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INCOMPATIBLE ELASTICITY AND THE IMMERSION OF NON-FLAT RIEMANNIAN MANIFOLDS IN EUCLIDEAN SPACE

BY

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ABSTRACT

We study a geometric problem that originates from theories of nonlinear elasticity: given a non-flat n-dimensional Riemannian manifold with boundary, homeomorphic to a bounded subset of $\mathbb{R}^n$, what is the minimum amount of deformation required in order to immerse it in a Euclidean space of the same dimension? The amount of deformation, which in the physical context is an elastic energy, is quantified by an average over a local metric discrepancy. We derive an explicit lower bound for this energy for the case where the scalar curvature of the manifold is non-negative. For $n = 2$ we generalize the result for surfaces of arbitrary curvature.

1. Introduction

Let $(\mathcal{M}, \bar{\mathfrak{g}})$ be an n-dimensional smooth, oriented, simply connected Riemannian manifold, homeomorphic to a bounded subset of $\mathbb{R}^n$. Let $\mathcal{F}$ be the set of orientation-preserving immersions $f : (\mathcal{M}, \bar{\mathfrak{g}}) \to (\mathbb{E}^n, \mathfrak{e})$, where $(\mathbb{E}^n, \mathfrak{e})$ is a

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Euclidean space (for the moment let us consider smooth immersions). The topology of the manifold implies that $\mathcal{F}$ is not empty. Every mapping $f \in \mathcal{F}$ induces on $\mathcal{M}$ a pullback metric, commonly denoted by $\mathfrak{g} = f^*\mathfrak{e}$, and defined by

$$\mathfrak{g}(u, v) = \langle df(u), df(v)\rangle_{\mathfrak{e}}, \quad u, v \in T\mathcal{M}, \tag{1.1}$$

where $df : T\mathcal{M} \to T\mathbb{E}^n$ is the differential of f. As is well-known, there exists an isometric orientation-preserving immersion $f \in \mathcal{F}$, namely, an immersion for which $\mathfrak{g} = \bar{\mathfrak{g}}$, if and only if the manifold $(\mathcal{M}, \bar{\mathfrak{g}})$ is flat, i.e., has a vanishing Riemann curvature tensor (simple-connectedness is needed for the "if" part; a torus, for example, is flat but cannot be immersed isometrically into $\mathbb{E}^2$). See, e.g., Ciarlet [2], p. 26 for a proof of the $n = 3$ case.

A natural question is how "close" is the manifold $(\mathcal{M}, \bar{\mathfrak{g}})$ to being flat. Such a quantification relies on a choice of a distance between the Riemannian metric $\bar{\mathfrak{g}}$ and the set of induced metrics $f^*\mathfrak{e}$. To this end we define a "mismatch" function, $E : \mathcal{F} \to [0, \infty)$, and quantify the extent by which $(\mathcal{M}, \bar{\mathfrak{g}})$ fails to be isometrically immersible into $(\mathbb{E}^n, \mathfrak{e})$ by

$$E^0 = \inf\{E[f] : f \in \mathcal{F}\}. \tag{1.2}$$

Problems of such type arise in the theory of elasticity, and notably in theories of so-called "incompatible elasticity", where the metric $\bar{\mathfrak{g}}$ represents the *local* equilibrium state of an elastic material (the rest distance between "neighboring" material elements), whereas the set $\mathcal{F}$ corresponds to the set of all actual configurations that the material can assume (Wang [12] and Kröner [7] are classical expository references for incompatible elasticity in the context of defects in materials; see also Efrati et al. [3] and Yavari [13] for more recent applications). The mismatch function E is interpreted as an elastic energy, and the equilibrium state of the material is postulated to correspond to the orientation-preserving immersion that minimizes this elastic energy.

Elastic materials are said to be hyper-elastic (Truesdell [11]) if the elastic energy associated with the configuration f can be represented as a volume integral of the form

$$E[f] = \int_{\mathcal{M}} W(f(x), df(x), x)\, \mathrm{vol}_{\bar{\mathfrak{g}}}(x),$$

where W is an energy density function and $\mathrm{vol}_{\bar{\mathfrak{g}}}$ is the volume form of $(\mathcal{M}, \bar{\mathfrak{g}})$. In the absence of external forces or external constraints, there is no explicit dependence of the energy density on f, hence $W = W(df(x), x)$.

The energy density W is commonly subject to the following requirements:

1. It is non-negative, i.e., $W(df(x), x) \geq 0$ for all $f \in \mathcal{F}$ and $x \in \mathcal{M}$.
2. It satisfies $W(df(x), x) = 0$ if and only if $f^*\mathfrak{e}(x) = \bar{\mathfrak{g}}(x)$.
3. It is invariant under rotations, namely $W(df(x), x) = W(Q\, df(x), x)$ for all $f \in \mathcal{F}$ and $Q \in \mathrm{SO}(n)$ (a condition known as "frame invariance").
4. It is coercive in the sense that

$$W(df(x), x) \geq c\, |f^*\mathfrak{e}(x) - \bar{\mathfrak{g}}(x)|^2_{\bar{\mathfrak{g}}}, \tag{1.3}$$

for some constant $c > 0$, where $|\cdot|_{\bar{\mathfrak{g}}}$ is the norm on tensors, $T^*\mathcal{M}\otimes T^*\mathcal{M}$, induced by the inner product $\bar{\mathfrak{g}}$.

The frame invariance condition is automatically fulfilled if the energy density is represented as a function of $\mathfrak{g}(x)$ rather than as a function of $df(x)$. There is, however, one caveat: the metric $\mathfrak{g}$ is invariant under reflections of f, i.e., under improper rotations, hence the energy density remains explicitly dependent on df through the requirement that it preserves orientation. From the point of view of the mathematical modeling there are two alternatives: the first is, as above, to restrict from the outset the configurations f to orientation-preserving immersions, in which case the energy density can be represented as $W = W(\mathfrak{g}, x)$, with $\mathfrak{g} = f^*\mathfrak{e}$, which, in particular, vanishes if and only if $\mathfrak{g}(x) = \bar{\mathfrak{g}}(x)$ (see, e.g., [3]). The second alternative, which, due to numerous technical reasons, is somewhat more popular, is to relax the restriction that f be an immersion, and define an energy density $W = W(df(x), x)$, which is only invariant under proper rotations, and, in particular, does not vanish if $\mathfrak{g}(x) = \bar{\mathfrak{g}}(x)$ but df reverses orientation (see e.g., Friesecke et al. [6] and Lewicka and Pakzad [9]).

The second alternative can be implemented, for example, as follows: There exists a (non-unique) smooth orientation-preserving bundle map $\bar{\mathfrak{q}} : T\mathcal{M} \to T\mathbb{E}^n$ (i.e., $\pi \circ \bar{\mathfrak{q}} = f \circ \pi$, where π denote the projections $T\mathcal{M} \to \mathcal{M}$ and $T\mathbb{E}^n \to \mathbb{E}^n$), such that $\bar{\mathfrak{g}} = (\bar{\mathfrak{q}}^* \otimes \bar{\mathfrak{q}}^*)\mathfrak{e}$, or explicitly,

$$\bar{\mathfrak{g}}(u, v) = \langle \bar{\mathfrak{q}}(u), \bar{\mathfrak{q}}(v)\rangle_{\mathfrak{e}}, \quad u, v \in T\mathcal{M}. \tag{1.4}$$

The tensor $\bar{\mathfrak{q}}$ is a "square root" (with respect to the Euclidean inner product) of the metric $\bar{\mathfrak{g}}$, in the same sense as df is a "square root" of the metric $\mathfrak{g}$. The condition that $W(df(x), x)$ be zero if and only if $\mathfrak{g}(x) = \bar{\mathfrak{g}}(x)$ and df be orientation-preserving can now be restated as follows: $W(df(x), x)$ vanishes if and only if there exists a proper rotation $Q \in \mathrm{SO}(n)$, such that $df(x) = Q\bar{\mathfrak{q}}(x)$. In this framework, the coercivity condition (1.3) is replaced by a similar condition,

$$W(df(x), x) \geq c_1 \, W_{\text{quad}}(df(x), x), \tag{1.5}$$

for some constant $c_1 > 0$, where

$$W_{\text{quad}}(df(x), x) = \min_{Q \in \mathrm{SO}(n)} |df(x) - Q\bar{\mathfrak{q}}(x)|^2_{\bar{\mathfrak{g}},\mathfrak{e}}, \tag{1.6}$$

and $|\cdot|_{\bar{\mathfrak{g}},\mathfrak{e}}$ is the norm on bundle maps $T\mathcal{M} \to T\mathbb{E}^n$ induced by both inner products $\bar{\mathfrak{g}}$ and $\mathfrak{e}$. A minimizer for the right hand side of (1.6) exists since $\mathrm{SO}(n)$ is a compact group. The quadratic energy density (1.6) was defined in [9]; it is the square distance of $df(x)$ from the set of "square roots" of $\bar{\mathfrak{g}}(x)$.

The determination of E^0, given by (1.2), along with a minimizing state (if such exists) is in general a hard problem, even computationally. A subsidiary problem is the determination of a lower bound for the elastic energy in terms of the geometric properties of the manifold $(\mathcal{M}, \bar{\mathfrak{g}})$. This task relates back to our original statement of the problem: finding a quantitative measure for the lack of isometric immersibility in terms of the non-flatness (i.e., the curvature) of the manifold.

In a recent work by Lewicka and Pakzad [9], it was proved that E^0 is positive whenever the Riemannian curvature tensor associated with $\bar{\mathfrak{g}}$ does not vanish almost everywhere. A quantitative lower energy bound was simultaneously derived in [8] for the case of a two-dimensional manifold, $n = 2$. The energy density in [8] was taken to be

$$W(df(x), x) = |f^*\mathfrak{e}(x) - \bar{\mathfrak{g}}(x)|^2_{\bar{\mathfrak{g}}},$$

with f explicitly restricted to orientation-preserving immersions. Using techniques that are specific to two-dimensions (properties of Riemann surfaces and complex analysis), it was proved that E^0 can be bounded from below by a term which is essentially proportional to the square of the surface integral of the Gaussian curvature, *provided that this integral is non-negative.* No counterpart was found if this integral is strictly negative, and, in particular, if the surface is hyperbolic.

In this paper we extend the above analysis for arbitrary dimension, $n \geq 2$. The result is a generalization of [8]: E^0 can be bounded from below by a quantity which is monotonically increasing with the volume integral of the scalar curvature of the manifold, provided that the latter is non-negative (Theorem 4.1). We further generalize this result for two-dimensional manifolds with arbitrary Gaussian curvature. Deriving a similar result for hyperbolic n-dimensional manifolds, $n > 2$, remains an open problem. The role of the sign of the scalar curvature will be addressed in the Discussion section.

2. Notations and problem restatement

We start by introducing our notations and terminology. As common, we formulate everything in intrinsic suffix-free notation, recurring as needed to a suffix notation for the ease of calculations.

$(\mathcal{M}, \bar{\mathfrak{g}})$ is a smooth, oriented, simply-connected n-dimensional Riemannian manifold, which is homeomorphic to a bounded subset of $\mathbb{R}^n$. As standard, we denote by $T\mathcal{M}$ and $T^*\mathcal{M}$ the tangent and cotangent bundles of $\mathcal{M}$. The linear spaces of tensor fields, i.e., of sections of the tensor bundles

$$T^r_s\mathcal{M} = T\mathcal{M}^{\otimes r} \otimes T^*\mathcal{M}^{\otimes s}$$

are denoted by $\mathcal{T}^r_s(\mathcal{M}) = \Gamma_{\mathcal{M}}(T^r_s\mathcal{M})$. Likewise, we denote the spaces of differential forms by $E_k(\mathcal{M}) = \Gamma_{\mathcal{M}}(\Lambda_k(T^*\mathcal{M}))$.

We denote the Riemannian inner-product by

$$\langle u, v\rangle_{\bar{\mathfrak{g}}}, \quad u, v \in T\mathcal{M}.$$

The corresponding norm is

$$|v|_{\bar{\mathfrak{g}}} = \langle v, v\rangle_{\bar{\mathfrak{g}}}^{1/2}, \quad v \in T\mathcal{M}.$$

The metric $\bar{\mathfrak{g}}$ induces a so-called index lowering operator $\flat : T\mathcal{M} \to T^*\mathcal{M}$, which sends a vector field into a one-form via

$$v \mapsto \flat v = \langle v, \cdot\rangle_{\bar{\mathfrak{g}}}.$$

The index raising operator $\# : T^*\mathcal{M} \to T\mathcal{M}$ is the inverse of $\flat$. The metric $\bar{\mathfrak{g}}$ induces inner products on tensor products of $T\mathcal{M}$ and $T^*\mathcal{M}$ of all ranks and variance, as well as on differential forms.

Recall that every linear mapping $\ell : T\mathcal{M} \to T\mathcal{M}$ can be identified as a tensor field in $T^1_1\mathcal{M}$. The trace, or contraction of ℓ is defined by its action on decomposable tensors,

$$\operatorname{tr} : u^* \otimes v \mapsto u^*(v).$$

More generally, a contraction maps $T^r_s\mathcal{M}$ into $T^{r-1}_{s-1}\mathcal{M}$. For tensors $\ell \in T^0_2\mathcal{M}$ we define

$$\operatorname{tr}_{\bar{\mathfrak{g}}} \ell = \operatorname{tr}(\#\ell).$$

We denote by ∇ the Riemannian connection associated with the metric $\bar{\mathfrak{g}}$. The Riemann curvature tensor is a tensor field

$$\mathfrak{R}_{\bar{\mathfrak{g}}} \in \mathcal{T}^1_3(\mathcal{M}) \simeq \operatorname{Hom}(T\mathcal{M}^{\otimes 3}, T\mathcal{M}),$$

defined by

$$\mathfrak{R}_{\bar{\mathfrak{g}}}(u,v,w) = \nabla_v\nabla_u w - \nabla_u\nabla_v w - \nabla_{[u,v]}w.$$

The Ricci curvature tensor is a tensor field $\mathfrak{Ric}_{\bar{\mathfrak{g}}} \in \mathcal{T}^0_2(\mathcal{M})$, defined by

$$\mathfrak{Ric}_{\bar{\mathfrak{g}}}(u,v) = \text{trace of the endomorphism } w \mapsto \mathfrak{R}_{\bar{\mathfrak{g}}}(w,u,v).$$

Finally, the scalar curvature is the metric trace of the Ricci tensor, namely,

$$\mathfrak{s}_{\bar{\mathfrak{g}}} = \operatorname{tr}_{\bar{\mathfrak{g}}} \mathfrak{Ric}_{\bar{\mathfrak{g}}}.$$

The Hodge-star operator is a linear mapping $\star_{\bar{\mathfrak{g}}} : E_k(\mathcal{M}) \to E_{n-k}(\mathcal{M})$ defined by the relation

$$\lambda \wedge \omega = \langle \star_{\bar{\mathfrak{g}}}\lambda, \omega\rangle_{\bar{\mathfrak{g}}} \operatorname{vol}_{\bar{\mathfrak{g}}}, \quad \lambda, \omega \in: E_k(\mathcal{M}).$$

The co-derivative is defined by

$$\delta = (-1)^{nk+n+1} \star_{\bar{\mathfrak{g}}} d \star_{\bar{\mathfrak{g}}},$$

where d is the exterior derivative and k is the rank of the form on which the co-derivative operates. The deRham–Laplace operator is

$$\Delta_{\mathrm{dR}} = (d+\delta)^2 = d\delta + \delta d,$$

where we used the fact that $d^2 = \delta^2 = 0$. For scalar functions, i.e., zero-forms, f,

$$\Delta_{\mathrm{dR}} f = \delta df = -\star_{\bar{\mathfrak{g}}} d \star_{\bar{\mathfrak{g}}} df. \tag{2.1}$$

A scalar function is said to be harmonic if $\Delta_{\mathrm{dR}} w = 0$. For scalar functions, the deRham–Laplace operator coincides with the Bochner Laplacian, also known as the Laplace–Beltrami operator,

$$\Delta_{\mathrm{B}} = \nabla^*\nabla,$$

or in suffix notation, $\Delta_{\mathrm{B}} = \bar{\mathfrak{g}}^{ij}\nabla_i\nabla_j$. For higher rank k-forms Δ_{dR} and Δ_{B} are related by the Weitzenböck identity (e.g., Petersen [10], p. 181).

Let $f : (\mathcal{M}, \bar{\mathfrak{g}}) \to (\mathbb{E}^n, \mathfrak{e})$ be a differentiable mapping. The induced pullback metric $\mathfrak{g} = f^*\mathfrak{e}$ is defined by

$$\mathfrak{g}(\cdot,\cdot) = \langle df(\cdot), df(\cdot)\rangle_{\mathfrak{e}}.$$

As stated in the Introduction, every metric $\bar{\mathfrak{g}}$ has a (non-unique) smooth orientation-preserving square-root, which is a bundle map, $\bar{\mathfrak{q}} : T\mathcal{M} \to T\mathbb{E}^n$, satisfying $\bar{\mathfrak{g}} = (\bar{\mathfrak{q}} \otimes \bar{\mathfrak{q}})^*\mathfrak{e}$, or more explicitly,

$$\langle \bar{\mathfrak{q}}(\cdot), \bar{\mathfrak{q}}(\cdot)\rangle_{\mathfrak{e}} = \langle \cdot,\cdot\rangle_{\bar{\mathfrak{g}}}.$$

The inverse relation is

$$\langle \bar{\mathfrak{q}}^*(\cdot), \bar{\mathfrak{q}}^*(\cdot)\rangle_{\bar{\mathfrak{g}}} = \langle \cdot,\cdot\rangle_{\mathfrak{e}}.$$

For bundle maps $S, T : T\mathcal{M} \to T\mathbb{E}^n$, we define the inner product

$$\langle S, T\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = \mathrm{tr}_{\bar{\mathfrak{g}}}\langle S(\cdot), T(\cdot)\rangle_{\mathfrak{e}} = \mathrm{tr}_{\mathfrak{e}}\langle S^*(\cdot), T^*(\cdot)\rangle_{\bar{\mathfrak{g}}}, \tag{2.2}$$

along with the corresponding norm $|\cdot|_{\bar{\mathfrak{g}},\mathfrak{e}}$. Note that $|\bar{\mathfrak{q}}|_{\bar{\mathfrak{g}},\mathfrak{e}} = \sqrt{n}$ as seen using suffix notation,

$$\langle \bar{\mathfrak{q}}, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = \bar{\mathfrak{g}}^{ij}\mathfrak{e}_{\alpha\beta}\bar{\mathfrak{q}}_i^\alpha\bar{\mathfrak{q}}_j^\beta = \bar{\mathfrak{g}}^{ij}\bar{\mathfrak{g}}_{ij} = n.$$

A proper rotation $Q \in \mathrm{SO}(n)$ is a linear operator, $Q \in \mathrm{Hom}(T\mathbb{E}^n, T\mathbb{E}^n)$, that is orthogonal, namely,

$$\langle Q(\cdot), Q(\cdot)\rangle_{\mathfrak{e}} = \langle \cdot,\cdot\rangle_{\mathfrak{e}},$$

and satisfies $\det Q = 1$. The quadratic energy density W_{quad} was defined by

$$W_{\text{quad}}(df, x) = \min_{Q\in\mathrm{SO}(n)} |df - Q\bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}}. \tag{2.3}$$

We now turn to characterize the tensor $Q\bar{\mathfrak{q}}$ that minimizes the above expression:

Proposition 2.1: *Let f be twice differentiable, and let $\bar{\mathfrak{q}}$ be the square root of $\bar{\mathfrak{g}}$ that minimizes* (2.3). *Then for every vector field v,*

$$\langle df, \nabla_v\bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = 0 \quad \text{and} \quad \langle df, \Delta_B\bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \le 0.$$

Proof. Let v be an arbitrary vector field. Consider a "virtual" evolution equation for a one-parameter family of bundle maps $\bar{\mathfrak{p}}_t : T\mathcal{M} \to T\mathbb{E}^n$,

$$\frac{d}{dt}\bar{\mathfrak{p}}_t = \nabla_v\bar{\mathfrak{p}}_t, \quad \bar{\mathfrak{p}}_0 = \bar{\mathfrak{q}}.$$

We show that $\bar{\mathfrak{p}}_t$ remains a square root of $\bar{\mathfrak{g}}$ for all t, namely, $\bar{\mathfrak{p}}_t \in \mathrm{SO}(n)$, or

$$\langle \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}} = \langle \cdot,\cdot\rangle_{\bar{\mathfrak{g}}}.$$

This is true because

$$\frac{d}{dt}\langle \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}} = \langle \nabla_v \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}} + \langle \bar{\mathfrak{p}}_t(\cdot), \nabla_v \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}} = \nabla_v \langle \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}}.$$

Set then $A_t = \nabla_v \langle \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}}$. It follows that

$$\frac{dA_t}{dt} = \nabla_v A_t, \quad A_0 = 0,$$

whose unique solution is $A_t = 0$, i.e., $\langle \bar{\mathfrak{p}}_t(\cdot), \bar{\mathfrak{p}}_t(\cdot)\rangle_{\mathfrak{e}}$ is time-invariant.

By the defining property of $\bar{\mathfrak{q}}$ as a minimizer it follows that

$$\frac{d}{dt}\langle df, \bar{\mathfrak{p}}_t\rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\bigg|_{t=0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}\langle df, \bar{\mathfrak{p}}_t\rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\bigg|_{t=0} \le 0,$$

i.e.,

$$\langle df, \nabla_v \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = 0 \quad \text{and} \quad \langle df, \nabla_v \nabla_v \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \le 0.$$

This proves the first statement. The second statement follows from the fact that at every point we may express Δ_{B} as a linear combination

$$\Delta_{\mathrm{B}} = \sum_i c_i \nabla_{v_i} \nabla_{v_i},$$

with $c_i > 0$. ∎

The elastic energy is given by

$$E[f] = \int_{\mathcal{M}} W(df, x)\, \mathrm{vol}_{\bar{\mathfrak{g}}}.$$

Our goal is to find a positive lower bound for $E[f]$ over all mappings $f \in \mathcal{F}$. The set of admissible functions is some subset of $W^{1,2}(\mathcal{M}, \bar{\mathfrak{g}})$, depending on the precise form of the energy density (note we we only impose coercivity conditions, but no growth conditions). Since we aim for a lower bound, we may consider any function space that is dense in $W^{1,2}(\mathcal{M}, \bar{\mathfrak{g}})$, for example, smooth functions.

Without loss of generality (as we explain shortly), we make an additional coercivity assumption, whereby there exists a positive constant c_2, such that

$$W(df, x) \ge c_2 |(\star_{\bar{\mathfrak{g}}} - \star_{\mathfrak{g}}) df|^2_{\bar{\mathfrak{g}},\mathfrak{e}}, \tag{2.4}$$

where $\star_{\mathfrak{g}}$ is the Hodge-star operator with respect to the induced metric $\mathfrak{g} = f^*\mathfrak{e}$. We make this assumption to simplify some of the derivations in the next section; however, such coercivity condition can be obviated as follows (see [9]): using an approximation theorem [5, 6] one can show that there exists a constant $M > 0$ (that depends on the metric of the manifold but not on its derivatives), such that every function $f \in W^{1,2}(\mathcal{M}, \bar{\mathfrak{g}})$ can be approximated by a function $\bar{f} \in W^{1,\infty}(\mathcal{M}, \bar{\mathfrak{g}})$, with $\|d\bar{f}\|_\infty < M$ and $E[\bar{f}] \leq 10\, E[f]$. This implies that the analysis can be limited to mappings f that are uniformly bounded in $W^{1,\infty}(\mathcal{M}, \bar{\mathfrak{g}})$, in which case it is easy to see that there exists a constant c_2 (which depends on M) such that (2.4) holds. In any case, condition (2.4) is plausible in any realistic elastic model.

Our derivation of a lower energy bound is based on the following steps: (i) Using harmonic analysis, we show that the analysis can be restricted to bounding from below the quadratic energy

$$E_{\text{quad}}[f] = \int_{\mathcal{M}} W_{\text{quad}}(df, x)\, \text{vol}_{\bar{\mathfrak{g}}}, \tag{2.5}$$

over the set of *harmonic* maps $f : \mathcal{M} \to \mathbb{E}^n$. (ii) For the case where the scalar curvature is non-negative, we derive a lower bound for $E_{\text{quad}}[f]$, with f harmonic.

3. Harmonic analysis

A mapping $u : (\mathcal{M}, \bar{\mathfrak{g}}) \to (\mathcal{N}, \mathfrak{h})$ between two Riemannian manifolds is said to be harmonic if it is a critical point of the Dirichlet functional,

$$\int_{\mathcal{M}} |du|^2_{\bar{\mathfrak{g}},\mathfrak{h}}\, \text{vol}_{\bar{\mathfrak{g}}}.$$

When the target space is Euclidean, and we endow $(\mathbb{E}^n, \mathfrak{e})$ with the canonical parametrization, x^α, $\alpha = 1, \dots, n$, then harmonicity amounts to the component-wise conditions $\Delta_{\text{B}} f^\alpha = \Delta_{\text{dR}} f^\alpha = 0$.

When $f : (\mathcal{M}, \bar{\mathfrak{g}}) \to (\mathbb{E}^n, \mathfrak{e})$ is an isometry, namely, $f^*\mathfrak{e} = \bar{\mathfrak{g}}$, then f is harmonic. This follows from the pullback property,

$$\star_{f^*\mathfrak{e}} df^\alpha = \star_{f^*\mathfrak{e}}(f^*\, dx^\alpha) = f^*(\star_{\mathfrak{e}} dx^\alpha),$$

which implies that

$$d \star_{f^*\mathfrak{e}} df^\alpha = f^*(d \star_{\mathfrak{e}} dx^\alpha) = 0, \tag{3.1}$$

the latter being a consequence of the vanishing of $d \star_{\mathfrak{e}} dx^\alpha$. In elastic contexts, this identity is known as the vanishing of the divergence of the cofactor matrix of df. Thus, when $\bar{\mathfrak{g}} = f^*\mathfrak{e}$,

$$\Delta_{\text{dR}} f^\alpha = -\star_{\bar{\mathfrak{g}}} d \star_{\bar{\mathfrak{g}}} df^\alpha = -\star_{\bar{\mathfrak{g}}} d \star_{f^*\mathfrak{e}} df^\alpha = 0.$$

In this section, we prove that a lower bound for $E[f]$ can be derived by restricting f to the set of harmonic maps, by showing that when the elastic energy is small, then f is close in L^2 to being harmonic. This analysis parallels that in [9], however within a more general context.

Given $f^\alpha : \mathcal{M} \to \mathbb{R}^n$, we decompose it as

$$f^\alpha = w^\alpha + z^\alpha,$$

where $\Delta_{\mathrm{B}} w^\alpha = 0$ and $w^\alpha = f^\alpha$ on $\partial\mathcal{M}$. Such a decomposition is uniquely defined due to the well-posedness of the Laplace–Beltrami equation with Dirichlet boundary conditions.

The coercivity condition (1.5) implies that

$$E[f] \geq c_1 \int_{\mathcal{M}} |df - \bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}.$$

Thus,

$$\begin{aligned} E[f] &\geq c_1 \int_{\mathcal{M}} |dw - \bar{\mathfrak{q}} + dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \\ &\geq \frac{c_1}{2} \int_{\mathcal{M}} |dw - \bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} - c_1 \int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \\ &\geq \frac{c_1}{2} E_{\mathrm{quad}}[w] - c_1 \int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}, \end{aligned} \tag{3.2}$$

where in the passage from the first to the second line we used the inequality $|a+b|^2 \geq \frac{1}{2}a^2 - b^2$, and in the passage from the second to the third line we used the fact that

$$W_{\mathrm{quad}}(dw, x) \leq |dw - \bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}}.$$

We proceed to bound from above the L^2-norm of dz. For that, we use the fact that (2.1) and (3.1) imply

$$\Delta_{\mathrm{dR}} f = \star_{\bar{\mathfrak{g}}} d(\star_{\mathfrak{g}} - \star_{\bar{\mathfrak{g}}}) df. \tag{3.3}$$

Thus,

$$\begin{aligned}\int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} &= \sum_\alpha \int_{\mathcal{M}} \langle dz^\alpha, dz^\alpha \rangle_{\bar{\mathfrak{g}}} \operatorname{vol}_{\bar{\mathfrak{g}}} = \sum_\alpha \int_{\mathcal{M}} (\star_{\bar{\mathfrak{g}}} dz^\alpha) \wedge dz^\alpha \\ &= \sum_\alpha \int_{\mathcal{M}} (\star_{\bar{\mathfrak{g}}} \Delta_{\mathrm{dR}} z^\alpha)\, z^\alpha \\ &= \sum_\alpha \int_{\mathcal{M}} (\star_{\bar{\mathfrak{g}}} \Delta_{\mathrm{dR}} f^\alpha)\, z^\alpha \\ &= \sum_\alpha \int_{\mathcal{M}} [\star_{\bar{\mathfrak{g}}} \star_{\bar{\mathfrak{g}}} d(\star_{\mathfrak{g}} - \star_{\bar{\mathfrak{g}}}) df^\alpha]\, z^\alpha \\ &= \sum_\alpha \int_{\mathcal{M}} [\star_{\bar{\mathfrak{g}}} \delta(I - \star_{\bar{\mathfrak{g}}}{}^{-1} \star_{\mathfrak{g}}) df^\alpha]\, z^\alpha \\ &= \sum_\alpha \int_{\mathcal{M}} (\star_{\bar{\mathfrak{g}}} - \star_{\mathfrak{g}}) df^\alpha \wedge dz^\alpha \\ &\le \left(\int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \right)^{1/2} \left(\int_{\mathcal{M}} |(\star_{\mathfrak{g}} - \star_{\bar{\mathfrak{g}}}) df|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \right)^{1/2},\end{aligned}$$

namely,

$$\int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \le \int_{\mathcal{M}} |(\star_{\mathfrak{g}} - \star_{\bar{\mathfrak{g}}}) df|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}.$$

In the above, we used sequentially the integration by parts identity

$$\int d\omega \wedge \star\eta = \int \omega \wedge \star\delta\eta, \tag{3.4}$$

which is valid if the form ω vanishes on $\partial\mathcal{M}$, along with the fact that $\star_{\bar{\mathfrak{g}}}\star_{\bar{\mathfrak{g}}} = (-1)^{k(n-k)}$ for k-forms, the fact that $\Delta_{\mathrm{dR}} z = \Delta_{\mathrm{dR}} f$, eq. (3.3), once again (3.4), and the Cauchy–Schwarz inequality. By the coercivity condition (2.4),

$$\int_{\mathcal{M}} |dz|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}} \le \frac{1}{c_2} E[f],$$

which together with (3.2) yields

$$E[f] \ge \frac{c_1 c_2}{2(c_1 + c_2)} E_{\mathrm{quad}}[w].$$

This implies that we can focus henceforth our analysis on deriving a lower bound for

$$E^0_{\mathrm{quad}} = \inf\{E_{\mathrm{quad}}[w] : w : \mathcal{M} \to \mathbb{R}^n,\ \Delta_{\mathrm{B}} w^\alpha = 0\}, \tag{3.5}$$

as $E^0 \ge c_1 c_2 / 2(c_1 + c_2)\, E^0_{\mathrm{quad}}$.

4. A lower bound for $E^0_{\mathbf{quad}}$

Let χ be a smooth, non-negative, compactly supported test function, and consider

$$I = \int_{\mathcal{M}} \langle dw - \bar{\mathfrak{q}}, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \Delta_{\rm B}\chi \,\mathrm{vol}_{\bar{\mathfrak{g}}}, \quad \Delta_{\rm B} w^\alpha = 0.$$

Using the self-adjointness of the Laplace–Beltrami operator,

$$I = \int_{\mathcal{M}} \Delta_{\rm B} \langle dw, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}}.$$

At this stage it is useful to write the Laplace–Beltrami operator in suffix notation, $\Delta_{\rm B} = \nabla^k \nabla_k$. Using both assertions of Proposition 2.1,

$$\begin{aligned}
I &= \int_{\mathcal{M}} \nabla^k \langle \nabla_k dw, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} \\
&= \int_{\mathcal{M}} \langle \Delta_{\rm B} dw, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} + \int_{\mathcal{M}} \langle \nabla_k dw, \nabla^k \bar{\mathfrak{q}},\, \rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} \\
&= \int_{\mathcal{M}} \langle \Delta_{\rm B} dw, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} - \int_{\mathcal{M}} \langle dw, \Delta_{\rm B} \bar{\mathfrak{q}},\, \rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} \\
&\geq \int_{\mathcal{M}} \langle \Delta_{\rm B} dw, \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}}.
\end{aligned}$$

Note that in the last passage we explicitly used the fact that χ is non-negative. We then observe that $\Delta_{\rm B} w^\alpha = 0$ implies that

$$\Delta_{\rm B} d_i w^\alpha = \nabla^k \nabla_i d_k w^\alpha = (\mathfrak{Ric}_{\bar{\mathfrak{g}}})_i^k d_k w^\alpha,$$

thus

$$\begin{aligned}
I &\geq \sum_\alpha \int_{\mathcal{M}} \mathfrak{Ric}_{\bar{\mathfrak{g}}}(dw^\alpha, \bar{\mathfrak{q}}^\alpha)\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} \\
&= \sum_\alpha \int_{\mathcal{M}} \mathfrak{Ric}_{\bar{\mathfrak{g}}}(dw^\alpha - \bar{\mathfrak{q}}^\alpha, \bar{\mathfrak{q}}^\alpha)\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}} + \int_{\mathcal{M}} \mathfrak{s}_{\bar{\mathfrak{g}}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}},
\end{aligned}$$

where we used the fact that

$$\sum_\alpha \mathfrak{Ric}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{q}}^\alpha, \bar{\mathfrak{q}}^\alpha) = \mathfrak{s}_{\bar{\mathfrak{g}}}.$$

Rearranging terms,

$$\int_{\mathcal{M}} \langle dw - \bar{\mathfrak{q}}, \Delta_{\rm B}\chi\, \bar{\mathfrak{q}} - \chi\, \mathfrak{Ric}_{\bar{\mathfrak{g}}} \bar{\mathfrak{q}} \rangle_{\bar{\mathfrak{g}},\mathfrak{e}} \,\mathrm{vol}_{\bar{\mathfrak{g}}} \geq \int_{\mathcal{M}} \mathfrak{s}_{\bar{\mathfrak{g}}}\, \chi \,\mathrm{vol}_{\bar{\mathfrak{g}}}.$$

Applying the Cauchy–Schwarz inequality on the left-hand side,

$$\left(\int_{\mathcal{M}} |dw - \bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}\right)^{1/2} \left(\int_{\mathcal{M}} |\Delta_{\mathrm{B}}\chi - \chi \mathfrak{s}_{\bar{\mathfrak{g}}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}\right)^{1/2} \geq \int_{\mathcal{M}} \mathfrak{s}_{\bar{\mathfrak{g}}}\, \chi \operatorname{vol}_{\bar{\mathfrak{g}}}.$$

For positive scalar curvature we may square this inequality and obtain

$$E_{\text{quad}}[w] \geq \frac{\left(\int_{\mathcal{M}} \mathfrak{s}_{\bar{\mathfrak{g}}}\, \chi \operatorname{vol}_{\bar{\mathfrak{g}}}\right)^2}{\int_{\mathcal{M}} |\Delta_{\mathrm{B}}\chi - \chi \mathfrak{s}_{\bar{\mathfrak{g}}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \operatorname{vol}_{\bar{\mathfrak{g}}}}. \tag{4.1}$$

Thus, E^0_{quad} is bounded from below by the right hand side of (4.1) for any choice of test function χ, and therefore by the supremum over all choices of χ.

To summarize:

Theorem 4.1: *Let $W(df, x)$ satisfy the coercivity conditions* (1.5) *and* (2.4). *Then,*

$$E^0 \geq \frac{c_1 c_2}{2(c_1 + c_2)} \frac{\left(\int_{\mathcal{M}} \mathfrak{s}_{\bar{\mathfrak{g}}}\, \chi\, vol_{\bar{\mathfrak{g}}}\right)^2}{\int_{\mathcal{M}} |\Delta_B\chi - \chi \mathfrak{s}_{\bar{\mathfrak{g}}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}}\, vol_{\bar{\mathfrak{g}}}}$$

for any non-negative test function $\chi : \mathcal{M} \to \mathbb{R}$, provided that the integral in the numerator is positive.

5. The case $n = 2$

In this section we consider the case of $n = 2$, and generalize the result of the previous section to surfaces of arbitrary Gaussian curvature. The notable difference between dimension $n = 2$ and higher dimensions is that in two dimensions harmonicity is closely related to analyticity. Thus, harmonic functions in two dimensions satisfy properties that are specific to the dimensionality, whence our inability to generalize these results for $n > 2$.

Two-dimensional Riemannian manifolds are conformally flat, namely, there exists a (non-unique) system of coordinates $x = (x^1, x^2)$ for which the reference metric has a representation

$$\bar{\mathfrak{g}}_{ij}(x) = e^{2\lambda(x)}\delta_{ij}, \tag{5.1}$$

where λ is the conformal factor of the metric $\bar{\mathfrak{g}}$ with respect to a Euclidean metric (such a system of coordinates is called isothermal [1]). We also endow $(\mathbb{E}^2, \mathfrak{e})$ with the canonical parametrization; the mapping $f : \mathcal{M} \to \mathbb{E}^2$ is then represented by a mapping

$$f : \Omega \to \mathbb{R}^2,$$

where Ω is the domain of parametrization of the manifold $\mathcal{M}$.

With respect to an isothermal parametrization, the inner-product of two vector fields a^i, b^i is given by

$$\langle a, b\rangle_{\bar{\mathfrak{g}}} = e^{2\lambda}(a^1 b^1 + a^2 b^2), \tag{5.2}$$

where we use superscripts and subscripts to denote components. The inner-product of one-forms a_i, b_i, is given by

$$\langle a, b\rangle_{\bar{\mathfrak{g}}} = e^{-2\lambda}(a_1 b_1 + a_2 b_2),$$

and the Laplace–Beltrami operator is given by

$$\Delta_{\mathrm{B}} = e^{-2\lambda}\left(\frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2}\right).$$

Finally, the Gaussian curvature of the manifold is related to the conformal factor λ via Liouville's formula,

$$K = -2\Delta_{\mathrm{B}}\lambda.$$

Our starting point in bounding E^0_{quad} from below is the identity,

$$\begin{aligned} E_{\mathrm{quad}}[f] &= \int_{\mathcal{M}} |df - \bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} \,\mathrm{vol}_{\bar{\mathfrak{g}}} = \int_{\mathcal{M}} \left(|df|^2_{\bar{\mathfrak{g}},\mathfrak{e}} - 2\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} + n\right) \mathrm{vol}_{\bar{\mathfrak{g}}} \\ &= \int_{\mathcal{M}} \left[n\left(\frac{1}{n}\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} - 1\right)^2 + \left(|df|^2_{\bar{\mathfrak{g}},\mathfrak{e}} - \frac{1}{n}\langle df, \bar{\mathfrak{q}}\rangle^2_{\bar{\mathfrak{g}},\mathfrak{e}}\right)\right] \mathrm{vol}_{\bar{\mathfrak{g}}}. \end{aligned}$$

By the Cauchy–Schwarz inequality and the fact that $|\bar{\mathfrak{q}}|^2_{\bar{\mathfrak{g}},\mathfrak{e}} = n$, the second term is non-negative, hence

$$E_{\mathrm{quad}}[f] \geq n\int_{\mathcal{M}} \left(\frac{1}{n}\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} - 1\right)^2 \mathrm{vol}_{\bar{\mathfrak{g}}}.$$

The next step is to show a property that is specific to two-dimensional surfaces.

Proposition 5.1: *Let $f : (\mathcal{M}, \bar{\mathfrak{g}}) \to (\mathbb{E}^2, \mathfrak{e})$ be harmonic. Then*

$$\Delta_B \log\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = \frac{1}{2}K,$$

where K is the Gaussian curvature of the manifold.

Thus we can lower bound E^0_{quad} by the infimum of

$$\mathcal{E}[\omega] = 2\int_{\mathcal{M}} (e^{\omega} - 1)^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}},$$

over all real-valued functions that belong to the set,

$$\mathcal{F}_K = \{\omega : \mathcal{M} \to \mathbb{R} : \Delta_{\mathrm{B}}\omega = \tfrac{1}{2}K\},$$

namely,

$$E^0_{\mathrm{quad}} \geq \mathcal{E}^0 = \inf\{\mathcal{E}[\omega] : \omega \in \mathcal{F}_K\}. \tag{5.3}$$

The final step is to obtain a lower bound for $\mathcal{E}^0$.

Proof of Proposition 5.1. The square roots $\bar{\mathfrak{q}}$ of $\bar{\mathfrak{g}}$ that have positive orientation are represented by the set of matrices,

$$\bar{\mathfrak{q}} = e^{\lambda}Q, \quad Q \in \mathrm{SO}(2),$$

that is, are parametrized by a single parameter $\theta \in [0, 2\pi)$. It takes a short calculation to show that the quadratic energy density $W_{\mathrm{quad}}(df, x)$ is given by

$$|df - \bar{\mathfrak{q}}|^2 = \min_{\theta\in[0,2\pi)} e^{-2\lambda}\Big[(\partial_1 f^1 - e^{\lambda}\cos\theta)^2 + (\partial_2 f_1 - e^{\lambda}\sin\theta)^2 \\ + (\partial_1 f^2 + e^{\lambda}\sin\theta)^2 + (\partial_2 f^2 - e^{\lambda}\cos\theta)^2\Big].$$

Straightforward algebra yields

$$\cos\theta = \frac{\partial_1 f^1 + \partial_2 f^2}{[(\partial_1 f^1 + \partial_2 f^2)^2 + (\partial_1 f^2 - \partial_2 f^1)^2]^{1/2}},$$
$$\sin\theta = \frac{\partial_2 f^1 - \partial_1 f^2}{[(\partial_1 f^1 + \partial_2 f^2)^2 + (\partial_2 f^1 - \partial_1 f^2)^2]^{1/2}}.$$

Thus,

$$\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = e^{-\lambda}\left[(\partial_1 f^1 + \partial_2 f^2)\cos\theta + (\partial_2 f^1 - \partial_1 f^2)\sin\theta\right] = e^{-\lambda}(a^2+b^2)^{1/2},$$

where

$$a = \partial_1 f^1 + \partial_2 f^2 \quad \text{and} \quad b = \partial_1 f^2 - \partial_2 f^1.$$

It is easy to verify that the harmonicity of f implies that the function $a + ib$ is analytic. Recall that the modulus of any analytic function is log-harmonic, namely, $\Delta_{\mathrm{B}} \log(a^2 + b^2)^{1/2} = 0$. Thus,

$$\Delta_{\mathrm{B}} \log\langle df, \bar{\mathfrak{q}}\rangle_{\bar{\mathfrak{g}},\mathfrak{e}} = -\Delta_{\mathrm{B}}\lambda + \Delta_{\mathrm{B}} \log(a^2+b^2)^{1/2} = \frac{1}{2}K. \qquad \blacksquare$$

Lower bound for $\mathcal{E}^0$. The lower bound $\mathcal{E}^0$ of the functional

$$\mathcal{E}[\omega] = 2\int_{\mathcal{M}} (e^{\omega} - 1)^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}}$$

over functions $\omega \in \mathcal{F}_K$ is a measure of the non-flatness of the manifold $(\mathcal{M}, \bar{\mathfrak{g}})$. Indeed, $\mathcal{E}[\omega]$ is zero if and only if $\omega = 0$ a.e., which is only possible if the manifold is flat. Suppose it were known that functions $\omega \in \mathcal{F}_K$ are uniformly bounded from below on some sub-manifold $\mathcal{M}' \subset \mathcal{M}$. In such case, there would exist a constant $c > 0$ such that

$$(e^{\omega} - 1)^2 \geq c\omega^2, \quad \forall x \in \mathcal{M}'.$$

We would then consider the functional

$$\hat{\mathcal{E}}[\omega] = 2\int_{\mathcal{M}'} \omega^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}}, \quad \omega \in \mathcal{F}_K.$$

Let $\phi : \mathcal{M}' \to \mathbb{R}$ be a test function with support in a compact subset of $\mathcal{M}'$, and such that the product ϕK is non-negative in $\mathcal{M}'$. Then,

$$\int_{\mathcal{M}'} \omega\, \Delta_{\mathrm{B}}\phi \,\mathrm{vol}_{\bar{\mathfrak{g}}} = \int_{\mathcal{M}'} \Delta_{\mathrm{B}}\omega\, \phi \,\mathrm{vol}_{\bar{\mathfrak{g}}} = \frac{1}{2}\int_{\mathcal{M}'} K\,\phi \,\mathrm{vol}_{\bar{\mathfrak{g}}}.$$

By the Cauchy–Schwarz inequality, as the right-hand side is non-negative,

$$\frac{1}{2}\int_{\mathcal{M}'} |K\,\phi| \,\mathrm{vol}_{\bar{\mathfrak{g}}} \leq \left(\int_{\mathcal{M}'} \omega^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^{1/2} \left(\int_{\mathcal{M}'} (\Delta_{\mathrm{B}}\phi)^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^{1/2},$$

i.e.,

$$\hat{\mathcal{E}}[\omega] \geq \frac{1}{2}\frac{\left(\int_{\mathcal{M}'} K\,\phi \,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^2}{\int_{\mathcal{M}'} (\Delta_{\mathrm{B}}\phi)^2 \,\mathrm{vol}_{\bar{\mathfrak{g}}}}, \tag{5.4}$$

the right-hand side being a uniform bound on $\hat{\mathcal{E}}[\omega]$. (Note that the lower bound (5.4) holds for curvatures with arbitrary sign.)

The test function ϕ was chosen such that the sign of $(\Delta_{\mathrm{B}}\omega)\phi$ is everywhere non-negative, for every $\omega \in \mathcal{F}_K$. In general, there is no test function ϕ such that $(\Delta_{\mathrm{B}} e^{\omega})\,\phi$ has a fixed sign, unless K is non-negative. The "quadratic" functional $\hat{\mathcal{E}}$ is used as an auxiliary functional to circumvent this difficulty.

We proceed as follows: We show that for every $\epsilon > 0$ there exists a sub-manifold $\mathcal{M}'$ (which varies with ϵ) and a constant c_ϵ, such that

$$\mathcal{E}[\omega] \leq \epsilon \quad \text{implies that} \quad \omega \geq c_\epsilon, \quad \forall x \in \mathcal{M}',$$

in which case $(e^\omega - 1)^2 \geq e^{2c_\epsilon}\omega^2$, and

$$\mathcal{E}[\omega] \geq e^{2c_\epsilon}\hat{\mathcal{E}}[\omega] \geq \frac{e^{2c_\epsilon}}{2}\frac{\left(\int_{\mathcal{M}'}|K\,\phi|\,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^2}{\int_{\mathcal{M}'}(\Delta_{\mathrm{B}}\phi)^2\,\mathrm{vol}_{\bar{\mathfrak{g}}}},$$

which further implies that for every $\epsilon > 0$,

$$\mathcal{E}^0 \geq \min\left(\epsilon, \frac{e^{2c_\epsilon}}{2}\frac{\left(\int_{\mathcal{M}'}|K\,\phi|\,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^2}{\int_{\mathcal{M}'}(\Delta_{\mathrm{B}}\phi)^2\,\mathrm{vol}_{\bar{\mathfrak{g}}}}\right). \tag{5.5}$$

One can then maximize the right-hand side over all choices of $\epsilon > 0$ and test functions ϕ.

Thus, given $\epsilon > 0$ it remains to choose the set $\mathcal{M}'$ and uniformly bound ω from below in $\mathcal{M}'$ in terms of ϵ. In the rest of this section we will assume a given isothermal parametrization with domain Ω and conformal factor λ. Denote by $B_e(x,r)$ the open *Euclidean* ball of radius $r > 0$ centered at a point $x \in \Omega$. Given a function $\omega \in \mathcal{F}_K$ (which we now view as a function on Ω), we consider the functional

$$A[\omega](x,r) = \omega(x) - \fint_{B_e(x,r)} \omega(y)\,d\ell(y), \tag{5.6}$$

where $d\ell$ denotes the Lebesgue measure, and $\fint$ denotes an average with respect to the domain of integration. The function $A[w]$ quantifies the extent by which ω fails to satisfy the mean-value property in $B_e(x,r)$ (with respect to the parametrization-dependent Lebesgue measure).

Proposition 5.2: *The functional A given by* (5.6) *is constant on $\mathcal{F}_K$, namely, independent of ω. Furthermore, there exists a function $\psi_{x,r}$, independent of ω, such that*

$$A[w](x,r) = \int_{B_e(x,r)} K\psi_{x,r}\,\mathrm{vol}_{\bar{\mathfrak{g}}}. \tag{5.7}$$

Proof. Recall that the Gaussian curvature is given by $K = -2\Delta_{\mathrm{B}}\lambda$, with

$$\Delta_{\mathrm{B}} = e^{-2\lambda}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

hence the function $\omega + \lambda$ satisfies $\Delta_{\mathrm{B}}(\omega + \lambda) = 0$, i.e, is Harmonic with respect to the Euclidean geometry. By the mean-value property,

$$A[\omega](x,r) = \fint_{B_e(x,r)} \lambda(y)\,d\ell(y) - \lambda(x)$$

is, as claimed, independent of ω. The second part of the proof, which relates the extent by which a function fails to satisfy the mean-value property to its Laplacian, is classical, and can be found, for example, in Evans [4], p. 26. ∎

Let $r > 0$, and let Ω_r be a compact subset of Ω, with a Euclidean distance $\operatorname{dist}(\partial\Omega_r, \partial\Omega) > r$.

Proposition 5.3: *Let $y \in \Omega_r$ and let $B_e(x, r/2) \subset B_e(y, r) \subset \Omega$. Then there exist constants $a(r), b(r)$, such that for every $\omega \in \mathcal{F}_K$,*

$$\omega(y) - \frac{1}{4}\omega(x) \le a(r)\sqrt{\mathcal{E}[\omega]} + b(r). \tag{5.8}$$

Proof. By the definition of $A(y, r)$ and $A(x, r/2)$, we have

$$\begin{aligned} \omega(y) - A(y,r) &= \frac{1}{\pi r^2} \int_{B(y,r)} \omega \, d\ell, \\ \omega(x) - A(x,r/2) &= \frac{4}{\pi r^2} \int_{B(x,r/2)} \omega \, d\ell. \end{aligned}$$

Using the identity $d\ell = e^{-2\lambda}\mathrm{vol}_{\bar{\mathfrak{g}}}$, the inequality $t \le e^t - 1$ and the Cauchy–Schwarz inequality,

$$\begin{aligned} \pi r^2 \left[\omega(y) - A(y,r)\right] - \frac{\pi r^2}{4}\left[\omega(x) - A(x,r/2)\right] &= \int_{B(y,r)\setminus B(x,r/2)} \omega \, d\ell \\ &= \int_{B(y,r)\setminus B(x,r/2)} \omega e^{-2\lambda} \, \mathrm{vol}_{\bar{\mathfrak{g}}} \\ &\le \int_{B(y,r)\setminus B(x,r/2)} (e^{\omega} - 1)\, e^{-2\lambda} \, \mathrm{vol}_{\bar{\mathfrak{g}}} \\ &\le \left(\int_{B_e(y,r)} e^{-4\lambda} \, \mathrm{vol}_{\bar{\mathfrak{g}}}\right)^{1/2} \sqrt{\mathcal{E}(\omega)}. \end{aligned}$$

or

$$\omega(y) - \frac{1}{4}\omega(x) \le \frac{1}{\pi r^2}\left(\int_{B_e(y,r)} e^{-4\lambda} \, \mathrm{vol}_{\bar{\mathfrak{g}}}\right)^{1/2} \sqrt{\mathcal{E}(\omega)} + \left[A(y,r) - \frac{1}{4}A(x,r/2)\right],$$

which implies (5.8) with $a(r), b(r)$ given by

$$\begin{aligned} a(r) &= \max_{y\in\Omega_r} \frac{1}{\pi r^2}\left(\int_{B_e(y,r)} e^{-4\lambda}\,\mathrm{vol}_{\bar{\mathfrak{g}}}\right)^{1/2}, \\ b(r) &= \max_{x\in\Omega_r}\int_{B(x,r)} K(z)\psi_{x,r}(z)\,\mathrm{vol}_{\bar{\mathfrak{g}}}(z) \\ &\quad - \frac{1}{4}\min_{x\in\Omega_{r/2}}\int_{B(x,r/2)} K(z)\psi_{x,r}(z)\,\mathrm{vol}_{\bar{\mathfrak{g}}}(z). \end{aligned} \tag{5.9}$$ ∎

Note that the constants $a(r), b(r)$ are parametrization dependent. This is not surprising as the above statement holds for every pair of points x, y that are sufficiently close with respect to the Euclidean metric, which is parametrization dependent.

Let C_r be defined by $\frac{1}{C_r} = 2\min_{x\in\Omega_r}|B_e(x,r)|$.

Proposition 5.4: *Let $\omega \in \mathcal{F}_K$ satisfy $\mathcal{E}(\omega) \le \epsilon$, where $C_r\epsilon < \frac{1}{4}$. Then for every point $x \in \Omega_r$ there exists a point $y \in B_e(x,r)$ at which*

$$\omega(y) \ge -\frac{1}{4}.$$

Proof. Suppose this were not the case. Then there exists a point $x \in \Omega_r$ such that

$$\omega(y) < -\frac{1}{4} \quad \forall y \in B_e(x,r),$$

which implies that

$$\begin{aligned} \epsilon \ge& 2\int_{\mathcal{M}} (e^{\omega}-1)^2\,\mathrm{vol}_{\bar{\mathfrak{g}}} > 2\int_{B_e(x,r)} \left(e^{-1/4}-1\right)^2\,\mathrm{vol}_{\bar{\mathfrak{g}}} \\ =& 2|B_e(x,r)|\left(e^{-1/4}-1\right)^2. \end{aligned}$$

Dividing both sides by $2|B_e(x,r)|$ and taking square roots yields

$$1-e^{-1/4} < \sqrt{\frac{\epsilon}{2|B_e(x,r)|}} \le \sqrt{C_r\epsilon} < \frac{1}{2},$$

which is a contradiction. ∎

Given a point $x \in \Omega_{3r/2}$, then (5.8) holds for every $y \in B(x, r/2)$. Applying Proposition 5.3, we obtain the following result:

COROLLARY 5.1: *Let $\omega \in \mathcal{F}_K$ satisfy $\mathcal{E}(\omega) \leq \epsilon$, and suppose that $C_r\epsilon < \frac{1}{4}$. Then*

$$\omega(x) \geq -4a(r)\sqrt{\epsilon} - 4b(r) - 1 \tag{5.10}$$

for every $x \in \Omega_{3r/2}$, and $a(r), b(r)$ are given by (5.9).

We have thus reached the goal of uniformly bounding from below functions in $\mathcal{F}_K$ on some subset of $\mathcal{M}$ in terms of ϵ, given that $\mathcal{E}[\omega] \leq \epsilon$. Inequality (5.10) holds for any choice of parametrization λ, provided that the Euclidean distance of x from the boundary of $\mathcal{M}$ is greater than $3r/2$. In principle, one can maximize the right-hand side with respect to all choices of parametrizations for which x belongs to $\Omega_{3r/2}$.

6. Discussion

We derived a lower bound on a measure of deformation associated with the immersion of a non-flat n-dimensional Riemannian manifold in the n-dimensional Euclidean space. This problem is motivated by issues that originate from the theory of non-linear elasticity, but is of interest also as a purely geometric problem.

For dimensions larger than two, a positive lower bound for the deformation is obtained only if the scalar curvature of the manifold is positive (more precisely, if the scalar curvature is positive on a sub-domain of $\mathcal{M}$). A positive scalar curvature implies that the volume of a geodesic ball in $(\mathcal{M}, \bar{\mathfrak{g}})$ is smaller that its Euclidean counterpart. Thus, low deformation maps of $(\mathcal{M}, \bar{\mathfrak{g}})$ into Euclidean space are dominated by stretching, rather than contraction. The qualitative difference between positive and negative scalar curvature in our problem is due to the fact that the quadratic energy density W_{quad} is not symmetric with respect to stretching and contraction; it diverges at large stretching, but remains bounded for large contractions. Yet, it was proved in [9] that the energy has a positive lower bound regardless of the sign of the curvature, hence the derivation of an explicit lower bound in the general case remains an open issue.

For surfaces where the Gaussian curvature K associated with the metric $\bar{\mathfrak{g}}$ changes sign, we obtain a low energy bound as the test function ϕ has to change sign accordingly, and as a result the numerical prefactor obtained in Section 5 is small (it is inversely proportional to the norm of the Laplacian of ϕ). One may wonder whether a more careful analysis could not yield a lower energy bound of the form

$$E^0 \geq C \int_{\mathcal{M}'} |K| \operatorname{vol}_{\bar{\mathfrak{g}}},$$

where the constant C is independent of the curvature. The following theorem shows that changes in sign of the Gaussian curvature have an inevitable effect on the energy bound:

Theorem 6.1: *Let $R > 0$ be given. For every $\epsilon > 0$ there exists a Riemannian disc $(\mathcal{M}, \bar{\mathfrak{g}})$ of radius R with Gaussian curvature satisfying $|K| = 1$, and a configuration $f : \mathcal{M} \to \mathbb{R}^2$ such that*

$$E_{quad}[f] < \epsilon.$$

In other words, we can generate discs that can be immersed in the plane at arbitrarily small energetic cost, even though the Gaussian curvature is everywhere non-small in absolute value.

Proof. Consider a disc with domain of parametrization

$$\Omega = [0, R] \times [0, 2\pi),$$

i.e., with polar coordinates (r, θ), and a metric that only depends on the radial coordinate,

$$\bar{\mathfrak{g}}(r) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2(r) \end{pmatrix},$$

with $\phi(r) > 0$ to be determined a posteriori. This two-dimensional Riemannian manifold is a metric disc of radius R, with Gaussian curvature

$$K = -\frac{\phi''}{\phi}.$$

The manifold is elliptic, with $K = 1$, for functions ϕ of the form

$$\phi_+(r) = A \sin(r + B),$$

and hyperbolic, with $K = -1$, for functions ϕ of the form

$$\phi_-(r) = A \sinh(r + B).$$

Take now the map $f : \Omega \to \mathbb{R}^2$,

$$f(r, \theta) = r\,(\cos\theta, \sin\theta).$$

An easy calculation gives

$$W_{\text{quad}}(r, \theta) = (1 - r/\phi(r))^2.$$

To obtain an arbitrarily small energy $E_{\text{quad}}[f]$, we uniformly bound the energy density $W_{\text{quad}}(r,\theta)$. For that, we need ϕ to be uniformly close to the identity map. It is possible to uniformly approximate the identity map by a $W^{2,2}$ function defined piecewise, where each piece is either of the form ϕ_+ or of the form ϕ_-. Thus, there exists a $W^{2,2}(\Omega)$ metric with $|K|=1$ and a map f such that $E_{\text{quad}}[f]<\epsilon$. ∎

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