. Mardare, S. Mardare]; none rules out this possiblity.
Partial results

**Theorem I:**
1. Let $\bar{g}_{ij}$ be a smooth **non-flat** metric defined on a bounded **two-dimensional** domain. The distance of $\bar{g}_{ij}$ from any metric induced by an immersion of this manifold in the plane is bounded from below by a positive quantity, which is expressible in terms of the Gaussian curvature of $\bar{g}_{ij}$.

   [The same problem in 3D remains open.]

2. (Still in 2D) A minimizer does not necessarily exist (we found counter examples).

   [The model was intended for small strains; stronger nonlinearity required to ensure a minimizer.]
Non-Euclidean plates

**Definition:** A **plate** is an elastic medium for which there exists a parametrization of the form

\[
\bar{g}_{ij} = \begin{pmatrix}
\bar{g}_{11} & \bar{g}_{12} & 0 \\
\bar{g}_{21} & \bar{g}_{22} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and the \(g_{ij}\) are independent of \(x^3\).

The plate is called **even** if domain = \(S \times [-t/2,t/2]\).

It is called **thin** if thickness \(t\) small compared to all other lengths.

3D reference metric non-immersible iff 2D reference metric (of mid-surface) has a non-zero Gaussian curvature (recall it is a metric property).
Reduced plate model

For thin plates we would like to obtain a reduced 2D model: express the energy as a (mid-)surface integral.

A long history...... Recent success using Γ-convergence [Friesecke et al. 2002-6]--but requires assumptions about scaling of equilibrium energy on $\xi$.

Additional modeling assumptions (Kirchhoff-Love):
- Stress tangent to mid-surface.
- Metric of actual configuration of the form

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Substituting assumptions + algebra....
Reduced plate model (cont.)

**Elastic energy:**

\[ E = \int_S w(x^1, x^2) \sqrt{|\bar{g}|} \, dx^1 dx^2 \]

---

**2D energy density:**

\[ w = \frac{t}{2} A^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} + \frac{t^3}{24} A^{\alpha\beta\gamma\delta} h_{\alpha\beta} h_{\gamma\delta} \]

Greek letters for 2D indices

Energy density is a sum of **stretching** + **bending** (cf. Föppl-von Kármán theory)

---

**Elastic tensor:**

\[ A^{\alpha\beta\gamma\delta} = \frac{Y}{1 - \nu^2} \left[ \nu \bar{g}^{\alpha\beta} \bar{g}^{\gamma\delta} + \frac{1}{2} (1 - \nu) \left( \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta} \bar{g}^{\beta\gamma} \right) \right] \]

Young’s modulus  
Poisson’s ratio
Reduced plate model (cont.)

Elastic energy:

\[ E = \int_S w(x^1, x^2) \sqrt{|\bar{g}|} \, dx^1 \, dx^2 \]

Recall: a 2D metric does not uniquely determine conformation of surface. Metric \( g \) and second fundamental form \( h \) satisfy Gauss-Mainardi-Codazzi relations (PDEs).

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Reduced plate model (cont.)

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2D energy density:

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Energy density is a sum of stretching + bending (cf. Föppl-von Kármán theory).

Elastic tensor:

\[ A^{\alpha \beta \gamma \delta} = \frac{1}{2} \left[ \frac{\partial^2 w(x^1, x^2)}{\partial x^\alpha \partial x^\beta} \frac{\partial^2 w(x^1, x^2)}{\partial x^\gamma \partial x^\delta} + g^{\alpha \delta} g^{\beta \gamma} \right] \]

Equilibrium configuration: 1st and 2nd quadratic forms that minimize energy functional, subject to the satisfaction of the GMC relations.

Greek letters for 2D indices
Derivation of 2D model as a thin-plate limit of the 3D model remains an open problem. First requires the well-posedness of the 3D model (in particular, a lower bound on energy is needed to apply the $\Gamma$-convergence technique of Frieseke-James-Müller).
Comments:

• We named such structures non-Euclidean plates. Like plates, homogeneous along the thin direction. Non-Euclidean because the reference metric is non-immersible in Euclidean space.

• Only the 3D reference metric is non-immersible. The 2D reference metric can usually be immersed (with finite bending!) in $\mathbb{R}^3$ (and non-uniquely).

• If $g_{ij}(x) = \bar{g}_{ij}(x)$ we call the configuration an isometric immersion. In this case the bending energy density is

$$w = \frac{Y t^3}{12(1 - \nu^2)} \left[ 2 H^2 + (1 - \nu)K \right]$$

(Willmore functional)

Mean curvature  \quad Gaussian curvature
Example
Making non-Euclidean plates


- A thin sheet of gel is injected in a Hele-Shaw cell.
- Upon heating the gel shrinks; shrinkage inversely proportional to polymer concentration.
- By modulating the polymer concentration, a prescribed (radially symmetric) reference metric can be “programmed”.

![Diagram](image)
Making non-Euclidean plates

- A thin sheet of gel is injected in a Hele-Shaw cell.
- Upon heating the gel shrinks; shrinkage inversely proportional to polymer concentration.
- By modulating the polymer concentration, a prescribed (radially symmetric) reference metric can be “programmed”.

This experimental system (E. Sharon’s lab) allows controlled experiments with non-Euclidean plates.
Polymer concentration increasing (decreasing) with radius generates a hyperbolic (elliptic) metric.

Equilibrium configuration determined by target metric and plate thickness.

hyperbolic (thick and thin)
Reversible shrinkage of a “programmed” (hyperbolic) plate
Modeling of the shrinking gels

Domain of parametrization: a **punctured disc** \((x^1=r, x^2=\theta)\)

**Reference metric:**

\[
\bar{g}_{\alpha\beta}(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & \Phi^2(r) \end{pmatrix}
\]

**Elliptic** metrics (spherical caps); \(K>0\)

\[
\Phi(r) = \frac{1}{\sqrt{K}} \sin \sqrt{K} r \quad \text{perimeter grows slower than linearly}
\]

**Hyperbolic** metrics; \(K<0\)

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\[
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\text{perimeter grows faster than linearly}
\]

Recall: we are looking for the 1st and second fundamental forms of a surface that (i) satisfy the GMC compatibility relations, and (ii) minimize the 2D energy functional.
Numerical results

Elliptic reference metric \((K=1)\), variable thickness \(\ell\):

- Configuration always **axisymmetric**.
- **Buckling** transition: above thickness \(\ell_b\) flat surface.
- For very thin plates metric close to reference metric.
- Boundary layers.

Elastic energy versus plate thickness

- **Isometric immersion** (bending dominated)
- **Flat** (stretching dominated)
Numerical results (cont.)

**Hyperbolic reference metric** \((K=-1)\)

In the hyperbolic case buckling must break axisymmetry. Full 2D simulations can be done but become difficult for thin plates (details later).

\[
i=80000 \quad E_s=0.00052604 \quad E_b=0.011251
\]

\[
i=8000 \quad E_s=1.8809e-05 \quad E_b=0.00062963
\]

\[
t=0.05
\]

\[
t=0.01
\]

Thinner hyperbolic plates are more convoluted
How do thin sheets “shape themselves”?
How do thin sheets “shape themselves”?

By controlling their reference metric (and then relying on Gauss’ theorem...)
Analysis
Buckling

Theorem II

Given a reference (2D) metric, if the plate thickness is sufficiently large then the configuration is flat (plane-stress solution).

Unless reference metric is flat, a buckling transition (transition to a non-flat configuration) occurs at a critical thickness $t_b$. An infinitely thin body cannot sustain compression without buckling.

Transition may be either super-critical (i.e., continuous) or sub-critical (i.e., discontinuous).

Proof: Tedious but quite standard...
Very thin plates

**Theorem III** (infinitely thin plate)

Let $f(x^1, x^2; t)$ be the equilibrium configuration for a plate of thickness $t$. If there exists a limiting configuration (in $H^2$) as $t \to 0$, then this limit is an isometric immersion that minimizes the bending energy.

**Conjecture**

If the 2D reference metric admits a finite bending isometric immersion then the $t \to 0$ limit exists.

One could approach this question from two different starting points: (1) the full 3D model, or (2) the reduced 2D model. Both cases seem to follow from $\Gamma$-convergence arguments once we have a lower bound on the elastic energy.
**Very thin plates (cont.)**

**Assuming a limit exists:** for thin but finite thickness, deviation from isometric immersion maximal at boundaries:

- In the bulk, the metric deviates by $O(t^2)$.

- An $O(t)$ deviation in a **boundary layer** of size $\Theta\left(\sqrt{|h|t}\right)$.
Conclusions
Summary

We developed a mathematical model for thin elastic sheets that are homogeneous along the thin direction, and have no stress-free rest configuration.

We named such structures **non-Euclidean plates**.

When the (3D) reference metric is immersible the model reduces to existing models [e.g., Ciarlet 2005].

General results about the buckling transition, and deviation from an isometric immersion for very thin sheets.
A lot of work to be done

Mathematics:
- Open questions.
- Characterization of infinitely-thin limit. **How thin is thin?**
- Study *isometric immersions*.
- Why do torn plastics generate **fractal** patterns?

Modeling:
- Non-Euclidean shells.
- Loaded/constrained non-Euclidean **plates**.
- Growth processes and reference metrics [Goriely and Ben Amar].

Experiments:
- How to **measure the target metric** of a body?
Geometry and elasticity

Our conjecture (i.e., give me 2 more weeks to prove it :) ) is that a very thin plate will tend to the/an isometric immersion that minimizes the bending content [closest to a minimal surface]. (“Purely geometric” problem)

What if the 2D metric does not have an immersion with finite bending content? [Nash-Kuiper theorem only guarantees a $C^1$ immersion; we need $W^{2,2}$].

Then no matter how thin, deviation from isometry is necessary. Could a lack of finite-energy immersion be at the heart of fractal-like patterns?