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ON STRAIN MEASURES AND THE GEODESIC DISTANCE TO SO_n IN THE GENERAL LINEAR GROUP

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Abstract. We consider various notions of strains—quantitative measures for the deviation of a linear transformation from an isometry. The main approach, which is motivated by physical applications and follows the work of [12], is to select a Riemannian metric on GL_n, and use its induced geodesic distance to measure the distance of a linear transformation from the set of isometries. We give a short geometric derivation of the formula for the strain measure for the case where the metric is left-GL_n-invariant and right-O_n-invariant. We proceed to investigate alternative distance functions on GL_n, and the properties of their induced strain measures. We start by analyzing Euclidean distances, both intrinsic and extrinsic. Next, we prove that there are no bi-invariant distances on GL_n. Lastly, we investigate strain measures induced by inverse-invariant distances.

1. **Introduction.** In various physical and mathematical contexts, a natural question arises: how to quantify the distortion of an invertible linear transformation $A \in \mathrm{GL}_n$? That is, how far is A from being an isometry? In material science, the local distortion of a map between two manifolds is known as a *strain measure*.

One can investigate various notions of strain measures. A natural approach is to choose a distance function d on GL_n, and define the strain measure as follows:

$$\mathrm{Strain}(A) = \mathrm{dist}(A, \mathrm{SO}_n) = \inf_{Q \in \mathrm{SO}_n} d(A, Q).$$

Since SO_n is compact, the distance is realized for some $Q \in \mathrm{SO}_n$.

In material science, the configuration of a body is a map f from a body manifold $\mathcal{B}$ to a space manifold $\mathcal{S}$. If both manifolds are endowed with Riemannian metrics, then one can define a local strain measure at every point p of the body manifold,

$$\mathrm{Strain}(df) = \mathrm{dist}(df, \mathrm{SO}_n),$$

where SO_n here refers to the space of pointwise orientation-preserving isometries. By choosing orthonormal frames at both p and $f(p)$, invertible linear maps between tangent spaces can be identified with GL_n, whence the relevance of the proposed framework to general Riemannian settings.

The notion of strain measure depends on the choice of a distance function d. In physical applications, one expects this distance to satisfy certain symmetries with respect to left- and right-multiplication—the former is related to symmetries of

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the ambient space whereas the latter is related to material symmetries. The most common symmetry assumptions are frame-indifference, which is left-O_n-invariance, and material isotropy, which is right-O_n-invariance.

Left- and right-O_n-invariance do not determine a unique distance on GL_n, nor do they determine a unique strain measure. The most common distance d is the so-called Frobenius, or Euclidean distance,

$$d^{\mathrm{Euc}}(A,B) = \|A - B\|_F,$$

where $\|A\|_F^2 = \mathrm{tr}(A^T A)$. The Euclidean distance results in a strain measure given by

$$\mathrm{Strain}^{\mathrm{Euc}}(A) = \|\sqrt{A^T A} - I\|_F.$$

The Euclidean strain measure suffers from well-known drawbacks. From a physical point of view, the main drawback is that $\mathrm{Strain}^{\mathrm{Euc}}(A)$ remains finite as A tends toward singularity. The Euclidean strain "penalizes" extreme expansions, but does not "penalize" extreme contractions.

The space GL_n is a smooth submanifold of the space M_n of $n \times n$ matrices. Thus, a natural way to define a distance d on GL_n is via a Riemannian metric g. In this context, the Euclidean distance d^{Euc} is induced by the Euclidean metric on M_n,

$$g_Z^{\mathrm{Euc}}(X,Y) = \mathrm{tr}(X^T Y), \qquad \text{where } X, Y \in T_Z M_n \simeq M_n$$

For $A, B \in M_n$, $d^{\mathrm{Euc}}(A,B)$ is the length of the segment $[A,B]$ with respect to the metric g^{Euc}.

A note about terminology: to avoid confusion, we will use the term "distance", rather than "metric" in the context of a metric space. The term "metric" will be reserved for Riemannian metrics.

From a mathematical point of view, a drawback of $d^{\mathrm{Euc}}|_{\mathrm{GL}_n}$ as a distance function on GL_n is that it is not an intrinsic distance. Since GL_n is not convex, segments $[A,B]$, $A, B \in \mathrm{GL}_n$ may not be contained in GL_n.

The drawbacks of the Euclidean strain measure are at the heart of a series of papers by Neff and co-workers [14, 7, 9, 12]. They endow GL_n with a metric that possesses an additional symmetry: in addition to the bi-O_n-invariance, they assume left-GL_n-invariance; this is perhaps the most symmetric choice, as it is well-known that there are no bi-invariant metrics on GL_n. This additional symmetry restricts drastically the set of possible metrics. The left-GL_n invariance implies that the metric is fully determined by its value at the identity. The addition of right-O_n-invariance yields a family of metrics depending only on three parameters.

In a prior related work by Mielke and co-workers [11, 6, 10], the authors analyze the same family of left-invariant metrics on SL_n. In particular they prove a formula for the geodesics on SL_n which is identical to 1

Another work on the geodesics appears in [1], where the authors derive a formula for the geodesics for a family of left-invariant metrics intersecting the family mentioned above, which is the main focus of this paper. (They took the standard p-norm on $T_I\mathrm{GL}_n$ and used its left-translation as the metric. This includes only the standard metric from our family, however it also contains some Finsler metrics that are not Riemannian).

It was shown in [12] that the unique matrix in SO_n that is the closest to $A \in \mathrm{GL}_n^+$ is its orthogonal polar factor O, where $A = OP$, with $O \in \mathrm{SO}_n$ and $P \in \mathrm{Psym}_n$. Moreover, a closed formula for the strain measure was derived,

$$\mathrm{Strain}(A) = \mathrm{dist}(A, \mathrm{SO}_n) = \|\log\sqrt{A^T A}\|, \tag{1}$$

where the logarithm of a symmetric positive-definite matrix is its unique symmetric logarithm, and the norm $\|\cdot\|$ depends on two out of the three parameters mentioned above (see (4) below). This strain measure diverges in singular limits. In particular, it is *inverse-invariant*, i.e

$$\mathrm{Strain}(A) = \mathrm{Strain}(A^{-1}).$$

In this paper we provide an elementary derivation of formula (1) for the strain measure. Using geometric insights, the set of all possible minimizing paths from a given $A \in \mathrm{GL}_n^+$ to SO_n is narrowed considerably. This helps determining the minimal distance in an elementary way. In particular, our analysis clarifies the different roles played by the various symmetries of the metrics.

In section 2, we introduce the family of left-GL_n- right-O_n-invariant metrics. We state a key property satisfied by these metrics—orthogonality relations—which play a central role in the forthcoming analysis. We also describe the form of the geodesics. Section 3 contains the derivation of the corresponding strain measure. In Subsection 8, we shed light on the reasons for assuming O_n-invariance, rather than SO_n-invariance, which might have seemed a more natural assumption.

In Section 4, we turn to analyze extrinsic versus intrinsic distances, first in a general Riemannian setting and then applied to the case of GL_n viewed as a submanifold of M_n endowed with the Euclidean metric. The main result is that while the intrinsic distance differs from the extrinsic distance, the strain measures are the same in both cases. In particular, we give a very short derivation of Grioli's optimality theorem [5] (see also [13]), which says that for a given $A \in \mathrm{GL}_n^+$, its orthogonal polar factor is the closest matrix to A in SO_n with respect to the Frobenius norm.

In Section 5, we investigate how other natural symmetries on distance functions affect the strain measure. We start by showing there are no bi-invariant distance functions on GL_n, hence there is an "upper limit" to the amount of symmetries a distance function can possess (see Subsection 5.1).

Next, we show that an inverse-invariant strain measure is obtained if the distance/metric is inverse-invariant. We then describe two different techniques for obtaining such distances/metrics via symmetrization, and analyze the resulting strain measures. In the case of symmetrizing a distance, we investigate the two families of distances considered thus far: the Euclidean (intrinsic and extrinsic) distance, and the (intrinsic) distances induced by the metrics considered in Section 2. In the case of the Euclidean distance, the result is an improved strain measure, which penalizes expansions and contractions equally. In the other cases, the strain measure is essentially the same as without the symmetrization.

Finally, we discuss the symmetrizations of all the metrics considered in Section 2. The resulting strain measure is also essentially the same as the original. The proof contains an analysis of metrics that are expressed as sums of two metrics, and also sheds light on the key ingredients in the derivation of the strain measure in Subsection 3.2.

2. Symmetries and geodesics.

2.1. **Left-GL_n- and right-O_n-invariant metrics.** Throughout this paper, we use the following notations:

GL_n is the group of $n \times n$ invertible real matrices, GL_n^+ and GL_n^- are the connected components of GL_n, i.e., GL_n^+ is the subgroup of $n \times n$ invertible matrices with positive determinant, and GL_n^- is the subset of matrices with negative determinant. We denote by

$$\mathrm{O}_n = \{Q \in \mathrm{GL}_n \mid Q^T Q = I\} \subset \mathrm{GL}_n$$

the subgroup of orthogonal matrices, whereas $\mathrm{SO}_n \subset \mathrm{GL}_n^+$ is the subgroup of special orthogonal matrices, i.e those with determinant 1.

We will denote by M_n the vector space of $n \times n$ real matrices, and by $\mathrm{Psym}_n \subset M_n$ the cone of symmetric positive-definite matrices.

For readability, we will try to stick to the following choice of symbols:

$$\begin{aligned}&A, B \in \mathrm{GL}_n\\&O, U \in \mathrm{O}_n\\&Q \in \mathrm{SO}_n\\&X, Y \in M_n\\&P \in \mathrm{Psym}_n.\end{aligned}$$

Let g be a left-GL_n- and right-O_n-invariant metric on GL_n. A left-invariant metric g on a Lie group G is determined by its restriction at the identity. For $A \in \mathrm{GL}_n$, let $L_A : \mathrm{GL}_n \to \mathrm{GL}_n$ denote left multiplication by A, i.e $L_A(B) = AB$. L_A is a diffeomorphism and its differential $(dL_A)_I : T_I\mathrm{GL}_n \to T_A\mathrm{GL}_n$ is a vector space isomorphism. For all $X, Y \in T_I\mathrm{GL}_n$,

$$g_I(X, Y) = g_A\left((dL_A)_I X, (dL_A)_I Y\right). \tag{2}$$

Since GL_n is an open subset of M_n, its tangent space at each point is canonically identified with M_n as follows: Given $A \in \mathrm{GL}_n$, the identification $i_A : M_n \to T_A\mathrm{GL}_n$ is $i_A(X) = [t \mapsto A + tX]$.

The action of the differential dL_A on a tangent vector at B is

$$(dL_A)_B i_B(X) = (dL_A)_B([B + tX]) = [AB + tAX] = \iota_{AB}(AX).$$

Using the above identification,

$$(dL_A)_B X = AX,$$

where the dependence of the right-hand side on B is implicit via the identification of M_n with $T_{AB}\mathrm{GL}_n$.

Substituting this last identify for $B = I$ into (2) we obtain that left-GL_n-invariance implies,

$$g_I(X, Y) = g_A(AX, AY) \qquad \forall A \in \mathrm{GL}_n.$$

Similarly, right-O_n-invariance implies

$$g_A(X, Y) = g_{AO}(XO, YO) \qquad \forall O \in \mathrm{O}_n.$$

An immediate consequence of both left- and right-O_n-invariance, is that g_I is isotropic. For every $U \in \mathrm{O}_n$:

$$g_I(X, Y) = g_{U^T}(U^T X, U^T Y) = g_{U^T U}(U^T XU, U^T YU) = g_I(U^T XU, U^T YU). \tag{3}$$

In fact, the same argument shows that for any Lie group G and subgroup $H \subseteq G$, a left-invariant metric g is right-H-invariant if and only if g_e is invariant under conjugation with elements in H.

From a representation theorem for isotropic operators [2], it follows that there exist constants $\alpha, \beta \geq 0$ and $\gamma \leq 0$, such that

$$g_I(X, Y) = \alpha \operatorname{tr}(X) \operatorname{tr}(Y) + \beta \operatorname{tr}(\mathrm{sym}X\, \mathrm{sym}Y) + \gamma \operatorname{tr}(\mathrm{skew}\, X\ \mathrm{skew}\, Y), \tag{4}$$

where $\mathrm{sym}X$ and $\mathrm{skew}\, X$ denote respectively the symmetric and skew-symmetric parts of X.

Let $\mathrm{sym} \subset M_n$ and $\mathfrak{O}_n \subset M_n$ denote the subspaces of symmetric and anti-symmetric matrices in $M_n \cong T_I\mathrm{GL}_n$. The following lemma asserts that these sets are orthogonally complementary with respect to g_I:

Lemma 2.1. *Let g_I satisfy the isotropy condition (3). Then, sym and $\mathfrak{O}_n$ are orthogonally complementary.*

Proof. The orthogonality of sym and $\mathfrak{O}_n$ can be shown by an explicit substitution in the form (4) of the metric, hence $\mathrm{sym} \subseteq \mathfrak{O}_n^\perp$. The fact the these subspaces are complementary follows from a dimensional argument,

$$\dim \mathrm{sym} + \dim \mathfrak{O}_n = n^2 = \dim T_I\mathrm{GL}_n.$$

□

2.2. **Geodesics.** In this section we review the properties of geodesic curves in (GL_n, g).

Proposition 1 (g-geodesics starting at the identity)**.** *Let g be left-GL_n, right-O_n-invariant. Let g_I be given by (4) and denote $\kappa = (\beta - \gamma)/2\beta$. Let $\gamma : I \to GL_n$ be the g-geodesic, satisfying the initial conditions*

$$\gamma(0) = I \qquad \text{and} \qquad \dot\gamma(0) = X_0.$$

Then,

$$\gamma(t) = \exp((1-\kappa)tX_0 + \kappa t X_0^T)\,\exp(\kappa t(X_0 - X_0^T)).$$

Proof. This was proved in [9] using an argument based on variations of energy. A shorter alternative proof using Cartan's moving frame method is given in Appendix A. □

Corollary 1 (g-geodesics)**.** *Under the same assumptions as above, let $\gamma : I \to GL_n$ be the g-geodesic satisfying the initial conditions*

$$\gamma(0) = A \qquad \text{and} \qquad \dot\gamma(0) = AX_0.$$

Then,

$$\gamma(t) = A\exp((1-\kappa)tX_0 + \kappa t X_0^T)\,\exp(\kappa t(X_0 - X_0^T)).$$

Proof. This follows from the fact that left multiplication is an isometry of (GL_n, g). It is a general property of Riemannian manifolds that isometries map geodesics into geodesics. □

Corollary 2. *Let $\gamma : I \to GL_n$ be the g-geodesic satisfying the initial conditions*

$$\gamma(0) = Q \qquad \text{and} \qquad \dot\gamma(0) = QV,$$

where $Q \in O_n$ and $V \in$ sym. Then,

$$\gamma(t) = Q\exp(tV).$$

3. **Geodesic distance from SO_n.** Every Lie group endowed with a left-invariant metric is complete as a Riemannian manifold. That is, every geodesic extends indefinitely. This follows from the fact that its isometry group acts transitively; see [3, p. 154, Example 12]. By the Hopf-Rinow theorem, [3, p. 146] the length-distance between any two points is realized by a minimizing geodesic.

Generally, there doesn't seem to exist any explicit expression for the (possibly many) geodesics connecting any two elements $A, B \in \mathrm{GL}_n^+$, nor for the resulting distance between these elements. Yet, we are only interested in the distance of an element $A \in \mathrm{GL}_n^+$ from the subgroup SO_n of isometries. As demonstrated in [12], an explicit expression can be derived for that distance. In this section we offer a simplified derivation of that expression.

3.1. **Reduction to diagonal positive-definite matrices.** The first step in calculating the distance of $A \in \mathrm{GL}_n^+$ from SO_n is to show that it is sufficient to obtain a formula for diagonal matrices. The following proposition holds for any bi-SO_n-invariant distance on GL_n^+—not necessarily a distance induced by a Riemannian metric.

Proposition 2. *Let d be a bi-SO_n-invariant distance on GL_n^+; we denote the corresponding distance between sets by* dist. *Let $A \in GL_n^+$. If $A = U\Sigma V^T$ is a singular value decomposition (SVD) of A with $U, V \in SO_n$, then*

$$\mathit{dist}(A, SO_n) = \mathit{dist}(\Sigma, SO_n).$$

Moreover, if Q is a matrix closest to Σ in SO_n, then UQV^T is a matrix closest to A in SO_n.

Proof. We first note that for every $A \in \mathrm{GL}_n^+$, there exists an SVD such that $U, V \in \mathrm{SO}_n$ (see the comment after the proof of Corollary 3). Moreover, Σ is unique (up to permutation), i.e., the singular values do not depend on the particular decomposition.

Assuming $U, V \in \mathrm{SO}_n$ and using the bi-SO_n-invariance,

$$\begin{aligned}\operatorname{dist}(A, \mathrm{SO}_n) &= \min_{Q \in \mathrm{SO}_n} d(A, Q) = \min_{Q \in \mathrm{SO}_n} d(U\Sigma V^T, Q) \\ &= \min_{Q \in \mathrm{SO}_n} d(\Sigma, U^T Q V) = \operatorname{dist}(\Sigma, \mathrm{SO}_n).\end{aligned} \tag{5}$$

The last equality holds since $\{U^T Q V \mid Q \in \mathrm{SO}_n\} = \mathrm{SO}_n$. Equation (5) implies that $Q \in \mathrm{SO}_n$ is a matrix closest to Σ in SO_n if and only if UQV^T is a matrix closest to A in SO_n. ∎

3.2. **Geodesic distance for diagonal matrices.** By Proposition 2, we can focus our attention on finding the distance from SO_n for diagonal positive-definite matrices, Σ. Since GL_n^+ is complete, we look for a minimizing geodesic from Σ to SO_n. To do so, we are going to exploit the fact that any geodesic minimizing the distance of a point to a submanifold intersects that submanifold perpendicularly. More precisely:

Lemma 3.1. *Let M be a complete Riemannian manifold. Let $S \subseteq M$ be a submanifold, and let $p \in M \setminus S$. Assume $q \in S$ is a point on S satisfying $d(p, q) = \operatorname{dist}(p, S)$ (there is always such a point q if S is compact). Let α be a minimizing geodesic connecting p and q. Then α is orthogonal to S at q.*

See C.1 for a proof.

Proposition 3. *Let $\Sigma = diag(\sigma_1, \dots \sigma_n)$ be a diagonal matrix with positive entries. Then,*

$$dist(\Sigma, SO_n) = d(\Sigma, I).$$

Moreover, I is the unique element of SO_n minimizing the distance from Σ.

Proof. Let $Q \in \mathrm{SO}_n$ satisfy

$$d(\Sigma, Q) = \mathrm{dist}(\Sigma, \mathrm{SO}_n).$$

By the completeness of GL_n^+, there exists a minimizing geodesic $\alpha : [0,1] \to \mathrm{GL}_n^+$ from Q to Σ, i.e.,

$$\alpha(0) = Q, \quad \alpha(1) = \Sigma \quad \text{and} \quad L(\alpha) = \|\dot\alpha(0)\|_Q = d(\Sigma, Q).$$

Lemma 3.1 implies that $\dot\alpha(0) \perp T_Q\mathrm{SO}_n$.

Denoting $\dot\alpha(0) = QV$ (where we think of V as an element of $T_I\mathrm{GL}_n^+ \cong M_n$, and QV is identified with $d(L_Q)_I(V)$), we obtain for any antisymmetric matrix $X \in \mathfrak{O}_n = T_I\mathrm{SO}_n$:

$$g_I(X, V) = g_Q(QX, QV) = g_Q(QX, \dot\alpha(0)) = 0,$$

where the last equality is valid since $d(L_Q)_I(T_I\mathrm{SO}_n) = T_Q\mathrm{SO}_n$, hence $QX = d(L_Q)_I(X) \in T_Q\mathrm{SO}_n$.

Thus, $V \in T_I\mathrm{GL}_n^+$ is orthogonal to every anti-symmetric matrix, and by Lemma 2.1, $V \in \mathrm{sym}$. It follows from Corollary 2 that

$$\alpha(t) = Qe^{tV}.$$

By the definition of α, $\alpha(1) = \Sigma = Qe^V$. Since V is symmetric, e^V is symmetric positive-definite, hence we obtain two polar decompositions of Σ,

$$\Sigma = I\Sigma \qquad \text{and} \qquad \Sigma = Qe^V.$$

By the uniqueness of polar decomposition for invertible matrices, we conclude that $Q = I$, which completes the proof. □

We proceed to derive an explicit formula for the distance of Σ from SO_n. Since $Q = I$,

$$e^V = \Sigma = e^{\log\Sigma},$$

where $\log\Sigma = \mathrm{diag}(\log\sigma_i)$. Since V and $\log\Sigma$ are symmetric, and since the matrix exponential is injective on the space of symmetric matrices, it follows that $V = \log\Sigma$. Hence,

$$\mathrm{dist}(\Sigma, \mathrm{SO}_n) = \|\dot\alpha(0)\|_I = \|V\|_I = \|\log\Sigma\|_I.$$

Substituting the explicit form (4) of the metric g_I,

$$\mathrm{dist}(\Sigma, \mathrm{SO}_n) = \sqrt{\alpha\left(\sum\log\sigma_i\right)^2 + \beta\sum(\log\sigma_i)^2}. \tag{6}$$

As a corollary, we get that $\alpha(t) = e^{t\log\Sigma}$ is the unique minimizing geodesic connecting I to Σ.

3.3. **Geodesic distance for arbitrary matrices.** Let $A \in \mathrm{GL}_n^+$ be an arbitrary matrix. If $A = U\Sigma V^T$ is an SVD of A, then $\sqrt{A^TA} = V\Sigma V^T$, hence $\log\sqrt{A^TA} = V \log\Sigma\, V^T$. By Proposition 2,

$$\mathrm{dist}(A, \mathrm{SO}_n) = \|\log\Sigma\|_I = \|V \log\Sigma\, V^T\|_I = \|\log\sqrt{A^TA}\|_I, \tag{7}$$

where the second equality follows from the invariance (3). Whenever we write log of a symmetric positive-definite matrix, we refer to its unique symmetric logarithm. Since the exponential map is a diffeomorphism from sym_n to Psym_n there is no ambiguity here. We have thus obtained an explicit expression for the distance of any matrix $A \in \mathrm{GL}_n^+$ from SO_n by elementary means.

We have shown that for a diagonal positive-definite matrix Σ, $Q = I$ is the unique element in SO_n satisfying $\mathrm{dist}(\Sigma, \mathrm{SO}_n) = d(\Sigma, Q)$. By Proposition 2, if $A = U\Sigma V^T$ is an SVD of $A \in \mathrm{GL}_n^+$ with $U, V \in \mathrm{SO}_n$, then UV^T is the unique matrix in SO_n that is closest to A.

Moreover:

Corollary 3 (The orthogonal polar factor is the minimizer)**.** *Let $A \in GL_n^+$. Let $A = OP$ be the polar decomposition of A, $O \in SO_n$ and $P \in Psym_n$. Then O is the matrix closest to A in SO_n.*

Proof. By orthogonally diagonalizing P with $P = \tilde{U}\Sigma\tilde{U}^T$, we obtain an SVD,

$$A = O\tilde{U}\Sigma\tilde{U}^T = U\Sigma V^T,$$

where $U = O\tilde{U}$ and $V = \tilde{U}$. Note that by interchanging two columns if necessary, we can assume $\tilde{U} \in \mathrm{SO}_n$, hence $V, U \in \mathrm{SO}_n$. By the above discussion, $UV^T = O\tilde{U}\tilde{U}^T = O$ is the matrix closest to A. $\square$

Please note: the above argument shows that for $A \in \mathrm{GL}_n^+$ there always exists an SVD where both orthogonal matrices are in SO_n.

3.4. **O_n versus SO_n-invariance.** The analysis presented in Sections 2 and 3 assumes that the metric g is left-GL_n- and right-O_n invariant. Since we are interested in intrinsic distances in GL_n^+ from the subgroup SO_n, it may seem as if we could perform the whole analysis in GL_n^+ rather than in GL_n. In such case, it only makes sense to require the Riemannian metric to be left-GL_n^+ and right-SO_n invariant—right O_n-invariance, for example, is meaningless. A natural question is the following: would we obtain the same geodesic distances and the same strain measures if we considered left-GL_n^+ and right-SO_n-invariant metric on GL_n^+?

An inner-product g_I satisfying condition (3) is called **isotropic**. In contrast, an inner-product g_I satisfying

$$g_I(X, Y) = g_I(S^TXS, S^TYS), \qquad \forall S \in \mathrm{SO}_n \tag{8}$$

is called **hemitropic**. If every hemitropic inner-product is isotropic, then our entire analysis extends as is to SO_n-invariant metrics on GL_n^+. If, however, isotropy and hemitropy are not equivalent, then our analysis has to be revisited, as the representation of the inner-product (4) relies explicitly on the isotropic nature of the inner-product g_I.

Our analysis relies on the specific form (4) of the inner-product g_I in two crucial aspects: (i) in the derivation of an explicit formula for the geodesics, and (ii) in obtaining the orthogonality of symmetric and anti-symmetric matrices. Since an inner-product is of the form (4) if and only if it is isotropic, any hemitropic, but non-isotropic inner-product is not of that form, hence our analysis is not applicable.

It turns out that for all dimensions $n \neq 4$, there are no hemitropic non-isotropic inner-products. For odd n this is trivial to see since $-I \in \mathrm{O}_n \setminus \mathrm{SO}_n$ commutes with every other matrix. The analysis for even dimensions is less trivial. A proof can be found in [4] . Thus, our work holds as is with isotropy replaced by hemitropy in any dimension other than 4. Understanding the implications of an hemitropy assumption for $n = 4$ remains an open question.

4. **Intrinsic versus extrinsic distances.** Endowing GL_n^+ with distances induced by Riemannian metrics is one type of choice for measuring the distortion of a linear map. Another popular choice is the distance induced by the Frobenius inner-product on M_n, or equivalently, the Euclidean metric on M_n identified with $\mathbb{R}^{n^2}$. In fact, one of the motivations in [12] for considering distances induced by Riemannian metrics was the claimed inadequacy of the Euclidean metric. Note that the Euclidean metric gives rise to two distinct distances on GL_n^+: (i) an *extrinsic* distance, obtained by restricting the Euclidean distance function to the subset GL_n^+ of M_n, and (ii) an *intrinsic* length-distance determined by paths in GL_n^+.

In this section we explore the distinction between extrinsic and intrinsic distances, first in a general Riemannian context, and second, in the original Euclidean context.

4.1. **The general Riemannian case.** Let (M, g) be a Riemannian manifold. Denote the induced Riemannian distance function by d^M. Let $S \subset M$ be an embedded connected submanifold.

There are two natural ways to induce a distance on S:

1. Intrinsic: Consider S as a Riemannian submanifold of M, i.e., endow S with the pullback metric i^*g along the inclusion $i : S \to M$. Denote by d_S^{int} the Riemannian distance function induced by i^*g.
2. Extrinsic: Consider S as a subspace of the metric space (M, d^M). Denote by d_S^{ext} the restriction of d^M to $S \times S$.

An immediate observation is that $d_S^{\mathrm{int}} \geq d_S^{\mathrm{ext}}$. The question we pose is under what conditions, $d_S^{\mathrm{int}} = d_S^{\mathrm{ext}}$.

In general, both equality and inequality may hold: For $M = \mathbb{S}^2$ endowed with the round metric and S a great circle, $d_S^{\mathrm{int}} = d_S^{\mathrm{ext}}$. For $M = \mathbb{R}^2$ with the standard Euclidean metric and $S = \mathbb{S}^1$, $d_S^{\mathrm{int}} > d_S^{\mathrm{ext}}$.

To state our results we need the following classical definitions:

Definition 4.1. Let (M, g) be a Riemannian manifold. A subset C of M is said to be a *geodesically convex* if, given any two points in C, there is a minimizing geodesic (in M) contained within C joining these two points.

Definition 4.2. A submanifold S of a Riemannian manifold (M, g) is called *totally geodesic* if any geodesic on the submanifold S with its induced Riemannian metric is also a geodesic on the Riemannian manifold (M, g). (This condition is equivalent to the vanishing of the second fundamental form of S in M.)

To prove our results we shall need the following lemma which roughly says that paths that are close to being length-minimizers are within a narrow tubular neighborhood of a (minimizing) geodesic.

Lemma 4.3 (Nearly length-minimizing paths are close to geodesics)**.** *Let (M, g) be a complete Riemannian manifold, and let $p, q \in M$. Then, for any $\epsilon > 0$ there* *exists a $\delta > 0$ (possibly dependent on p and q) such that any path α joining p and q satisfying*

$$L(\alpha) < d(p, q) + \delta$$

is in an ϵ-neighborhood of a minimizing geodesic $\gamma : I \to M$ joining p and q. In particular, there exists a reparametrization $\alpha \circ \varphi : I \to M$ of α satisfying

$$\sup_{t \in I} d\left(\alpha \circ \varphi(t), \gamma(t)\right) < \epsilon.$$

In this lemma there is no submanifold, so the distance has only one possible meaning— the Riemannian distance on M.

Proof. Assume by contradiction that the claim is false. Denote $r = d(p, q)$. Then, there exists an $\epsilon > 0$ and a sequence of paths $\alpha_n : I \to M$ joining p and q, satisfying

$$L(\alpha_n) \le d(p, q) + \frac{1}{n},$$

and α_n is not in an ϵ-neighborhood of any minimizing geodesic.

Since $L(\alpha_n) \to r$, we can assume $L(\alpha_n) \le 2r$, thus $\text{Image}(\alpha_n) \subseteq \bar{B}^M(p, 2r)$ (the closed ball of radius $2r$ around p). By the completeness of M, it follows from the Hopf-Rinow theorem that $\bar{B}^M(p, 2r)$ is compact.

Reparametrize α_1 by arclength, i.e assume $I = [0, L(\alpha_1)]$, and that $\alpha_1 : I \to M$ has a constant speed. For every $n \in \mathbb{N}$, reparametrize $\alpha_n : I \to M$ such that it has a constant speed $c_n = \|\dot{\alpha}_n\|$. Then

$$2L(\alpha_1) \ge 2r \ge L(\alpha_n) = c_n L(\alpha_1) \quad \Rightarrow \quad c_n \le 2,$$

which implies that the α_n are equicontinuous, since

$$d(\alpha(t), \alpha(s)) \le L(\alpha_{[t,s]}) \le 2(s - t).$$

By the Arzela-Ascoli theorem, there exists a subsequence (also denoted α_n) converging uniformly to a path $\alpha : I \to M$. By the lower-semicontinuity of the length functional we deduce:

$$L(\alpha) \le \lim_{n \to \infty} L(\alpha_n) = d(p, q).$$

This implies that α is a length-minimizing curve between p and q, hence its reparametrization by arclength $\alpha \circ \varphi$ is a geodesic.

Finally, the uniform convergence $\alpha_n \to \alpha$ yields a contradiction: there exists an N such that for all $n > N$,

$$\sup_{t \in I} d\left((\alpha_n \circ \varphi)(t), (\alpha \circ \varphi)(t)\right) < \epsilon.$$

□

We next prove the following:

Proposition 4. *Let S be a submanifold S of a Riemannian manifold (M, g). Then:*

1. *If S is a geodiscally convex subset of M, then $d_S^{int} = d_S^{ext}$. The reverse implication does not hold in general. The next assertion shows that the only obstruction for the reverse direction to hold, is topological.*
2. *If S is topologically closed in M, then $d_S^{int} = d_S^{ext}$ if and only if S is a geodesically convex subset of M.*
3. *If $d_S^{int} = d_S^{ext}$ then S is a totally geodesic submanifold of M. The reverse implication does not hold in general.*
4. *Let $p, q \in S$. Assume there exists a* ***unique*** *minimizing geodesic $\gamma : I \to M$ connecting p and q. If $\gamma \cap (M \setminus \bar{S}) \neq \emptyset$, Then $d_S^{int}(p, q) > d_S^{ext}(p, q)$.*

The last two statements hold also if we replace the existence of a unique geodesic with the existence of finitely many geodesics.

Proof. 1. The fact that geodesic convexity implies equality of the distances is immediate. A counter-example for the reverse implication is $M = \mathbb{R}^2, S = \mathbb{R}^2 \setminus (0,0)$.

2. Assume $d_S^{Int} = d_S^{Ext}$. Let $p, q \in S$. Let $\alpha_n : I \to S$ be a sequence of paths satisfying $L(\alpha_n) \to d(p,q)$. Then, by a similar argument to the one in the proof of Lemma 4.3, there is a subsequence α_n converging uniformly to a path α. By the lower-semicontinuity of the length, $L(\alpha) = d(p,q)$, so α is minimizing, hence it is a reparametrization of a geodesic. Closedness of S implies $\text{Image}(\alpha) \subseteq S$.
3. Suppose that $d_S^{\text{int}} = d_S^{\text{ext}}$, and let α be a geodesic in S. Then, it is locally length-minimizing in S, and for small enough t,

$$L(\alpha|_{[0,t]}) = d_S^{\text{int}}(\alpha(0), \alpha(t)) = d_S^{\text{ext}}(\alpha(0), \alpha(t)) = d^M(\alpha(0), \alpha(t)).$$

So $\alpha|_{[0,t]}$ is length-minimizing path between $\alpha(0), \alpha(t)$ in M with parameter proportional to arc length, hence it is a geodesic in M.

A counter-example for the reverse implication is $M = \mathbb{S}^1, S = \mathbb{S}^1 \setminus \{p\}$ (where p is an arbitrary point in $\mathbb{S}^1$). We will see another counter-example in the next section: $M = M_n$ and $S = \mathrm{GL}_n^+$.
4. By assumption, there exists $t_0 \in I$ such that $\gamma(t_0) \in M \setminus \bar{S}$. Since $M \setminus \bar{S}$ is open, there is some open ball of d^M-radius ϵ, $\gamma(t_0) \in B_\epsilon \subset M \setminus \bar{S}$.

By Lemma 4.3, $\exists \delta > 0$ such that if α is a path between p, q, $L(\alpha) < d^M(p,q) + \delta$ then α is in an ϵ-neighborhood of some minimizing geodesic joining p and q. By our assumption, there is only one minimizing geodesic between p and q in M, namely γ.

Thus, there exists a reparametrization of α, $\alpha \circ \varphi : I \to M$, such that for every t,

$$d\big((\alpha \circ \varphi)(t), \gamma(t)\big) < \epsilon.$$

In particular, $d\big((\alpha \circ \varphi)(t_0), \gamma(t_0)\big) < \epsilon$ implies that $(\alpha \circ \varphi)(t_0) \in B_\epsilon \subseteq M \setminus \bar{S}$.

This shows that any path α which is δ-close to being a minimizer intersect $M \setminus \bar{S}$. Hence $d_S^{Int}(p,q) \geq d_S^{Ext}(p,q) + \delta$.

□

4.2. **Euclidean distances in $\mathbf{GL}_n^+$.** Next, we consider the particular case where the Riemannian manifold is the vector space of $n \times n$ matrices endowed with the Euclidean metric. This is the case considered classically in the context of elastic strain measures, and whose shortcomings has motivated, in part, the consideration of alternative measures of strain.

We start with a few definitions:

Definition 4.4 (Euclidean metric on M_n)**.** We denote by (M_n, d^{Euc}) the space of $n \times n$ real matrices endowed with the Euclidean distance. Note that the distance d^{Euc} can be derived from a Riemannian metric g given by

$$g_Z(X,Y) = \operatorname{tr}(X^T Y).$$

Definition 4.5 (Extrinsic Metric on GL_n^+)**.** We denote by $(\mathrm{GL}_n^+, d^{\text{ext}})$ the metric space of $n \times n$ invertible matrices with positive determinant, where d^{ext} is the restriction of d^{Euc}, with GL_n^+ viewed as a subset of $\mathbb{R}^{n^2}$.

Definition 4.6 (Intrinsic Metric on GL_n^+)**.** Consider $(\mathrm{GL}_n^+, g|_{\mathrm{GL}_n^+})$ as an open sub-manifold of the Riemannian manifold (M_n, g). That is, we endow GL_n^+ with the pullback metric i^*g of the Euclidean metric g along the inclusion $i : \mathrm{GL}_n^+ \to (M_n, g)$. We denote by d^{int} the distance function induced by the Riemannian metric i^*g.

We first observe that $d^{\mathrm{int}} > d^{\mathrm{ext}}$ for some pair of matrices. Indeed, for any $X, Y \in M_n$, the unique minimizing geodesic is the segment

$$[X, Y] = \{X + t\,(Y - X) \;:\; t \in [0, 1]\}.$$

Since the sub-manifold GL_n^+ is not convex, there exist $A, B \in \mathrm{GL}_n^+$, such that the segment $[A, B]$ intersects GL_n^-. By Item 4 in Proposition 4,

$$d^{\mathrm{ext}}(A, B) < d^{\mathrm{int}}(A, B).$$

However, we note the following:

Lemma 4.7. *Let $A, B \in GL_n^+$. If $[A, B] \subset GL_n^+$, then $d^{int}(A, B) = d^{ext}(A, B)$.*

Proof. This is obvious, since $[A, B]$ is an extrinsic length-minimizing path which stays in the submanifold. □

We next observe that both right- and left-multiplications by elements of SO_n are isometries of (GL_n^+, i^*g). Hence, they are isometries of the metric space $(GL_n^+, d^{\mathrm{int}})$ (any Riemannian isometry is an isometry of the induced distance function). It follows that d^{int} is both left- and right-SO_n invariant.

In particular, let $A = U\Sigma V^T$ be an SVD of $A \in \mathrm{GL}_n^+$. By Proposition 2,

$$d^{\mathrm{int}}(A, \mathrm{SO}_n) = d^{\mathrm{int}}(\Sigma, \mathrm{SO}_n),$$

hence, as before, the problem of computing the distance of $A \in \mathrm{GL}_n^+$ from SO_n (and finding the minimizer) can be reduced to positive-definite diagonal matrices Σ. We now give a short proof that I is the unique matrix closest to Σ with respect to the extrinsic distance, that is,

$$d^{\mathrm{ext}}(\Sigma, \mathrm{SO}_n) = d^{\mathrm{ext}}(\Sigma, I) = \|\Sigma - I\|_F.$$

Indeed, since

$$\|A - Q\|_F^2 = \|A\|_F^2 + \|Q\|_F^2 - 2\langle A, Q\rangle_F = \|A\|_F^2 + n - 2\,\mathrm{tr}(A^T Q), \tag{9}$$

it follows that given $A \in M_n$, minimizing $\|A - Q\|_F^2$ over $Q \in \mathrm{SO}_n$ is equivalent to maximizing the linear functional $\varphi_A(Q) = \langle A, Q\rangle_F = \mathrm{tr}(A^T Q)$. For diagonal and positive-definite Σ,

$$\varphi_\Sigma(Q) = \mathrm{tr}(\Sigma^T Q) = \sum_{i=1}^n \sigma_i Q_{ii} \le \sum_{i=1}^n \sigma_i = \mathrm{tr}(\Sigma^T) = \varphi_\Sigma(I),$$

where the inequality follows from the fact that Q is orthonormal, hence $|Q_{ij}| \le 1$. The unique maximizer is $Q = I$, hence I is the unique matrix closest to Σ.

Theorem 4.8. *Let $\Sigma \in GL_n^+$ be a diagonal matrix with strictly positive entries on the diagonal. Then the unique minimizer of the intrinsic distance of Σ from SO_n is I, and*

$$d^{int}(\Sigma, SO_n) = d^{ext}(\Sigma, SO_n).$$

Proof. Since Psym_n is closed under convex combinations, it follows that $[\Sigma, I] \subseteq GL_n^+$. By Lemma 4.7, it follows that $d^{\mathrm{int}}(\Sigma, I) = d^{\mathrm{ext}}(\Sigma, I)$.

The inequality $d^{\mathrm{int}} \geq d^{\mathrm{ext}}$ implies that

$$\mathrm{dist}^{int}(\Sigma, \mathrm{SO}_n) \geq \mathrm{dist}^{ext}(\Sigma, \mathrm{SO}_n).$$

Hence,

$$\mathrm{dist}^{ext}(\Sigma, \mathrm{SO}_n) = d^{\mathrm{ext}}(\Sigma, I) = d^{\mathrm{int}}(\Sigma, I) \geq \mathrm{dist}^{int}(\Sigma, \mathrm{SO}_n) \geq \mathrm{dist}^{ext}(\Sigma, \mathrm{SO}_n).$$

So, I is a minimizer and $\mathrm{dist}^{int}(\Sigma, \mathrm{SO}_n) = \mathrm{dist}^{ext}(\Sigma, \mathrm{SO}_n)$.

The uniqueness of the minimizer for dist^{int} follows from the uniqueness of the minimizer for dist^{ext}: Let $Q \in \mathrm{SO}_n$ be a minimizer for the distance dist^{int} of Σ from SO_n. Then,

$$\mathrm{dist}^{ext}(\Sigma, \mathrm{SO}_n) = \mathrm{dist}^{int}(\Sigma, \mathrm{SO}_n) = d^{\mathrm{int}}(\Sigma, Q) \geq d^{\mathrm{ext}}(\Sigma, Q),$$

hence Q is also a minimizer for dist^{ext}, and by the uniqueness of the extrinsic minimizer, $Q = I$. □

Corollary 4. *Let $A \in GL_n^+$. Then*

$$d^{int}(A, SO_n) = d^{ext}(A, SO_n),$$

and the closest matrix to A in SO_n with respect to both distances is the same—it is the orthogonal polar factor of A. Moreover,

$$d^{int}(A, SO_n) = \|\sqrt{A^T A} - I\|_F. \tag{10}$$

Proof. This is an immediate consequence of Proposition 2 applied to both distances, together with Theorem 4.8. The fact that the orthogonal polar factor is the unique minimizer follows from the same considerations as in the proof of Corollary 3. □

5. **Additional results concerning invariant distances.**

5.1. **Bi-invariance.** As always in mathematics, the most symmetric structures are the most easy to handle. In [12], the authors consider either left- and right-GL_n-invariance as natural requirements on a metric on GL_n. Their choice is motivated by physical considerations.

A natural question is: does there exist a distance function on GL_n that is more symmetric than the ones we have considered? In this section we show here that there are no bi-invariant distance functions on GL_n that are compatible with the standard topology. (The fact there is no bi-invariant Riemannian metric is common knowledge.)

Theorem 5.1. *There is no bi-invariant distance function on GL_n generating the standard topology on GL_n (the subspace topology induced by the inclusion $GL_n \to \mathbb{R}^{n^2}$).*

Proof. The essential point is the existence of a non-trivial conjugacy class whose closure contains the identity. Assume, by contradiction, there is a bi-invariant distance function d compatible with the standard topology. Consider the following matrices:

$$D = \begin{pmatrix} 2^{-1} & 0 & \cdots & 0 \\ 0 & 2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{-n} \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

An explicit calculation yields,

$$D^{-n}AD^n = \begin{pmatrix} 1 & 2^{-n} & 0 & \cdots & 0 \\ 0 & 1 & 2^{-n} & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since d generates the standard topology on GL_n, it follows that

$$\lim_{n\to\infty} d(D^{-n}AD^n, I) = 0.$$

On the other hand, bi-invariance implies that for every n,

$$d(D^{-n}AD^n, I) = d(D^{-n}AD^n, D^{-n}ID^n) = d(A, I) \neq 0,$$

which is a contradiction. □

As an immediate corollary we obtain the classical result:

Corollary 5. *There is no bi-invariant Riemannian metric on GL_n.*

5.2. **Inverse-invariance.** When it comes to physical applications, a major drawback of the Euclidean strain measure is that it does not diverge in the limit where the linear map is singular. From a physical viewpoint, we expect a strain measure $\operatorname{dist}(A, \mathrm{SO}_n)$ to diverge when A either tends to infinity (expansion), or when it tends toward singularity (contraction). The strain measure is said to be *inverse-invariant* if

$$\operatorname{Strain}(A) = \operatorname{Strain}(A^{-1}), \tag{11}$$

for every $A \in \mathrm{GL}_n^+$.

As noted in the Introduction, the strains obtained via the metrics considered in Section 2 are all inverse-invariant. The essential reason behind this phenomenon, is the extreme symmetry of these metrics. Specifically, we have the following very general assertion:

Proposition 5. *Let G be a group and $H \subseteq G$ a subgroup. Let d be a left-G-right-H-invariant distance function on G. Then,*

$$\operatorname{dist}(g, H) = \operatorname{dist}(g^{-1}, H).$$

Moreover, if h is a closest element to g in H, then h^{-1} is a closest element to g^{-1} in H.

Proof. Using the assumed invariances,

$$\operatorname{dist}(g, H) = \inf_{h\in H} d(g, h) = \inf_{h\in H} d(e, g^{-1}h) = \inf_{h\in H} d(h^{-1}, g^{-1}) = \operatorname{dist}(g^{-1}, H)$$

The equality $d(g, h) = d(g^{-1}, h^{-1})$ (for $h \in H$) implies the correspondence between closest elements to g and g^{-1}. □

Note how this observation implies, without any computation, that certain strain measures are inverse-invariant (the actual form of the strain measures is irrelevant).

Another means for obtaining inverse-invariant strain measures is to consider distance functions d on GL_n that are inverse-invariant, i.e.,

$$d(A,B) = d(A^{-1}, B^{-1}). \tag{12}$$

Indeed, the inverse-invariance of the distance implies the inverse-invariance of the strain measure, as

$$\mathrm{dist}(A, \mathrm{SO}_n) = \mathrm{dist}(A^{-1}, \mathrm{SO}_n^{-1}) = \mathrm{dist}(A^{-1}, \mathrm{SO}_n).$$

As we will see, the requirement for inverse-invariance of the distance is far less restrictive than left-GL_n- and right O_n-invariance.

In a search for maximally-symmetric distance functions, we first look for inverse-invariant distances (or metrics) possessing additional symmetries. In Propositions 9–10 we prove that in any Lie group G, a left-invariant metric (distance) is inverse-invariant if and only if it is bi-invariant. Since there are no bi-invariant metrics/distances on GL_n, it follows that no left-GL_n- and inverse-invariant metric/distance exists either.

In particular, it follows that the distance functions induced by the metrics considered in Sections 2 and 3 are not inverse-invariant. That is, the inverse-invariance of the strain measure does not result from the inverse-invariance of the distance, but rather from the left-GL_n and right-SO_n invariance of the metrics, as shown in Proposition 5 (also observed in [12, Section 3.2 Eq. (23)]).

There is a systematic way of constructing inverse-invariant distances from arbitrary distances. Denote the inverse automorphism by i. Since i is a diffeomorphism of finite order, given any distance d on GL_n, it is possible to construct an inverse-invariant distance via symmetrization,

$$\tilde{d}(A,B) = d(A,B) + d(A^{-1}, B^{-1}).$$

It is easy to see that $\tilde{d}$ generates the same topology as d, and it is of course inverse-invariant.

A similar construction can be carried out for Riemannian metrics on GL_n. Given any metric g, the metric $g + i^*(g)$ is inverse-invariant, and induces the standard topology on GL_n, as does any Riemannian metric.

In the following subsections, we analyze how these two different methods for generating inverse-invariant distances affect the strain measure. We will see that if we start from distances/metrics having certain symmetries, then their symmetrizations possess corresponding symmetries as well.

5.2.1. *Inverse-invariant distances.* The next lemma considers a setting that generalizes our treatment of left-GL_n- and right-O_n invariant distance functions.

Lemma 5.2. *Let G be a group and let $H \subseteq G$ be a subgroup. Let d be a left-G- and right-H-invariant distance function on G. Let $\tilde{d}$ be its symmetrization. Then,*

$$\mathrm{dist}_{\tilde{d}}(g, H) = 2\ \mathrm{dist}_d(g, H).$$

Moreover, an element $h \in H$ is a closest element in H to g with respect to d if and only if it is a closest element with respect to $\tilde{d}$.

Proof. The assumed symmetries imply $d(g,h) = d(g^{-1},h^{-1})$ for any $g \in G$ and $h \in H$ (see the proof of Proposition 5). Hence,

$$\begin{aligned}\operatorname{dist}_{\tilde{d}}(g,H) &= \inf_{h\in H} \tilde{d}(g,h) \\ &= \inf_{h\in H} d(g,h) + d(g^{-1},h^{-1}) \\ &= \inf_{h\in H} 2\,d(g,h) \\ &= 2\operatorname{dist}_d(g,H).\end{aligned}$$

To prove the second part, note that for $h \in H$, $\tilde{d}(g,h) = \operatorname{dist}_{\tilde{d}}(g,H)$ if and only if $2\,d(g,h) = 2\operatorname{dist}_d(g,H)$. □

The above lemma implies that there is not much interest in symmetrizing left-G- and right-H-invariant distance functions, since the symmetrizations give rise to essentially identical notions of distance from H.

We turn to analyze symmetrization within the context of intrinsic versus extrinsic Euclidean distances. As above, we provide a slightly more general treatment that considers the relevant symmetries. Since this setting possesses less symmetries than the one considered above, we will have to use properties that are specific to the Euclidean distance; in particular, SVD plays an important role.

Lemma 5.3. *Let G be a group and let $H \subseteq G$ be a subgroup. Let d be a bi-H-invariant distance on G. Then, its symmetrization $\tilde{d}$ is also bi-H-invariant.*

Proof. For every $h \in H, x, y \in G$,

$$\tilde{d}(hx,hy) = d(hx,hy) + d(x^{-1}h^{-1},y^{-1}h^{-1}) = d(x,y) + d(x^{-1},y^{-1}) = \tilde{d}(x,y).$$

Right-H-invariance is proved similarly. □

In our context, since both intrinsic and extrinsic Euclidean distances are bi-SO_n invariant, their symmetrizations are also bi-SO_n invariant.

By Proposition 2, $\operatorname{dist}_{\tilde{d}}(A,\mathrm{SO}_n) = \operatorname{dist}_{\tilde{d}}(\Sigma,\mathrm{SO}_n)$, where $A = U\Sigma V^T$ is any SVD of A. By the results in Section 4.2, I is the closest matrix to both Σ and Σ^{-1} with respect to both intrinsic and extrinsic Euclidean distances.

Hence, for every $Q \in \mathrm{SO}_n$,

$$\tilde{d}(\Sigma,Q) = d(\Sigma,Q) + d(\Sigma^{-1},Q^{-1}) \geq d(\Sigma,I) + d(\Sigma^{-1},I) = \tilde{d}(\Sigma,I),$$

from which follows that I is the matrix in SO_n that is the closest to Σ with respect to $\tilde{d}$, and

$$\begin{aligned}\operatorname{dist}_{\tilde{d}}(A,\mathrm{SO}_n) = \operatorname{dist}_{\tilde{d}}(\Sigma,\mathrm{SO}_n) = \tilde{d}(\Sigma,I) &= d(\Sigma,I) + d(\Sigma^{-1},I) \\ &= \operatorname{dist}_d(A,\mathrm{SO}_n) + \operatorname{dist}_d(A^{-1},\mathrm{SO}_n).\end{aligned}$$

Again, we obtain that the matrix in SO_n that is the closest to A is the orthogonal polar factor of A.

The symmetrization of the Euclidean distance gives a truly different notion of strain measure, as it penalizes equally both expansions and contractions. Thus, it can be considered an improved strain measure. At the same time, it preserves the symmetries pertinent to the Euclidean metric—frame invariance and material isotropy.

5.2.2. *Inverse-invariant metrics.* We now turn to the symmetrization of Riemannian metrics on GL_n. Given any metric g, the metric $\tilde{g} = g + i^*(g)$ is inverse-invariant. It is common knowledge (see sketch of proof in [15]) that the space of inverse-invariant metrics for any Lie group is infinite-dimensional, hence, we have to address the question of finding natural inverse-invariant metrics. We note that in general, the distance function induced by the symmetrized metric $\tilde{g}$ is *not* the symmetrization of the distance function induced by g. Hence, the analysis of the previous subsection is not applicable.

We start by showing that the symmetrized metric inherits some of the symmetries of the original metric.

Lemma 5.4. *Let G be a Lie group and let $H \subseteq G$ be a subgroup. Let g be a bi-H-invariant metric on G. Its symmetrization $\tilde{g}$ is also bi-H-invariant.*

Proof. Let $h \in H$. Then

$$L_h^*(g + i^*g) = L_h^*g + L_h^*(i^*g) = g + (i \circ L_h)^*g.$$

Since $i \circ L_h = R_{h^{-1}} \circ i$,

$$L_h^*(g + i^*g) = g + (R_{h^{-1}} \circ i)^*g = g + i^*(R_{h^{-1}}^*g) = g + i^*g.$$

The proof that $R_h^*(g + i^*g) = g + i^*g$ is similar. □

It follows that the symmetrizations of all the metrics considered in sections 2 and 4.2 are bi-O_n invariant.

In the remaining part of this section we study the symmetrization of the metrics considered in Section 2.

As mentioned above, the symmetrized metrics, unlike the original metrics, are not left-GL_n invariant, hence, our analysis of the geodesics is not applicable. However, the symmetrized metrics share three important properties with the original metrics:

1. $(\mathrm{GL}_n^+, \tilde{g})$ is complete.
2. The symmetric and the skew-symmetric matrices are orthogonal with respect to $\tilde{g}_I$.
3. $\alpha(t) = e^{tV}$ is a $\tilde{g}$-geodesic for any symmetric matrix V.

We start by showing the orthogonality of symmetric and skew-symmetric matrices. For any $A \in \mathrm{sym}_n$ and $B \in \mathfrak{O}_n$,

$$(i^*g)_I(A, B) = g_I(di_I(A), di_I(B)) = g_I(-A, -B) = 0,$$

hence

$$\tilde{g}_I(A, B) = g_I(A, B) + (i^*g)_I(A, B) = 0.$$

We proceed to prove the completeness of $(\mathrm{GL}_n^+, \tilde{g})$. First note that completeness of g implies completeness of i^*g, since $i : (\mathrm{GL}_n, i^*g) \to (\mathrm{GL}_n, g)$ is an isometry. So, it suffices to prove that if two Riemannian manifolds (M, g_1) and (M, g_2) are complete then so is $(M, g_1 + g_2)$.

Lemma 5.5. *Let M be a smooth manifold, and let g_1, g_2 be Riemannian metrics on M. If either (M, g_1) or (M, g_2) is complete, then $(M, g_1 + g_2)$ is complete.*

Proof. Let $p, q \in M$. For any path α from p to q,

$$
\begin{aligned}
L_{g_1+g_2}(\alpha) &= \int \sqrt{g_1(\dot\alpha(t),\dot\alpha(t)) + g_2(\dot\alpha(t),\dot\alpha(t))} \\
&\geq \int \sqrt{g_1(\dot\alpha(t),\dot\alpha(t))} \\
&= L_{g_1}(\alpha).
\end{aligned}
$$

Similarly $L_{g_1+g_2}(\alpha) \geq L_{g_2}(\alpha)$. Without loss of generality, Assume that (M, g_1) is complete. For any $p, q \in M$,

$$
d^{g_1}(p,q) = \inf_{\alpha:p\mapsto q} L_{g_1}(\alpha) \leq \inf_{\alpha:p\mapsto q} L_{g_1+g_2}(\alpha) = d^{g_1+g_2}(p,q)
$$

By the Hopf-Rinow theorem, a Riemannian manifold (M, g) is complete if and only if closed and g-bounded sets are compact. Let $A \subseteq M$ be a closed and $(g_1 + g_2)$-bounded set. Boundedness implies that there exists a point $p \in M$ and a number $R > 0$, such tha $d^{g_1+g_2}(a,p) \leq R$ for every $a \in A$.

Since $d^{g_i}(a,p) \leq d^{g_1+g_2}(a,p) \leq R$, it follows that A is also g_1-bounded. Since (M, g_1) is complete, A is compact, hence $g_1 + g_2$ is complete as well. □

It remains to show that $\alpha(t) = e^{tV}$ is a geodesic for symmetric V. We first note that $\alpha(t) = i(e^{tV}) = e^{-tV}$ is a geodesic of i^*g, since $i : (\mathrm{GL}_n, i^*g) \to (\mathrm{GL}_n, g)$ is an isometry. Reversing time, we get that $\alpha(t) = e^{tV}$ is a geodesic of both g and i^*g. Note that every geodesic is parametrized by a parameter proportional to arclength, i.e., its speed $\|\dot\alpha(t)\|$ is constant. In this particular case, the speeds are the same when measured with respect to both metrics, since $g_I = (i^*g)_I$, hence $\|\dot\alpha(0)\|$ is independent of the metric chosen. It turns out that in this particular situation, α is a geodesic with respect to the metric $g + i^*g$.

Lemma 5.6. *Let M be a smooth manifold, and let g_1, g_2 be Riemannian metrics on M. Assume $\beta(t)$ is a geodesic for both g_1 and g_2, and that its speed is the same with respect to both metrics. Then $\beta(t)$ is also a $(g_1 + g_2)$-geodesic.*

Proof. The inequality: $\sqrt{a+b} \geq \frac{1}{\sqrt{2}}(\sqrt{a} + \sqrt{b})$ implies that for any path α in M,

$$
\begin{aligned}
L^{g_1+g_2}(\alpha) &= \int \sqrt{g_1(\dot\alpha(t),\dot\alpha(t)) + g_2(\dot\alpha(t),\dot\alpha(t))} \\
&\geq \frac{1}{\sqrt{2}} \cdot \Big(\int \sqrt{g_1(\dot\alpha(t),\dot\alpha(t))} + \int \sqrt{g_2(\dot\alpha(t),\dot\alpha(t))}\Big) \qquad (13) \\
&= \frac{1}{\sqrt{2}} \cdot (L^{g_1}(\alpha) + L^{g_2}(\alpha)).
\end{aligned}
$$

Since β is a geodesic with respect to both g_1 and g_2, it is locally length-minimizing with respect to both metrics; for small enough t, $L^{g_i}(\beta|_{[0,t]}) = d^{g_i}(\beta(0), \beta(t)) = tc$, for some constant c. Let α be any path connecting $\beta(0), \beta(t)$. By our assumption, $\sqrt{g_1(\dot\beta(t),\dot\beta(t))} = \sqrt{g_2(\dot\beta(t),\dot\beta(t))} = c$. Note that

$$
L^{g_1+g_2}(\beta|_{[0,t]}) = \int_0^t \sqrt{g_1(\dot\beta(t),\dot\beta(t)) + g_2(\dot\beta(t),\dot\beta(t))}
$$

$$= \sqrt{2}ct = \frac{1}{\sqrt{2}} \cdot (tc + tc)$$
$$= \frac{1}{\sqrt{2}} \cdot \left(d^{g_1}(\beta(0), \beta(t)) + d^{g_2}(\beta(0), \beta(t))\right),$$
$$\leq \frac{1}{\sqrt{2}} \cdot (L^{g_1}(\alpha) + L^{g_2}(\alpha)) \leq L^{g_1+g_2}(\alpha)$$

where the last inequality uses (13). Thus, β locally minimizes length with respect to $g_1 + g_2$, hence it is a geodesic. (Note it is parametrized proportional to arclength with respect to $g_1 + g_2$.) □

Next, we imitate from Section 3 the argument for finding the geodesic distance from SO_n. By Lemma 5.4, we can use Proposition 2 to reduce again the question to diagonal matrices. Since the derivation of the strain measure uses only the three properties of the metric mentioned above, it works in exactly the same manner for the symmetrized metric.

There is just one delicacy. The proof hinges on the fact that $\alpha(t) = Qe^{tV}$ is a $\tilde{g}$-geodesic for any symmetric V and $Q \in \mathrm{SO}_n$ (note that at this stage we don't yet know that $Q = I$). From Lemma 5.6 follows that e^{tV} is a $\tilde{g}$-geodesic. Since (Lemma 5.4 again) $\tilde{g}$ is bi-O_n-invariant it follows that α is also a $\tilde{g}$ geodesic. Following the rest of the proof, the only difference is at the final stage, when evaluating the speed $\|\dot{\alpha}(0)\|_I$, where $\tilde{g}_I$-is scaled by $\sqrt{2}$,

$$\|V\|_I = \sqrt{g_I(V,V) + (i^*g)_I(V,V)} = \sqrt{2 \cdot g_I(V,V)}.$$

Hence, the strain measure is multiplied by a factor of $\sqrt{2}$.

Appendix A. Calculating the geodesics.

A.1. **Analysis of the geodesic equations.** Let g_I be an inner-product on $M_n = T_I\mathrm{GL}_n$ given by the form (4), and let g be the Riemannian metric on GL_n which is the left-translation of g_I.

First, we need the following result.

Proposition 6. *For every $X, Y \in T_I GL_n$,*

$$g_I(X, [Y, X]) = \frac{\beta - \gamma}{2\beta} g_I([X, X^T], Y).$$

(Here $[X, Y]$ denotes the standard matrix commutator, i.e $[X, Y] = XY - YX$.)

Proof. Note first that

$$g_I(X, [Y, X]) = \beta \operatorname{tr}(\operatorname{sym}X \operatorname{sym}[Y, X]) + \gamma \operatorname{tr}(\operatorname{skew} X \operatorname{skew}[Y, X]),$$

and

$$g_I([X, X^T], Y) = \beta \operatorname{tr}(\operatorname{sym}[X, X^T] \operatorname{sym}Y).$$

Now,

$$\operatorname{tr}(\operatorname{sym}X \operatorname{sym}[Y, X]) = \frac{1}{4} \operatorname{tr}((X + X^T)(YX - XY + X^TY^T - Y^TX^T))$$
$$= \frac{1}{2} \operatorname{tr}([X, X^T]Y) = \frac{1}{2\beta} g_I([X, X^T], Y),$$

and

$$\operatorname{tr}(\operatorname{skew} X \operatorname{skew}[Y,X]) = \frac{1}{4}\operatorname{tr}((X - X^T)(YX - XY - X^TY^T + Y^TX^T))$$
$$= -\frac{1}{2}\operatorname{tr}([X,X^T]Y) = -\frac{1}{2\beta} g_I([X,X^T],Y),$$

hence

$$g_I(X,[Y,X]) = \frac{\beta-\gamma}{2\beta} g_I([X,X^T],Y).$$

□

Let $\{\mathfrak{a}_\alpha\}$ be a g_I-orthonormal basis for $(T_I\mathrm{GL}_n, g)$. Since g is left-invariant, $e_\alpha = d(L_g)_e(\mathfrak{a}_\alpha)$ is an orthonormal frame for $(T\mathrm{GL}_n, g)$. Let $\{\vartheta^\alpha\}$ be the orthonormal co-frame,

$$\vartheta^\alpha|_A = g_A(e_\alpha|_A, \cdot) = g_A(i_A(A\mathfrak{a}_\alpha), \cdot),$$

which implies that

$$\vartheta^\alpha|_A(i_A(AX)) = g_I(\mathfrak{a}_\alpha, X).$$

The Riemannian connection is represented by an anti-symmetric matrix of 1-forms, $\{\omega^\alpha{}_\beta\}$, defined by

$$\nabla_{e_\alpha} e_\beta = \omega^\gamma{}_\beta(e_\alpha)\, e_\gamma,$$

and satisfying *Cartan's first structural equation*,

$$d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta = 0.$$

Noting that,

$$d\vartheta^\alpha(e_\mu, e_\nu) = (\vartheta^\alpha(e_\nu))\, e_\mu - (\vartheta^\alpha(e_\mu))\, e_\nu - \vartheta^\alpha([e_\mu, e_\nu])$$
$$= \delta^\alpha_\nu\, e_\mu - \delta^\alpha_\mu\, e_\nu - \vartheta^\alpha([e_\mu, e_\nu]) - g(e_\alpha, [e_\mu, e_\nu]),$$

and that

$$g(e_\alpha, [e_\mu, e_\nu]) = g_I(\mathfrak{a}_\alpha, [\mathfrak{a}_\mu, \mathfrak{a}_\nu]), \tag{14}$$

we get

$$-g_I(\mathfrak{a}_\alpha, [\mathfrak{a}_\mu, \mathfrak{a}_\nu]) + \omega^\alpha{}_\nu(e_\mu) - \omega^\alpha{}_\mu(e_\nu) = 0.$$

Equality (14) holds because Lie brackets of left-invariant vector fields are left-invariant, together with the well-known fact that the Lie algebra commutator of GL_n is merely the standard matrix commutator in M_n, i.e $[e_\mu, e_\nu](e) = [\mathfrak{a}_\mu, \mathfrak{a}_\nu]$ [8, p. 193].

Rotating the indexes,

$$-g_I(\mathfrak{a}_\mu, [\mathfrak{a}_\alpha, \mathfrak{a}_\nu]) + \omega^\mu{}_\nu(e_\alpha) - \omega^\mu{}_\alpha(e_\nu) = 0$$
$$-g_I(\mathfrak{a}_\nu, [\mathfrak{a}_\mu, \mathfrak{a}_\alpha]) + \omega^\nu{}_\alpha(e_\mu) - \omega^\nu{}_\mu(e_\alpha) = 0.$$

Adding the three equations and renaming the indexes,

$$2\omega^\gamma{}_\beta(e_\alpha) = g_I(\mathfrak{a}_\alpha, [\mathfrak{a}_\gamma, \mathfrak{a}_\beta]) + g_I(\mathfrak{a}_\gamma, [\mathfrak{a}_\alpha, \mathfrak{a}_\beta]) + g_I(\mathfrak{a}_\beta, [\mathfrak{a}_\gamma, \mathfrak{a}_\alpha]).$$

That is,

$$\nabla_{e_\alpha} e_\beta = \Gamma^\gamma_{\alpha\beta}\, e_\gamma,$$

where

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\left(g_I(\mathfrak{a}_\alpha, [\mathfrak{a}_\gamma, \mathfrak{a}_\beta]) + g_I(\mathfrak{a}_\gamma, [\mathfrak{a}_\alpha, \mathfrak{a}_\beta]) + g_I(\mathfrak{a}_\beta, [\mathfrak{a}_\gamma, \mathfrak{a}_\alpha])\right)$$

are constant coefficients.

Consider now a geodesic curve, $\gamma : I \to \mathrm{GL}_n$, where

$$\dot\gamma(t) = p^\alpha(t)\, e_\alpha|_{\gamma(t)}.$$

The geodesic equation for the coefficients $p^\alpha(t)$ is

$$\dot{p}^\gamma(t) + \Gamma^\gamma_{\alpha\beta}\, p^\alpha(t)\, p^\beta(t) = 0.$$

Exploiting the symmetries of Γ and the symmetry of the geodesic equation,

$$\dot{p}^\gamma(t) + g_I(\mathfrak{a}_\alpha, [\mathfrak{a}_\gamma, \mathfrak{a}_\beta])\, p^\alpha(t)\, p^\beta(t) = 0.$$

This is a set of n^2 quadratic equations with constant coefficients.

Multiplying this equation by $\mathfrak{a}_\gamma$, setting $X(t) = p^\alpha(t)\mathfrak{a}_\alpha$, which is a curve in $T_I\mathrm{GL}_n$,

$$\dot{X}(t) + g_I(X(t), [\mathfrak{a}_\gamma, X(t)])\mathfrak{a}_\gamma = 0,$$

which by Proposition 6,

$$0 = \dot{X} + g_I(X, [\mathfrak{a}_\gamma, X])\mathfrak{a}_\gamma = \dot{X} + \frac{\beta-\gamma}{2\beta} g_I([X, X^T], \mathfrak{a}_\gamma)\mathfrak{a}_\gamma,$$

namely,

$$\dot{X} = \kappa(X^T X - X X^T), \tag{15}$$

where

$$\kappa = \frac{\beta - \gamma}{2\beta}.$$

Equation (15) is an ordinary differential system in the vector space M_n.

The geodesic $\gamma : I \to \mathrm{GL}_n$ is related to $X(t)$ via,

$$\dot{\gamma}(t) = i_{\gamma(t)}(\gamma(t), \gamma(t)X(t)). \tag{16}$$

A.2. **Solution of geodesic equations.** The factor $(\beta-\gamma)/2\beta$ has for effect to rescale time. We start by ignoring it.

Proposition 7. *The solution to*

$$\dot{X} = X^T X - X X^T \qquad X(0) = X_0,$$

where $X : I \to M_n$ is

$$X(t) = \exp(t(X_0^T - X_0))X_0 \exp(t(X_0 - X_0^T)).$$

Proof. Clearly, the initial conditions are satisfied. Differentiating with respect to t we get

$$\begin{aligned}
\dot{X}(t) &= \exp(t(X_0^T - X_0))(X_0^T - X_0)X_0 \exp(t(X_0 - X_0^T)) \\
&\quad + \exp(t(X_0^T - X_0))X_0(X_0 - X_0^T)\exp(t(X_0 - X_0^T)) \\
&= \exp(t(X_0^T - X_0))(X_0^T X_0 - X_0 X_0^T)\exp(t(X_0 - X_0^T)).
\end{aligned}$$

It only remains to insert $\exp(t(X_0 - X_0^T))\exp(t(X_0^T - X_0))$ inside the products in the middle term to get

$$\dot{X}(t) = X^T(t)X(t) - X(t)X^T(t).$$

□

Please note that this exponential is the "standard" matrix exponential, i.e., the one obtained from integral curves of left-invariant vector fields. It is not the exponential map of the g-geodesics.

Corollary 6. *The solution to (15) is*

$$X(t) = \exp(\kappa t(X_0^T - X_0))X_0 \exp(\kappa t(X_0 - X_0^T)).$$

Proposition 8. *Let $\gamma : I \to GL_n$ be the $\mathfrak{g}$-geodesic,*

$$\dot\gamma(t) = i_{\gamma(t)}(\gamma(t), \gamma(t)X(t)).$$

satisfying the initial conditions

$$\gamma(0) = e \qquad \textit{and} \qquad \dot\gamma(0) = X_0.$$

Then,

$$\gamma(t) = \exp((1-\kappa)tX_0 + \kappa t X_0^T)\,\exp(\kappa t(X_0 - X_0^T)).$$

Proof. Clearly, the initial conditions are satisfied. Differentiating with respect to t we get

$$\begin{aligned}
\dot\gamma(t) &= (T\gamma)_t(\partial_t) \\
&= (T\gamma)_t([t+s]) \\
&= [\gamma(t+s)] \\
&= [\gamma(t) + \exp((1-\kappa)tX_0 + \kappa t X_0^T)X_0 \exp(\kappa t(X_0 - X_0^T))\, s] \\
&= i_{\gamma(t)}\left(\gamma(t), \exp((1-\kappa)tX_0 + \kappa t X_0^T)X_0 \exp(\kappa t(X_0 - X_0^T))\right) \\
&= i_{\gamma(t)}(\gamma(t), \gamma(t)X(t)).
\end{aligned}$$

□

Appendix B. Inverse-invariant metrics on Lie groups.

Proposition 9. *Let G be a Lie group. A left- (or right-)invariant metric on G is inverse-invariant if and only if it is bi-invariant.*

Proof. Note that $\mathrm{inv} = R_{s^{-1}} \circ \mathrm{inv} \circ L_{s^{-1}}$,

$$R_{s^{-1}} \circ \mathrm{inv} \circ L_{s^{-1}}(g) = R_{s^{-1}} \circ \mathrm{inv}(s^{-1}g) = R_{s^{-1}}(g^{-1}s) = g^{-1}.$$

By the chain rule,

$$(d\,\mathrm{inv})_s = (dR_{s^{-1}})_e \circ (d\,\mathrm{inv})_e \circ (dL_{s^{-1}})_s.$$

Since $(d\,\mathrm{inv})_e : T_eG \to T_eG$ is the additive inverse operation ($v \mapsto -v$),

$$(d\,\mathrm{inv})_s = -(dR_{s^{-1}})_e \circ (dL_{s^{-1}})_s.$$

It follows at once that a bi-invariant metric is inverse-invariant.

Conversely, assume the metric is both left- and inverse-invariant. Then $\forall s \in G$, $(dR_s)_e$ is an isometry, hence the metric is right-invariant as well. □

Proposition 10. *Let G be a Lie group. A left- (or right-)invariant distance function d on G is inverse-invariant if and only if it is bi-invariant.*

Proof. Assume d is left- and inverse-invariant. Since $\mathrm{inv} = R_{s^{-1}} \circ \mathrm{inv} \circ L_{s^{-1}}$, d is also right-invariant.

Conversely, assume d is bi-invariant. Then,

$$d(x,y) = d(1, x^{-1}y) = d(y^{-1}, x^{-1}) = d(x^{-1}, y^{-1}).$$

□

In fact, Proposition 10 implies Proposition 9 by virtue of the Myers-Steenrod theorem whereby every isometry of d is a Riemannian isometry.

Appendix C. **A minimizing geodesic from a point to a submanifold is orthogonal to the submanifold.** We prove the following lemma, which plays a central role in the derivation of the geodesic distance.

Lemma C.1. *Let M be a complete Riemannian manifold. Let $S \subseteq M$ be a submanifold, and let $p \in M \setminus S$. Assume $q \in S$ is a point on S satisfying $d(p,q) = \operatorname{dist}(p,S)$ (there is always such a point q if S is compact). Let α be a minimizing geodesic connecting p and q. Then α is orthogonal to S at q.*

Proof. The proof is based upon a "variation of energy" technique, and is essentially taken from a discussion in [16].

Assume $\alpha : [0,a] \to M$ is a minimizing geodesic joining the points p and q. Let $v \in T_qS$ and $\phi : (-\epsilon, \epsilon) \to S$ be a differentiable curve such that $\phi(0) = q$ and $\phi'(0) = v$.

Consider a variation $f : (-\epsilon,\epsilon) \times [0,a] \to M$, such that $f(0,\cdot) = \alpha$, $f(\cdot, 0) = p$ and $f(\cdot, a) = \phi$. Let

$$E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt$$

be the energy associated with f. We are going the use the following inequality, relating the length of a curve $c : [0,a] \to M$ and its energy (see [3, p. 194]),

$$L^2(c) \le a\, E(c).$$

Equality holds if and only if c is parametrized proportional to arclength.

In particular,

$$L^2(f_s) \le a\, E(f_s). \tag{17}$$

Note that since $\operatorname{Image}(\phi) \subseteq S$, then for any $s \in (-\epsilon,\epsilon)$,

$$L(f_0) = L(\alpha) = d(p,q) = d(p,S) \le d(p,\phi(s)) = d(f_s(0), f_s(a)) \le L(f_s). \tag{18}$$

Finally, since $\alpha = f_0$ is a geodesic, its parametrization is proportional to arclength, so by the above comment,

$$L^2(f_0) = a\, E(f_0). \tag{19}$$

Combining (17),(18) and (19), we obtain

$$a\, E(f_0) = L^2(f_0) \le L^2(f_s) \le a\, E(f_s).$$

Thus $E(f_s)$ as a function of s has a minimum at $s = 0$, therefore $E'(0) = 0$. By the first variation of energy formula (see [3, p. 195]),

$$\frac{1}{2} \left. \frac{dE'(f_s)}{ds} \right|_{s=0} = \langle v, \alpha'(a) \rangle.$$

Since v is an arbitrary vector in T_qS, the proof is complete. □

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