MEAN SQUARE APPROXIMATION OF A NON-FLAT RIEMANNIAN MANIFOLD BY A FLAT ONE: TWO-DIMENSIONAL CASE

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Abstract. A two-dimensional Riemannian manifold can be immersed isometrically in the plane if and only if its Gaussian curvature vanishes identically, i.e., if it is flat. Given a non-flat surface, we are looking for its best approximation by a flat surface, where the deviation between the given and approximate surfaces is the L^2 distance between the metrics. Questions of that type arise in nonlinear elasticity theory, and in particular, in a recent theory of so-called "incompatible elasticity"; in elasticity, the square of the L^2 distance between the metrics is an elastic energy. In this paper we prove that a non-flat two dimensional surface cannot be arbitrarily well approximated by flat surfaces, and for the case of non-negative curvature, we lower bound the elastic energy by quantities that depend on the geometric properties of the manifold, and notably the Gaussian curvature.

Key words. Riemannian surface, incompatible elasticity

AMS subject classifications.

1. Introduction. Consider the following problem: let $(\mathscr{M},\bar{g})$ be a two-dimensional Riemannian manifold diffeomorphic to a disc; the metric $\bar{g}$ is assumed to be of class C^2. As is well-known, this Riemannian manifold can be immersed isometrically in the Euclidean plane if and only if $\bar{g}$ is flat, that is, if the corresponding Gaussian curvature vanishes identically. If this weren't the case, then it is natural to ask how "close" is $(\mathscr{M},\bar{g})$ to manifolds $(\mathscr{M},g)$, where the metric g is flat. Equivalently, what is the minimum amount of deformation associated with an immersion of $\mathscr{M}$ in the Euclidean plane?

Such optimization problem depends on the choice of distance between manifolds. In this work, the distance is quantified by the mean-square difference between the two metrics g and $\bar{g}$. Specifically, let $F:\mathscr{M}\to\mathbb{R}^2$ be a C^3 diffeomorphism. If $\mathbb{R}^2$ is endowed with the standard Euclidean inner-product, the mapping F induces a metric $g = dF\otimes dF$ on $\mathscr{M}$. We define the *amount of deformation* as the square distance,

$$E_{\bar{g}}[g] = \|g-\bar{g}\|_{\bar{g}}^2, \tag{1.1}$$

where the norm is induced by the inner-product for tensors,

$$\langle h_1,h_2\rangle_{\bar{g}} = \int_{\mathscr{M}} \bar{g}(h_1,h_2)\,d\mu,$$

and μ is the surface measure induced by $\bar{g}$.

This differential geometric problem arises from a recent theory of so-called *incompatible elasticity* [4, 3]. Incompatible elasticity is a physical model for elastic bodies that do not possess a stress-free rest configuration, even when relieved from any external force or constraint. This is in contrast with standard elasticity theories whose premise is that an elastic body has, in the absence of external forces, a stress-free rest configuration. A lack of stress-free configuration occurs in situations where the body has an intrinsic structure—a metric—which is not isometrically immersible in the ambient Euclidean space. Thus, every actual configuration will necessarily induce

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a metric that differs from the intrinsic *reference metric*, i.e., will involve a *strain*, which in turn will induce stresses. Expression (1.1) is the elastic energy, expressed as a function of the actual metric g. For this energy to be zero, the metric g must almost-everywhere coincide with the reference metric $\bar{g}$. This, however, cannot happen if $\bar{g}$ is non-flat. The premise of the theory is that the configuration that the body assumes at equilibrium in the absence of external forces is one that minimizes the elastic energy (1.1).

Note that expression (1.1) is not symmetric with respect to $g, \bar{g}$. The reason for this asymmetry is that we view $\bar{g}$ as the intrinsic geometry of the manifold, and therefore define all norms with respect to it.

A number of questions arise from the outset, regarding the well-posedness of this new theory. First, it is assumed that an energy minimizer exists, that is, that there exists one (or more) configurations whose metric minimizes (1.1). Even before that, there is an implicit assumption that a non-immersible metric cannot be approached (in the above sense) by immersible metrics, i.e., that it is not possible to construct a sequence of immersions with induced metrics g_n, such that

$$\lim_{n\to\infty} E_{\bar{g}}[g_n] = 0.$$

This last statement, a forteriori the existence of an energy minimizer, is far from trivial. The energy $E_{\bar{g}}[g]$ is a weighted L^2 distance between metrics, but the condition that g be immersible, i.e., flat, is a constraint on second derivatives of g (the vanishing of the Gaussian curvature). As a rule, a sequence of functions could converge in L^2, while the corresponding sequence of second derivatives behaves very differently than the second derivative of the limit.

Physical bodies are three-dimensional, and indeed, incompatible elasticity was formulated in three dimensions. Note that the setting leading to (1.1) can be formulated in any dimension. In this paper we focus on the two-dimensional case; this corresponds to the problem of *plane stress* considered in [4, 3]. A notable difference between two and three dimensions is that every two-dimensional Riemannian manifold can be identified as a *Riemann surface* [1], i.e., it carries a conformal structure. Taking advantage of this analytical structure, we obtain the following results: (i) The energy (1.1) is bounded away from zero for every non-flat $\bar{g}$. (ii) If the Gaussian curvature of $\bar{g}$ is non-negative, then we lower bound the elastic energy in terms of geometric properties of the surface.

The role of the sign of the Gaussian curvature is in fact indicative of shortcomings of the elastic model derived in [4]. This is addressed in the Discussion, where we propose other models, which although physically more plausible, seem less analytically tractable from the point of view of analysis.

2. Preliminary analysis. The above optimization problem was presented in an intrinsic, parametrization-independent formulation. Much simplification is gained by the choice of a convenient parametrization. Two-dimensional simply-connected surfaces admit an *isothermal parametrization*, that is, there exists a surface parametrization $\phi : \mathscr{M} \to \Omega$, for which the metric induced by $\bar{g}$ on Ω, i.e., $\bar{g} \circ (d\phi^{-1} \otimes d\phi^{-1})$ has the matrix representation

$$[\bar{g}_{ij}](x, y) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix}, \tag{2.1}$$

where $\lambda = \lambda(x, y)$ is a C^2-function. In this parametrization, the mapping $F : \mathscr{M} \to \mathbb{R}^2$ is represented by a C^3 diffeomorphism $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$, given by $f = F \circ \phi^{-1}$, and the metric g is represented by a matrix

$$[g_{ij}](x,y) = \sum_{m=1}^{2} \partial_i f_m \, \partial_j f_m.$$

Henceforth, we identify the metrics $\bar{g}$ and g with their parametric representations $[\bar{g}_{ij}]$ and $[g_{ij}]$.

The condition that f be a diffeomorphism implies that $\det \nabla f$ has fixed sign; without loss of generality we take this sign to be positive. With this restriction, the matrix g determines f up to a rigid transformation. Since the energy functional (1.1) is invariant under rigid transformations, we may as well assume that $E_{\bar{g}}[g] = E_{\bar{g}}[f]$. Finally, the energy (1.1) takes the explicit form

$$E_{\bar{g}}[f] = \int_\Omega \sum_{i,j=1}^{2} \left(e^{-\lambda} \sum_{m=1}^{2} \partial_i f_m \, \partial_j f_m - \delta_{ij} \right)^2 d\mu, \tag{2.2}$$

where $d\mu = e^\lambda \, dxdy$.

The Gaussian curvature of an isothermal metric (2.1) is given by $K[\bar{g}] = -\frac{1}{2}\Delta\lambda$, where Δ is the Laplace-Beltrami operator,

$$\Delta = e^{-\lambda} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Thus, the condition that $\bar{g}$ be non-flat correspond to the condition that λ be non-harmonic in Ω.

We denote by $\mathscr{F}$ the set of C^3 diffeomorphisms $f : \Omega \to \mathbb{R}^2$ such that $\det \nabla f$ has fixed positive sign. The solution to the elastic problem corresponds to finding a mapping $f \in \mathscr{F}$ which minimizes the energy functional (2.2). In principle, C^3 diffeomorphisms are a too restrictive, and we should be content with homeomorphisms f for which the energy (2.2) is finite, namely, within the Sobolev space $W^{1,4}$. For the sake of the current analysis, it is sufficient to restrict the homeomorphisms to any function space that is dense in $W^{1,4}$, for example, C^3 functions.

We now take advantage of the conformal structure of the parametrization, and complexify the representation by setting $z = x + iy$ and $f = f_1 + i\, f_2$. Likewise, we define the standard complex derivatives,

$$f_z = \frac{1}{2}(f_x - i f_y) \qquad \text{and} \qquad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

The condition that f be orientation preserving is

$$|f_{\bar{z}}| < |f_z| \qquad \text{everywhere in } \Omega. \tag{2.3}$$

The matrix g takes the form

$$g = \begin{pmatrix} |f_z + f_{\bar{z}}|^2 & 2\Im(f_z \overline{f_{\bar{z}}}) \\ 2\Im(f_z \overline{f_{\bar{z}}}) & |f_z - f_{\bar{z}}|^2 \end{pmatrix},$$

and upon substitution into (2.2), we obtain

$$E_{\bar{g}}[f] = 2\int_\Omega \left(e^{-\lambda}|f_z|^2 + e^{-\lambda}|f_{\bar{z}}|^2 - 1 \right)^2 d\mu + 8\int_\Omega e^{-2\lambda}|f_z|^2|f_{\bar{z}}|^2 \, d\mu. \tag{2.4}$$

The fact that $E_{\bar{g}}[f] > 0$ for all $f \in \mathscr{F}$ can be easily seen from (2.4), which is a sum of two non-negative terms. For the energy to vanish, both terms have to be zero, and hence

$$f_{\bar{z}} = 0 \qquad \text{and} \qquad |f_z|^2 = e^{\lambda} \qquad \text{everywhere in } \Omega.$$

Thus, f is holomorphic and $\lambda = \log |f_z|^2$. Since the logarithm of the modulus of an holomorphic function is harmonic, this cannot hold, unless λ is harmonic (which contradicts our assumption on λ).

3. A lower bound. In the present and next sections we prove that the energy functional $E_{\bar{g}}[f]$ is uniformly bounded away from zero for all $f \in \mathscr{F}$. We do so in several steps: first, we bound the energy from below by a sum of two non-negative terms, one that depend only on f_z, and one that depends only on $f_{\bar{z}}$. For non-flat $\bar{g}$, the two terms cannot vanish simultaneously. We use harmonic analysis to show that if the second term is sufficiently small, then the first term has to exceed a certain bound. A uniform bound is obtained by a balance between the smallness of the second term and the magnitude of the first term. Specifically, denoting the surface area by

$$a = \int_{\Omega} d\mu,$$

we show in this section that the energy is bounded from below in terms of

$$d = \inf \left\{ \frac{1}{a} \int_{\Omega} \left(e^{-\lambda/2} |w| - 1 \right)^2 d\mu : w_{\bar{z}} = 0 \right\}, \tag{3.1}$$

which quantifies an L^2 distance between $e^{\lambda/2}$ and the set of analytic functions. In the next section we show that d is strictly positive, and obtain an explicit bound in the case where the Gaussian curvature is non-negative.

Applying the orientation preservation condition (2.3) on the second term of (2.4), we obtain

$$E_{\bar{g}}[f] \geq 2 \int_{\Omega} \left(e^{-\lambda} |f_z|^2 + e^{-\lambda} |f_{\bar{z}}|^2 - 1 \right)^2 d\mu + 8 \int_{\Omega} e^{-2\lambda} |f_{\bar{z}}|^4 \, d\mu.$$

Using the inequality $2(a+b)^2 + 8b^2 \geq a^2 + 6b^2$ we get

$$E_{\bar{g}}[f] \geq \int_{\Omega} \left(e^{-\lambda} |f_z|^2 - 1 \right)^2 d\mu + 6 \int_{\Omega} e^{-2\lambda} |f_{\bar{z}}|^4 \, d\mu. \tag{3.2}$$

The presence of L^4-like norms is inconvenient. Applying the Cauchy-Schwarz inequality gives

$$\int_{\Omega} e^{-2\lambda} |f_{\bar{z}}|^4 \, d\mu \geq \frac{1}{a} \left(\int_{\Omega} e^{-\lambda} |f_{\bar{z}}|^2 \, d\mu \right)^2.$$

Note also the pointwise bound

$$\left(e^{-\lambda} |f_z|^2 - 1 \right)^2 \geq \left(e^{-\lambda/2} |f_z| - 1 \right)^2.$$

Substituting these two inequalities into (3.2) gives the following lower bound

$$E_{\bar g}[f] \geq \int_\Omega \left(e^{-\lambda/2}|f_z| - 1\right)^2 d\mu + \frac{6}{a}\left(\int_\Omega e^{-\lambda}|f_{\bar z}|^2\, d\mu\right)^2. \tag{3.3}$$

THEOREM 3.1. *There exists a numerical constant C, such that for every $f \in \mathscr{F}$,*

$$\frac{E_{\bar g}[f]}{a} \geq C\, \min(d^4, 1)$$

where d is defined by (3.1).

Proof. We start with some preliminaries. First, note that

$$\overline{f_z} = \bar f_{\bar z} \qquad \text{and} \qquad \Delta f = e^{-\lambda} f_{z\bar z}. \tag{3.4}$$

Every function f has a unique decomposition,

$$f = u + v,$$

where u is harmonic and equals to f on $\partial\Omega$. It follows that v vanishes on the boundary. Integrating by parts, and using (3.4) and the fact that u is harmonic, gives

$$\int_\Omega e^{-\lambda} u_z \overline{v_z}\, d\mu = \int_\Omega e^{-\lambda} u_{\bar z} \overline{v_{\bar z}}\, d\mu = 0,$$

which implies that

$$\int_\Omega e^{-\lambda}|f_z|^2\, d\mu = \int_\Omega e^{-\lambda}\left(|u_z|^2 + |v_z|^2\right) d\mu. \tag{3.5}$$

Also, integrating twice by parts and using (3.4), we get that

$$\int_\Omega e^{-\lambda}|v_z|^2\, d\mu = \int_\Omega e^{-\lambda} v_z \bar v_{\bar z}\, d\mu = \int_\Omega e^{-\lambda} v_{\bar z} \bar v_z\, d\mu = \int_\Omega e^{-\lambda}|v_{\bar z}|^2\, d\mu.$$

Using the orthogonality (3.5) of u and v, the first term in the energy bound (3.3) can be further bounded from below as follows,

$$\begin{aligned}
\int_\Omega \left(e^{-\lambda/2}|f_z| - 1\right)^2 d\mu &= \int_\Omega \left(e^{-\lambda}|f_z|^2 - 2e^{-\lambda/2}|f_z| + 1\right) d\mu \\
&\geq \int_\Omega \left(e^{-\lambda}|u_z|^2 + e^{-\lambda}|v_z|^2 - 2e^{-\lambda/2}\left(|u_z| + |v_z|\right) + 1\right) d\mu \\
&\geq \int_\Omega \left(e^{-\lambda/2}|u_z| - 1\right)^2 d\mu + \int_\Omega e^{-\lambda}|v_z|^2\, d\mu - 2a^{1/2}\left(\int_\Omega e^{-\lambda}|v_z|^2\, d\mu\right)^{1/2},
\end{aligned} \tag{3.6}$$

where we have also used the triangle and the Cauchy-Schwarz inequalities. The second term in (3.3) can be lower bounded as follows

$$\frac{6}{a}\left(\int_\Omega e^{-\lambda}|f_{\bar z}|^2\, d\mu\right)^2 \geq \frac{6}{a}\left(\int_\Omega e^{-\lambda}|v_{\bar z}|^2\, d\mu\right)^2 = \frac{6}{a}\left(\int_\Omega e^{-\lambda}|v_z|^2\, d\mu\right)^2.$$

Substituting these inequalities back into (3.3), and noting that the left-hand side of (3.6) is non-negative, yields

$$\frac{E_{\bar{g}}[f]}{a} \geq \max\left(\frac{1}{a}\int_{\Omega}(e^{-\lambda/2}|u_z|-1)^2\,d\mu + \rho^2 - 2\rho, 0\right) + 6\rho^4, \tag{3.7}$$

where

$$\rho^2 = \rho^2(v) = \frac{1}{a}\int_{\Omega} e^{-\lambda}|v_z|^2\,d\mu.$$

The energy per unit area can now be bounded from below by the infimum of (3.7) over all *independent* choices of u and v, with for only constraint that u be harmonic. Denoting $w = u_z$, we observe that w analytic if and only if u is harmonic. Thus, minimizing (3.7) with respect to the choice of w,

$$\frac{E_{\bar{g}}[f]}{a} \geq \max\left(d + \rho^2 - 2\rho, 0\right) + 6\rho^4,$$

where d is given by (3.1).

It remains to minimize the right hand side with respect to the choice of v, or equivalently, with respect to the choice of ρ. For sufficiently small values of d, the minimum is obtained by taking ρ to be the smallest number for which the first term vanishes, i.e., $\rho = 1 - \sqrt{1-d}$. Then, $E_{\bar{g}}[f]/a \geq 6\rho^4$. For values of d above some threshold, the first term is positive, and the minimum is obtained by minimizing the polynomial $6\rho^4 + \rho^2 - 2\rho$. Omitting straightforward technicalities,

$$E[f] \geq a\phi(d),$$

where

$$\phi(d) = \begin{cases} d - 0.491 & 0.607 \leq d \\ 6(1-\sqrt{1-d})^4 & 0 \leq d \leq 0.607. \end{cases}$$

It is easily verified that, as claimed, there exists a constant C, such that $\phi(d) \geq C\min(d^4, 1)$. ∎

The definition (3.1) of d involves the isothermal parameter λ, and may therefore seem parametrization-dependent. Note, however, that λ is uniquely defined up to an additive harmonic function. Thus, $e^{-\lambda}$ is uniquely defined up to a multiplicative log-harmonic function. Since every log-harmonic function is a modulus of an analytic function, it follows that d is independent of parametrization.

The minimal energy can be upper bounded by the infimum of (2.4) over the set of holomorphic functions,

$$\inf_{\mathscr{F}} E_{\bar{g}}[f]/a \leq 2\inf\left\{\frac{1}{a}\int_{\Omega}\left(e^{-\lambda/2}|w|-1\right)^2 d\mu : w \text{ is holomorphic}\right\}.$$

As the set of holomorphic functions is not dense in the space of analytic functions, the above term may take a value strictly larger than d.

4. Approximation of log-harmonic functions. In this section we prove that d, given by (3.1), is strictly positive. For elliptic surfaces we derive a lower bound that only depends on the intrinsic geometry of the manifold $(\mathscr{M}, \bar{g})$ (the Gaussian curvature).

THEOREM 4.1. $d > 0$.

Proof. Suppose, by contradiction, that $d = 0$. Then there exists a sequence of analytic functions $w^{(n)}$ such that

$$\lim_{n\to\infty} \int_\Omega \left(e^{-\lambda}|w^{(n)}| - 1\right)^2 \, d\mu = 0.$$

As the sequence $w^{(n)}$ is bounded in L^2, it has a weakly converging subsequence (not relabeled) in L^2, $w^{(n)} \rightharpoonup w$.

Next, let $x \in \Omega$ and let $B(x,r)$ be a ball of radius $r > 0$ contained in Ω. By the mean-value property for analytic functions,

$$w^{(n)}(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} w^{(n)}(y) \, d\mu(y),$$

and as the right-hand side converges as $n \to \infty$, then the sequence $w^{(n)}$ converges pointwise to a function $\hat{w}$, defined by

$$\hat{w}(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} w(y) \, d\mu(y),$$

and the right-hand side is independent of r. Taking the limit $r \to 0$, by the Lebesgue differentiation theorem we obtain that $w(x) = \hat{w}(x)$ a.e, which implies

$$\hat{w}(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \hat{w}(y) \, d\mu(y).$$

It follows that $\hat{w}$ is harmonic in Ω. Finally,

$$\begin{aligned} 0 = \lim_{n\to\infty} \int_\Omega \left(e^{-\lambda/2}|w^{(n)}| - 1\right)^2 \, d\mu &= \int_\Omega \lim_{n\to\infty} \left(e^{-\lambda/2}|w^{(n)}| - 1\right)^2 \, d\mu \\ &= \int_\Omega \left(e^{-\lambda/2}|\hat{w}| - 1\right)^2 \, d\mu, \end{aligned}$$

or

$$\lambda = -2 \log |\hat{w}|,$$

which contradicts our assumption on λ. □

THEOREM 4.2. *Suppose that the Gaussian curvature of the manifold, $K = -\frac{1}{2}\Delta\lambda$, is non-negative. Let $G \subset \Omega$ be a compact subset. Then,*

$$d \geq \min \left\{ \frac{(1/2a)\left(\int_G K \, d\mu\right)^2}{C(G) + \int_\Omega K^2 \, d\mu}, 1 \right\},$$

where $C(G)$ is a constant that tends to infinity as $G \to \Omega$.

Proof. Let w be analytic in Ω. As is well known, if w has an infinite number of zeros in G, then it is identically zero in Ω. In which case,

$$\frac{1}{a} \int_\Omega \left(e^{-\lambda/2}|w| - 1\right)^2 \, d\mu = 1.$$

We now assume that w has a finite number of zeros in every compact subset of Ω. Then it is convenient to consider the function

$$h = \log |w|,$$

defined for every point $x \in \Omega$ for which $w(x) \neq 0$. We observe that w is analytic if and only if h is harmonic. By defining $h(x) = -\infty$ whenever $w(x) = 0$, h is a continuous function in the extended sense. Let $\phi : \Omega \to \mathbb{R}$ be a smooth non-negative function such that (i) $\phi = 1$ in G, (ii) $0 \leq \phi \leq 1$ in Ω and (iii) both ϕ and $\nabla\phi$ vanish on $\partial\Omega$. Integrating twice by parts we obtain

$$\int_\Omega \Delta\phi \left(e^{h-\lambda/2} - 1\right) d\mu = \int_\Omega \phi \Delta e^{h-\lambda/2} d\mu = \int_\Omega \phi \left(e^{-\lambda/2} |\nabla(h-\lambda)|^2 + K\right) e^{h-\lambda/2}\, d\mu. \tag{4.1}$$

Thus,

$$\begin{aligned}
\int_\Omega \Delta\phi \left(e^{h-\lambda/2} - 1\right) d\mu &\geq \int_\Omega \phi K e^{h-\lambda/2}\, d\mu \\
&= \int_\Omega \phi K \left(e^{h-\lambda/2} - 1\right) d\mu + \int_\Omega \phi K\, d\mu \\
&\geq \int_\Omega \phi K \left(e^{h-\lambda/2} - 1\right) d\mu + \int_G K\, d\mu,
\end{aligned}$$

where we have used the properties of ϕ and the non-negativity of K. It follows that

$$\int_\Omega (\Delta\phi - \phi K) \left(e^{h-\lambda/2} - 1\right) d\mu \geq \int_G K\, d\mu.$$

By the Cauchy-Schwartz inequality,

$$\left(\int_\Omega (\Delta\phi - \phi K)^2 d\mu\right)^{1/2} \left(\int_\Omega \left(e^{h-\lambda/2} - 1\right)^2 d\mu\right)^{1/2} \geq \int_G K\, d\mu. \tag{4.2}$$

Squaring and using the non-negativity of K,

$$\frac{1}{a} \int_\Omega \left(e^{h-\lambda/2} - 1\right)^2 d\mu \geq \frac{(1/a) \left(\int_G K\, d\mu\right)^2}{2 \int_\Omega (\Delta\phi)^2\, d\mu + 2 \int_\Omega (\phi K)^2\, d\mu}.$$

Since $0 \leq \phi \leq 1$ and K is non-negative, then

$$\frac{1}{a} \int_\Omega \left(e^{h-\lambda/2} - 1\right)^2 d\mu \geq \frac{(1/2a) \left(\int_G K\, d\mu\right)^2}{C(G) + \int_\Omega K^2\, d\mu},$$

where

$$C(G) = \sup \int_\Omega (\Delta\phi)^2\, d\mu,$$

and the supremum is over all functions ϕ that satisfy the required properties. Since the left-hand side of the above inequality is independent of h we recover the desired result. Note that $C(G)$ tends to infinity as G approaches Ω.
□

The assumption that K be non-negative was explicitly used in the above proof, which cannot be adapted to the case where, for example, K is everywhere negative. Indeed, inequality (4.2) is useless in this case. The fundamental difference between positive and negative Gaussian curvatures is discussed in the next section.

5. Discussion. In this paper we prove that a non-flat two-dimensional Riemannian manifold cannot be approximated, to arbitrary precision, by a flat manifold, where the distance is the mean square difference between the metrics. This problem (as well as its three dimensional counterpart) arises in elasticity theory, where bodies are assumed to have an intrinsic metric, which may be non-flat, whereas any actual immersion in the ambient Euclidean space is necessarily flat. In the physical context, the distance between the metrics is an elastic energy, and the minimization problem is that of finding a static equilibrium that minimizes the energy cost.

In the case where the manifold is elliptic, i.e., the Gaussian curvature is positive, we were able to derive an explicit lower bound for the elastic energy, which relates to the mean value of the Gaussian curvature. The question is why did we fail to obtain a similar result in the case of a hyperbolic surface?

We believe that this failure is not due to some technical shortcoming, but rather reflects an important property of the elastic energy (1.1). The immersion of an elliptic surface in the plane involves mainly elongation, whereas the immersion of a hyperbolic surface in the plane involves mainly contraction. The elastic energy density in (1.1) penalizes differently elongations versus contractions. The energy density remain finite even for extreme contractions, where $\det \nabla f$ tends to zero. On the other hand, it diverges for extreme elongations, where $\det \nabla f$ tends to infinity. In fact, even a sequence of immersions that tends to a point, such that $g \to 0$, involves a finite energetic cost, which from the physical point of view is absurd.

Indeed, quadratic models of the type proposed in [4] are intended to be valid only for (uniformly) small strains, i.e., when g is everywhere close enough to $\bar{g}$. For large strains, it is customary to require the energy density to diverge at the boundaries of the set $\mathscr{F}$, and in particular, as g tends to singularity. Clearly, there are infinitely ways of producing an elastic energy that is quadratic at small strains, and satisfies the physical requirements at large strains. Identifying the right choice is a modeling issue, which furthermore may vary from one material to another.

A closely related issue is that of showing the existence of an energy minimizer. Numerical investigations with the energy (1.1) indicate that for sufficiently non-flat reference metrics $\bar{g}$, minimizing sequences tend to the boundaries of the set $\mathscr{F}$, and no minimizer exists. This model shortcoming is also related to unphysical behavior at large strains (there is a large amount of literature on conditions that ensure the existence of a minimizer, see e.g., [2]).

Two alternative expressions for the elastic energy that might remedy the shortcomings associated with the energy (1.1) are (i) adding to (1.1) a term that measures the deviation of g from $\bar{g}$ with respect to the metric g (or equivalently, measures the distortion of the co-tangent bundle of $\mathscr{M}$),

$$E[g] = E_{\bar{g}}[g] + \int_{\mathscr{M}} g(g - \bar{g}, g - \bar{g})\, d\mu.$$

This correction almost symmetrizes the energy with respect to g and $\bar{g}$ (only the surface measure is in both terms induced by the manifold $(\mathscr{M}, \bar{g})$). (ii) Use a logarithmic energy density, say,

$$E[g] = \| \log(\bar{g}^{-1} g) \|_2^2.$$

Both alternatives exhibit the desired behavior at large strains. However, we were unable to adapt our analysis for these functionals. In particular, the harmonic projection, which was at the heart of our analysis cannot be applied with the modified energy functionals.

As a last comment, we note that our analysis uses explicitly the choice of an isothermal parametrization. Even though every two-dimensional Riemannian surface admits an isothermal parametrization, it is not clear how to repeat our analysis for an arbitrary parametrization. This observation is relevant to the three-dimensional setting where an isothermal parametrization not necessarily exists.

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