Homogenization of edge-dislocations as a weak limit of de-Rham currents

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Abstract We consider the geometric homogenization of edge-dislocations as their number tends to infinity. The material structure is represented by 1-forms and their singular counterparts, de-Rham currents. Isolated dislocations are represented by closed 1-forms with singularities concentrated on submanifolds of co-dimension one (the defect locus), whereas a continuous distribution of dislocations is represented by smooth, non-closed 1-forms. We prove that every smooth distribution of dislocations is a limit, in the sense of weak convergence of currents, of increasingly dense and properly scaled isolated edge-dislocations. We also define a notion of singular torsion-current (associated with isolated dislocations), and prove that the torsion currents converge, in the homogenization limit, to the smooth torsion field which is the continuum measure of the dislocation density.

1 Introduction

Models of dislocations. The study of material defects, and notably dislocations, is a central theme in material science. The modeling of solid bodies, with or without defects, often follows a paradigm in which the elemental object is that of a **body manifold**: solid bodies are modeled as geometric objects—manifolds—and their internal structure is represented by additional structures such as a frame field, a metric or an affine connection. The mechanical properties of the body enter through a **constitutive relation**, whose structure is correlated with the geometric structure of the body.

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There have been two distinct approaches to the modeling of body manifolds with dislocations:

1. **Isolated dislocations**: One starts with a defect-free body, which is either modeled as a subset of Euclidean space or as a perfect lattice[1]. Defects are introduced by Volterra cut-and-weld protocols [1] (see Figure 1) resulting in a locally flat manifold with singularities. The singularities are identified as the defect loci and the presence of dislocations is detected by measuring a non-trivial circulation, known as the **Burgers vector**, along closed paths encircling the defect loci.

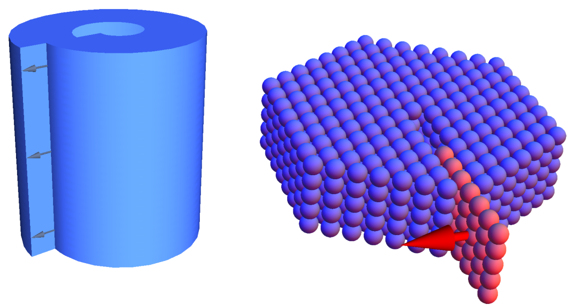


Fig. 1 Left: An edge-dislocation generated by a cut-and-weld protocol in a continuum setting. Right: An edge-dislocation generated by removing a half-plane in a lattice.

2. **Distributed dislocations**: The body is modeled as a smooth manifold endowed with a flat (curvature-free) affine connection. The density of the dislocations is identified with the torsion tensor of the affine connection [2, 3, 4, 5]. If, in addition, one adds a basis of the tangent space at one point, then the affine connection induces a smooth frame field, which is the kinematic model, for example, in [6]. In later literature [7], the continuum model is that of a Weitzenböck manifold, which is a smooth manifold endowed with a Riemannian metric and a metrically-consistent, curvature-free affine connection. Note that a frame field induces an intrinsic metric and a material connection, so that all three descriptions are essentially identical.

Homogenization. A longstanding problem has been to rigorously justify, in the spirit of homogenization theories, the continuum model of distributed dislocations as a dense limit of properly scaled isolated dislocations. In particular, one would like to understand how torsion, which is the continuum measure of the dislocation density, emerges in the homogenization limit.

[1] A perfect lattice may be related to a smooth Euclidean structure by assigning lengths and angles to inter-particle bonds and letting the lattice size tend to infinity with the inter-particle bonds scaled appropriately.

In order to obtain a body manifold endowed with a smooth geometric structure as a limit of body manifolds endowed with localized defects, we must first cast these two seemingly-distinct models into the same framework. One possible approach is to "remove" small neighborhoods of the isolated dislocations. Thus, bodies with isolated and smoothly distributed dislocations are both modeled as smooth Weitzenböck manifolds, where in the former case, the bodies are multiply-connected; see [8, 9] for a homogenization of defects using this approach.

Another possible approach is to account for the localized defects using singular geometric fields, which is the approach used in this work. As described above, the internal structure of a d-dimensional body $\mathcal{M}$ can be modeled by a frame field, $\{e_i\}_{i=1}^d$, or equivalently, by its dual coframe, which is a set of d 1-forms, $\{\vartheta^i\}_{i=1}^d$. Every smooth 1-form $\omega \in \Omega^1(\mathcal{M})$ induces a distribution for the tangent bundle

$$\ker(\omega) \subset T\mathcal{M}, \qquad \ker(\omega)_p = \ker(\omega_p) \leq T_pM.$$

Under certain integrability conditions [10, Chap. 19], $\ker(\omega)$ induces a foliation (or layering) of $\mathcal{M}$ as a union of *Bravais hypersurfaces*, which are tangent to $\ker(\omega)$ at every point. These surfaces represent the infinitesimal atomic/molecular layers composing the body. Henceforth, we call a 1-form inducing a foliation a **layering form**.

Bodies with localized defects are modeled using singular layering forms, which are represented by the distributional counterpart of differential forms—de Rham **currents**. As pointed out by Epstein and Segev [11], even a single layering form may detect the presence of defects. Following [11], we define:

A body with dislocations is a d-dimensional manifold $\mathcal{M}$ endowed with a possibly singular layering form ω on $\mathcal{M}$, viewed as a de-Rham $(d-1)$-current,

$$T_\omega : \Omega_c^{d-1}(\mathcal{M}) \to \mathbb{R}, \qquad T_\omega(\eta) = \int_\mathcal{M} \omega \wedge \eta, \tag{1}$$

where $\Omega_c^k(\mathcal{M})$ is the space of smooth, compactly-supported k-forms on $\mathcal{M}$.

The defect density associated with ω is represented by the **boundary current** ∂T_ω, which is defined in the next section.

A layering form ω models a density of Bravais surfaces. Given a vector $v_p \in T_p\mathcal{M}$, $\omega(v_p)$ is interpreted as the signed number of Bravais planes intersecting v_p. For a closed curve $C \subset \mathcal{M}$, the **Burgers scalar**

$$\oint_C \omega$$

is interpreted as the signed number of Bravais hyperplanes intersecting C. In particular if the Burgers scalar along C is non-vanishing, there is a discrepancy in the layering structure, that is, a defect. By Stoke's theorem, the defect density may be identified with the exterior derivative $d\omega$.

Since a defect-free structure is represented by a closed layering form, isolated dislocations are represented by layering forms ω that are closed everywhere except in a set $\Gamma \subset \mathcal{M}$, which we identify as the locus of the dislocations. Moreover, the existence of non-trivial Burgers scalars around Γ implies that ω must be singular at Γ.

To conclude, both isolated and smoothly-distributed dislocations are represented by de-Rham currents; in the smooth case, the currents are induced by smooth layering forms and in the isolated case, by closed forms with singularities. We may now state our main homogenization theorem (see Theorem 6.1 below) in terms of convergence of currents:

Let $\mathcal{M}$ be a compact, orientable two-dimensional surface, possibly with boundary. Let $\omega \in \Omega^1(\mathcal{M})$ be a (generally non-closed) layering form on $\mathcal{M}$. Then, there exist sequences ω_n and Γ_n such that

1. *Γ_n is a finite disjoint union of segments in $\mathcal{M}$ and is bounded away from $\partial\mathcal{M}$.*
2. *ω_n are closed C^1-bounded layering forms on $\mathcal{M} \setminus \Gamma_n$.*
3. *ω_n converge to ω in the sense of currents. That is, $T_{\omega_n} \to T_\omega$ as $n \to \infty$.*

We prove this homogenization theorem in three main steps:

Step I: A single dislocation. Given a (generally non-closed) layering form β on the unit square $\mathcal{M} = [0,1]^2$, we construct in Section 3 a closed layering form ν on $\mathcal{M} \setminus \Gamma$, where Γ is a segment. The layering form ν has the same circulation around $\partial\mathcal{M}$ as β. The layering form ν induces a 1-current T_ν on $\mathcal{M}$; its boundary is a 0-current supported on Γ. Thus, we may view the layering form ν as representing a singular edge-dislocation, whose locus is Γ, and whose intensity is equal to the integrated intensity of the layering form β.

Step II: Homogenization for the square. In Section 4, we prove that every (possibly non-closed) layering form $\beta \in \Omega^1(\mathcal{M})$ can be approximated by a sequence of closed layering forms ν_n, representing an n-by-n array of edge-dislocations ($\mathcal{M}$ is still the unit square). We construct ν_n by gluing together properly rescaled versions of the form ν constructed in Section 3. We then prove that T_{ν_n} converges as $n \to \infty$ to the 1-current T_β.

Step III: The general case. In Section 6, we prove a homogenization theorem for a general compact and orientable surface $\mathcal{M}$. We show that for every layering form $\beta \in \Omega^1(\mathcal{M})$, there exists a sequence ν_n of closed layering forms supported everywhere except for a lower-dimensional submanifold, such that T_{ν_n} converges to T_β. The proof relies on a classical classification theorem for two-dimensional manifolds, along with gluing techniques for 1-forms (presented in the appendix). The homogenization problem is thus reduced to two elemental building blocks: the closed disk and a "pair of pants" for which homogenization follows from the homogenization theorem for the square.

Singular torsion. In Section 5, we generalize the analysis to the case where $\mathcal{M}$ is a d dimensional manifold equipped with a full lattice structure, that is, a frame field $\{e_i\}_{i=1}^d$ or equivalently, the dual coframe of d layering forms $\{\vartheta^i\}_{i=1}^d$. A frame-coframe pair induces a path-independent parallel transport $\Pi_p^q : T_p\mathcal{M} \to T_q\mathcal{M}$ between every two points $p, q \in \mathcal{M}$. The corresponding material connection ∇ is flat but may be non-symmetric; the torsion tensor is given by

$$\tau = e_i \otimes d\vartheta^i,$$

and it is non-zero if the layering forms ϑ^i are not closed. In the case of isolated dislocations, the torsion is identically zero in the smooth set and not defined on the singular set. Note, that the above expression for τ cannot be interpreted as a de-Rham current on $\mathcal{M}$ (it behaves like a product of a Heaviside function and a delta function).

Using the distant parallelism induced by the frame field (defined also for isolated dislocations) we define a notion of singular torsion for a singular frame as a vector valued de-Rham current. We show how the singular torsion generalizes the notion of a smooth torsion field and prove a homogenization theorem for the torsion tensor; If a sequence of coframes $\{\vartheta_n^i\}$ converges in the sense of currents to a smooth coframe $\{\vartheta^i\}$, then the corresponding singular torsions converge to the smooth torsion associated with the limit.

There are several differences between the present work and the earlier work in [8, 9, 12, 13]: In the earlier work, the loci of the dislocations were "removed", yielding a geometric convergence of smooth multiply-connected manifolds to a smooth simply-connected limit. Furthermore, the mode of convergence was a strong L^p-convergence of frame fields, which is stronger than the weak convergence of currents; a stronger convergence is particularly important for obtaining a convergence of the associated mechanical models. On the other hand, the current approach is more physical, as it accounts explicitly for the singular region; also, our notion of singular dislocations chimes in with the classical case of cone singularities, i.e., disclinations. Finally, the emergence of torsion in the continuous case no longer occurs "out of the blue", but is shown to be a bone fide limit of singular torsion fields.

Three points should be emphasized: (i) This work focuses on the geometry of bodies with dislocations. There exists a wealth of literature addressing the mechanics of dislocations, which we don't mention here. (ii) A body manifold is our elemental object of consideration, and it should not be confused with a (deformed) configuration, which is an embedding of that manifold in the ambient space. Since the body manifold and the deformed configuration are diffeomorphic, the same defect structure would be observed in the deformed configuration. (iii) In our model, the locus of a dislocation is a submanifold of co-dimension one, whereas it is often described in the literature as a submanifold of co-dimension two, e.g., a point in $2d$. Geometrically, a dislocation is a curvature dipole, or a pair of disclinations of opposite signs (e.g., a 5-7 pair in a hexagonal lattice). Since the Frank vector of a positive disclination is bounded by 2π, one cannot obtain a non-zero point dislocation as a limit of disclination dipoles, as in the case of electrostatics.

This paper is organized as follows. In Section 2 we review the definition of de-Rham currents on manifolds, which are the kinematic variables of our model. Section 3 is devoted to the first step of our homogenization proof—the construction of a layering form representing a single dislocation. The second step—the homogenization construction for the square—is conducted in Section 4. We then consider in Section 5 the notion of singular torsion and its homogenization. Finally, we extend in Section 6 the homogenization proof to general compact orientable surfaces.

2 De-Rham currents

We start by reviewing the definition of de-Rham currents on manifolds. For a full introduction, see the classical monographs of Federer [14] or de-Rham [15]; see [16, 17] for more recent reviews.

Let $\mathcal{M}$ be a smooth, compact, orientable d-dimensional manifold with boundary. For every $1 \leq k \leq d$, let $\Omega^k(\mathcal{M})$ denote the space of smooth k-forms on $\mathcal{M}$ and let

$$\Omega^k_c(\mathcal{M}) = \left\{\omega \in \Omega^k(\mathcal{M}) \ : \ \operatorname{supp}(\omega) \Subset \mathcal{M}\right\}$$

denote the $C^\infty(\mathcal{M})$-module of smooth k-forms compactly-supported in $\mathcal{M}$. Choose a Riemannian metric g on $\mathcal{M}$, and define for every compact $K \Subset \mathcal{M}$ a family of seminorms $\phi^k_{K,j} : \Omega^k_c(\mathcal{M}) \to \mathbb{R}^+$ by

$$\phi^k_{K,j}(\omega) = \sup_{0\leq i\leq j} \|D^i\omega\|_K,$$

where $D^i\omega : \mathcal{M} \to \operatorname{Hom}(\otimes^i T\mathcal{M}, \Lambda^k T^*\mathcal{M})$ is the i-th differential of ω (not to be confused with the exterior derivative), and

$$\|D^i\omega\|_K = \sup_{p\in K} \|(D^i\omega)_p\|,$$

where $\|\cdot\|$ is the norm on $\operatorname{Hom}(\otimes^i T\mathcal{M}, \Lambda^k T^*\mathcal{M})$ induced by the metric g. Since $\mathcal{M}$ is compact, a different choice of g gives equivalent seminorms. As a result, it makes sense to say that a k-form is C^j-bounded without reference to any particular metric (recall that in a topological vector space a set is bounded if every open neighborhood of zero can be inflated to include that set).

The seminorms $\{\phi^k_{K,j}\}_{j=1}^\infty$ turn

$$\Omega^k_K(\mathcal{M}) = \{\omega \in \Omega^k_c(\mathcal{M}) \ : \ \operatorname{supp}(\omega) \subset K\}$$

into a Fréchet space, that is, a locally-convex topological vector space which is complete with respect to a translationally-invariant metric [18, p. 9]. Endow $\Omega^k_c(\mathcal{M})$ with the finest topology for which the inclusion maps

$$\Omega^k_K(\mathcal{M}) \hookrightarrow \Omega^k_c(\mathcal{M})$$

are continuous for all compact $K \Subset \mathcal{M}$. A sequence $\omega_n \in \Omega^k_c(\mathcal{M})$ converges in this topology to 0 if and only if there exists a compact set $K \Subset \mathcal{M}$ such that $\operatorname{supp}(\omega_n) \subset K$ for all n large enough, and $\omega_n \to 0$ in the $\Omega^k_K(\mathcal{M})$ topology.

Definition 2.1 (de-Rham current). A de-Rham k-current is a continuous linear functional on $\Omega^k_c(\mathcal{M})$. The vector space of de-Rham k-currents is denoted by $\mathscr{D}_k(\mathcal{M})$.

A linear functional $T : \Omega^k_c(\mathcal{M}) \to \mathbb{R}$ is a k-current if and only if there exists for every $K \Subset \mathcal{M}$ an $N_K \in \mathbb{N}$ and a constant $C_K > 0$, such that for every $\omega \in \Omega^k_K(\mathcal{M})$,

$$|T(\omega)| \le C_K \, \phi^k_{K,N_K}(\omega).$$

(See e.g. [18, Th. 6.8] in the context of distributions in $\mathbb{R}^d$.) We endow $\mathscr{D}_k(\mathcal{M})$ with the weak-star topology: a sequence of k-currents T_n converges to a k-current T if

$$\lim_{n\to\infty} T_n(\omega) = T(\omega)$$

for every $\omega \in \Omega^k_c(\mathcal{M})$. The **support** of a k-current $T \in \mathscr{D}_k(\mathcal{M})$ is defined by $\operatorname{supp}(T) = \mathcal{M} \setminus A(T)$, where $A(T)$ is the annihilation set of T, i.e., the union of all open subsets $U \subset \mathcal{M}$ for which $T(\alpha) = 0$ whenever $\operatorname{supp}(\alpha) \subset U$.

Example 2.1. Every locally-integrable k-form β on Ω defines a $(d-k)$-current $T_\beta \in \mathscr{D}_{d-k}(\mathcal{M})$ by

$$T_\beta(\alpha) = \int_{\mathcal{M}} \beta \wedge \alpha, \qquad \alpha \in \Omega^{d-k}_c(\mathcal{M}).$$

In other words, currents may be viewed as generalized differential forms.

Example 2.2. Let $S \subset \mathcal{M}$ be a k-dimensional oriented submanifold. Then, S induces a k-current $[S] \in \mathscr{D}_k(\mathcal{M})$ given by

$$[S](\alpha) = \int_S \alpha, \qquad \alpha \in \Omega^k_c(\mathcal{M}).$$

In other words, currents also generalize the concept of a submanifold.

Definition 2.2 (Boundary of current). The **boundary operator** of a k-current is a map $\partial : \mathscr{D}_k(\mathcal{M}) \to \mathscr{D}_{k-1}(\mathcal{M})$, defined by

$$\partial T(\alpha) = T(d\alpha), \qquad \alpha \in \Omega_c^{k-1}(\mathcal{M}).$$

Since $d^2 = 0$, it immediately follows by duality that $\partial^2 = 0$. Moreover, it follows from integration by parts and Stokes theorem that

$$\partial T_\beta = (-1)^{k-1} T_{d\beta}$$

for every smooth k-form β.

3 Layering form for an edge-dislocation

Let V be a vector space. A covector $\omega \in V^*$ induces a family of hyperplanes (Bravais planes),

$$H_t = \{v \in V \;:\; \omega(v) = t\}, \qquad t \in \mathbb{R}$$

foliating V (i.e., forming a disjoint cover of V). The action of ω on a vector $v \in V$ can be interpreted as the "number of hyperplanes intersected by v". In a smooth manifold $\mathcal{M}$, the role of the covector is played by a 1-form foliating $\mathcal{M}$: given a 1-form ν and an oriented curve $C \subset \mathcal{M}$, the integral

$$\int_C \nu$$

can be interpreted as the (signed) "number" of ν-hyperplanes intersected by C.

Definition 3.1 (Layering form). Let $\mathcal{M}$ be a smooth manifold. A 1-form $\nu \in \Omega^1(\mathcal{M})$ is called a layering form if it foliates $\mathcal{M}$. That is, if $\mathcal{M}$ is the disjoint union of smooth hypersurfaces—leaves—such that the tangent bundle of each leaf coincides with the kernel of ν.

A sufficient and necessary condition for a 1-form ν to induce a smooth layering structure is that locally

$$d\nu = \alpha \wedge \nu$$

for some $(d-1)$-form α [10, Chap. 19]. In particular, for a simply-connected two-dimensional manifold, every non-vanishing 1-form induces a smooth layering structure.

If ν is a closed layering form, $d\nu = 0$, it follows from Stokes' theorem that for every simple, oriented, closed curve $C \subset \mathcal{M}$, the "number" of hyperplanes intersected by C vanishes,

$$\int_C \nu = \int_{\Sigma_C} d\nu = 0, \tag{2}$$

where $\Sigma_C \subset \mathcal{M}$ is any two-dimensional submanifold of $\mathcal{M}$ bounded by C. In other words, there are no "extra" layers, and the layering structure is defect-free. In view of (2), we may interpret $d\nu$ as a **defect density** associated with the layering form ν.

Definition 3.2 (Continuously-distributed dislocations). Let $\mathcal{M}$ be a smooth simply-connected manifold. A smooth layering form is said to represents a continuous distribution of dislocations if there exists a closed curve C, such that

$$\int_C \nu \neq 0.$$

The quantity on the left-hand side is called the **Burgers scalar**, or the circulation of the layering form ν around the loop C. We consider Burgers scalars, rather than Burgers vectors, since we account for only one layering form. When representing the structure by a co-frame, one obtains d Burgers scalars, which are the components in the local frame of the Burgers vector.

Clearly, ν represents a continuous distribution of dislocations if and only if it is non-closed.

Definition 3.3 (Singular dislocation). Let $\mathcal{M}$ be a smooth manifold and let $\Gamma \subset \mathcal{M}$ be a hyper-surface. A layering form ν on $\mathcal{M} \setminus \Gamma$ is said to represent a dislocation concentrated on Γ, if ν is closed and there exists a closed curve $C \in \mathcal{M} \setminus \Gamma$, such that

$$\int_C \nu \neq 0.$$

Suppose that $\nu \in \Omega^1(\mathcal{M} \setminus \Gamma)$ represents a dislocation concentrated on Γ. Since ν is closed, its Burgers scalar vanishes for every contractible loop. Therefore, $\mathcal{M} \setminus \Gamma$ is necessarily not simply-connected, i.e., the removal of the dislocation locus Γ changes the topology of the manifold. Let C be a loop in $\mathcal{M}$ encircling Γ (Figure 2), such that

$$\int_C \nu \neq 0.$$

Since ν is closed on $\mathcal{M} \setminus \Gamma$, the circulation remains unchanged under homotopic variations of C, and in particular, as C shrinks to Γ. Hence, ν is necessarily singular at Γ.

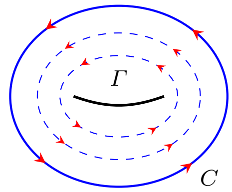


Fig. 2 A body endowed with a layering form ν with a singular dislocation located on a hypersurface Γ. The circulation of ν is homotopic-invariant for loops encircling the locus of the dislocation.

We next consider a two-dimensional manifold $\mathcal{M}$ endowed with a non-closed smooth layering form β (representing a continuous distribution of dislocations). We construct a layering form ν representing a singular dislocation concentrated on a curve $\Gamma \subset \mathcal{M}$, which approximates β is a sense made precise. In a sense, this construction concentrates the "defectiveness" of β onto the submanifold Γ. This construction will be used in the next section to prove the homogenization theorem.

Consider then a topological rectangle, i.e., a manifold that can be parametrized as follows:

$$\mathcal{M} = [0,1]^2 = \{(x,y) \ : \ 0 \le x, y \le 1\}.$$

We denote the left, right, top and bottom edges of $\mathcal{M}$ by $\mathcal{M}_{\text{left}}$, $\mathcal{M}_{\text{right}}$, $\mathcal{M}_{\text{top}}$ and $\mathcal{M}_{\text{bottom}}$, respectively. The locus of the singular dislocation will be the closed parametric segment

$$\Gamma_a = [1/2 - a/2, 1/2 + a/2] \times \{1/2\} \subset \mathcal{M}, \tag{3}$$

where $0 < a < 1$ is a parameter (see Figure 3).

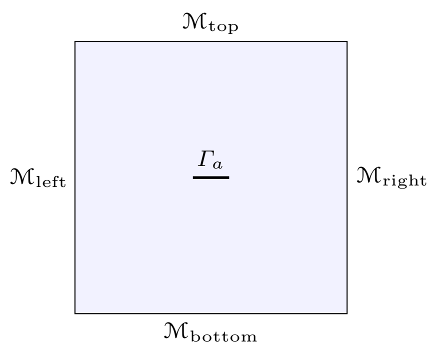


Fig. 3 The topological rectangle $\mathcal{M}$ and the locus Γ_a of the dislocation.

Proposition 3.1. *Let $\beta \in \Omega^1(\mathcal{M})$ be a nowhere-vanishing (generally non-closed) layering form. Then, there exists a continuously-differentiable layering form ν_a on $\mathcal{M} \setminus \Gamma_a$ satisfying the following properties:*

(a) ν_a is C^1-bounded (see definition in Section 2).

(b) ν_a is closed.

(c) ν_a coincides with β on $\mathcal{M}_{left}$ and $\mathcal{M}_{right}$.

(d) ν_a has the same circulation as β around $\partial\mathcal{M}$,

$$\int_{\partial\mathcal{M}} \nu_a = \int_{\partial\mathcal{M}} \beta.$$

(e) The horizontal components of ν_a and β coincide,

$$\nu_a(\partial_x) = \beta(\partial_x),$$

whenever $|x - 1/2| > a/2$.

Proof. We construct ν_a as the (continuous) differential of a discontinuous function f. First, define $f_0 : \partial\mathcal{M} \to \mathbb{R}$ by fixing $q_0 = (1, 1/2)$ and letting

$$f_0(q) = \int_{q_0}^{q} \beta,$$

where the integration from q_0 to q is counter-clockwise along $\partial\mathcal{M}$. If the circulation of β around $\partial\mathcal{M}$ is non-zero, then f_0 is discontinuous at q_0. However, its differential is well-defined and smooth at q_0 as it coincides with the tangential component of β (see Figure 4).

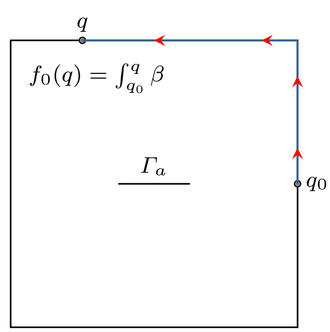


Fig. 4 The first stage in the construction of f: the 1-form β is integrated along $\partial\mathcal{M}$.

Next, consider the vertical strip of width a,

$$\mathcal{M}_a = \{(x,y) \in [0,1]^2 \ : \ |x - 1/2| < a/2\},$$

and define $\bar{f} : \mathcal{M} \setminus \mathcal{M}_a \to \mathbb{R}$ by integrating β horizontally, from the boundaries inward,

$$\bar{f}(x,y) = \begin{cases} f_0(0,y) + \int_{[(0,y),(x,y)]} \beta, & x < 1/2 - a/2 \\ f_0(1,y) + \int_{[(1,y),(x,y)]} \beta, & x > 1/2 + a/2 \end{cases}$$

(see Figure 5).

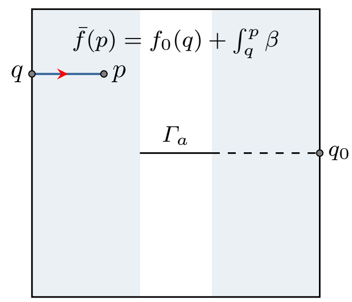


Fig. 5 The second stage in the construction of f: $\bar{f}$ is defined on the set $|x - 1/2| > a/2$ by integrating the horizontal component of β from the nearest vertical boundary point. The dashed segment connecting Γ_a to q_0 is the discontinuity line of f.

It remains to define f on $\mathcal{M}_a/\Gamma_a$. Denote by $p_L, p_R : \mathcal{M} \to \mathbb{R}$ the second-order Taylor expansions of $\bar{f}$ about $x_L = 1/2 - a/2$ and $x_R = 1/2 + a/2$ along the x-direction, i.e.,

$$\begin{aligned} p_L(x,y) &= \bar{f}(x_L,y) + \frac{\partial \bar{f}}{\partial x}(x_L,y)(x - x_L) + \frac{1}{2}\frac{\partial^2 \bar{f}}{\partial x^2}(x_L,y)(x - x_L)^2 \\ p_R(x,y) &= \bar{f}(x_R,y) + \frac{\partial \bar{f}}{\partial x}(x_R,y)(x - x_R) + \frac{1}{2}\frac{\partial^2 \bar{f}}{\partial x^2}(x_R,y)(x - x_R)^2. \end{aligned}$$

Let $r \in C^\infty(\mathbb{R})$ be a monotonically-increasing function satisfying,

$$r(t) = 0 \quad \forall t \le -1/2 \qquad \text{and} \qquad r(t) = 1 \quad \forall t \ge 1/2.$$

We extend $\bar{f}$ to $\mathcal{M} \setminus \Gamma_a$ by interpolating between p_L and p_R, using the smooth "connecting" function r (see Figure 8),

$$f(x,y) = \begin{cases} \bar{f}(x,y) & |x - 1/2| \ge a/2 \\ (1 - r(\frac{x-1/2}{a}))p_L(x,y) + r(\frac{x-1/2}{a})p_R(x,y) & |x - 1/2| < a/2. \end{cases} \tag{4}$$

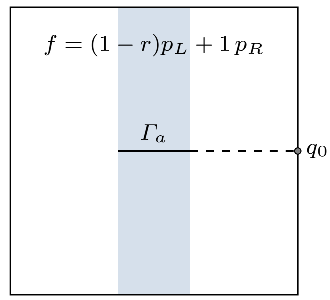


Fig. 6 The third stage in the construction of f: f is extended from $\bar{f}$ to the set $|x - 1/2| \le a/2$ by interpolation.

We obtain $\nu_a = df$ by differentiating (4). For $x > a/2 + 1/2$, an explicit calculation yields

$$df_{(x,y)} = \beta_1(x,y)\,dx + \left(\beta_2(1,y) + \int_1^x \frac{\partial\beta_1}{\partial y}(x',y)dx'\right)dy, \tag{5}$$

where β_1 and β_2 are the components of β,

$$\beta = \beta_1\,dx + \beta_2\,dy.$$

Similarly, for $x < 1/2 - a/2$,

$$df_{(x,y)} = \beta_1(x,y)\,dx + \left(\beta_2(0,y) + \int_0^x \frac{\partial\beta_1}{\partial y}(x',y)dx'\right)dy. \tag{6}$$

While f has a discontinuity along the segment $[1/2 + a, 1] \times \{1/2\}$, its one-sided derivatives along this segment are continuous, as they are expressed in terms of the smooth layering form β. Moreover,

$$df|_{\mathcal{M}_{\text{left}}} = \beta|_{\mathcal{M}_{\text{left}}} \qquad \text{and} \qquad df|_{\mathcal{M}_{\text{right}}} = \beta|_{\mathcal{M}_{\text{right}}},$$

proving Property (c). Likewise, for $|x - 1/2| \ge a/2$,

$$df(\partial_x) = \beta_1 = \beta(\partial_x),$$

proving Property (e).

For $(x,y) \in \mathcal{M}_a$,

$$
\begin{aligned}
df_{(x,y)} &= \frac{1}{a} r'\left(\tfrac{x-1/2}{a}\right)(p_R(x,y) - p_L(x,y))dx \\
&+ \left[\left(1 - r\left(\tfrac{x-1/2}{a}\right)\right)\frac{\partial p_L}{\partial x}(x,y) + r\left(\tfrac{x-1/2}{a}\right)\frac{\partial p_R}{\partial x}(x,y)\right] dx \\
&+ \left[\left(1 - r\left(\tfrac{x-1/2}{a}\right)\right)\frac{\partial p_L}{\partial y}(x,y) + r\left(\tfrac{x-1/2}{a}\right)\frac{\partial p_R}{\partial y}(x,y)\right] dy.
\end{aligned}
\tag{7}
$$

The layering form df is continuous at $x = 1/2 \pm a/2$. For example,

$$
\begin{aligned}
\lim_{x \nearrow 1/2 + a/2} df_{(x,y)} &= \frac{\partial p_R}{\partial x}(1/2 + a/2, y)\, dx + \frac{\partial p_R}{\partial y}(1/2 + a/2, y)\, dy \\
&= \frac{\partial \bar{f}}{\partial x}(1/2 + a/2, y)\, dx + \frac{\partial \bar{f}}{\partial y}(1/2 + a/2, y)\, dy \\
&= d\bar{f}(1/2 + a/2, y).
\end{aligned}
$$

A second differentiation shows that ν_a is continuously-differentiable at $x = 1/2 \pm a/2$. This together with (7) proves Property (a) and consequently also Property (b).

It remains to prove Property (d). From our construction of f_0 on $\partial\mathcal{M}$,

$$
\begin{aligned}
\int_{\partial\mathcal{M}} df &= \lim_{\varepsilon \to 0} \left(f(1, 1/2 - \varepsilon) - f(1, 1/2 + \varepsilon)\right) \\
&= \lim_{\varepsilon \to 0} \left(f_0(1, 1/2 - \varepsilon) - f_0(1, 1/2 + \varepsilon)\right) \\
&= \int_{\partial\mathcal{M}} \beta,
\end{aligned}
$$

which concludes the proof.

□

Regardless of the particular construction of ν_a, since ν_a is closed in $\mathcal{M} \setminus \Gamma_a$, it follows that

$$
\oint_C \nu_a = 0
$$

along every contractible loop C in $\mathcal{M} \setminus \Gamma_a$. Let g be a metric on $\mathcal{M}$, and denote by Γ_a^ε, $\varepsilon > 0$, a family of ε-tubular neighborhoods of Γ_a. By Stokes' law, for every small enough $\varepsilon > 0$,

$$
0 = \int_{\mathcal{M} \setminus \Gamma_a^\varepsilon} d\nu_a = \int_{\partial\mathcal{M}} \nu_a - \int_{\partial\Gamma_a^\varepsilon} \nu_a.
$$

Since ν_a has the same circulation as β along $\partial\mathcal{M}$,

$$
\int_{\partial\Gamma_a^\varepsilon} \nu_a = \int_{\partial\mathcal{M}} \beta.
$$

Letting $\varepsilon \to 0$, we obtain

$$\int_{\Gamma_a} [\nu_a] = \int_{\partial\mathcal{M}} \beta, \tag{8}$$

where $[\nu_a]$ is the discontinuity jump of ν_a along Γ_a, whose sign is determined by the orientation of $\mathcal{M}$ (hence of Γ_a^ε) and Γ_a. Note that the one-sided limits of ν_a at Γ_a exist since ν_a is C^1-bounded. Moreover, since $\mathcal{M}$ is compact, the limit leading to Identity (8) does not depend on the choice of the metric g. We conclude that if β (hence, ν_a) has non-vanishing circulation along $\partial\mathcal{M}$ then $[\nu_a] \neq 0$, that is, ν_a is discontinuous along Γ_a.

Remark 3.1. The singular set Γ_a of ν_a is uncountable. Generally, if $\mathcal{M}$ is a compact two-dimensional manifold with or without boundary, Γ is a submanifold of $\mathcal{M}$, and ν is a C^0-bounded closed 1-form on $\mathcal{M} \setminus \Gamma$, such that there exists a closed curve C for which

$$\oint_C \nu \neq 0,$$

then Γ cannot be a finite set. Suppose, by contradiction that $\Gamma = \{p_1, p_2, \dots, p_k\}$ is finite, and assume without loss of generality that all the points in Γ are enclosed by the curve C. Assuming as above a metric g, setting $\Gamma^\varepsilon = \cup_i B_\varepsilon(p_i)$, and performing the same calculation,

$$\sum_{i=1}^k \oint_{\partial B_\varepsilon(p_i)} \nu = -\oint_C \nu.$$

If ν is bounded, then the left-hand side vanishes as $\varepsilon \to 0$, yielding a contradiction. The physical interpretation of this observation is that in our setting *there is no such thing as an edge-dislocation supported at a point* (or on a line in three dimensions).

The 1-form ν_a (which is only defined on $\mathcal{M} \setminus \Gamma_a$) induces a 1-current on $\mathcal{M}$,

$$T_{\nu_a}(\alpha) = \int_{\mathcal{M}} \nu_a \wedge \alpha \qquad \alpha \in \Omega_c^1(\mathcal{M}).$$

Its boundary is the 0-current,

$$\partial T_{\nu_a}(f) = T_{\nu_a}(df) = \int_{\mathcal{M}} \nu_a \wedge df \qquad f \in C_c^\infty(\mathcal{M}).$$

Integrating by parts on $\mathcal{M} \setminus \Gamma_a^\epsilon$ snd taking $\epsilon \to 0$ (as above), we obtain

$$\partial T_{\nu_a}(f) = \int_{\Gamma_a} f[\nu_a],$$

where for $|x - 1/2| < a/2$,

$$\begin{aligned}[\nu_a](x) &= \lim_{\varepsilon\to 0}\left(df(x,1/2+\varepsilon) - df(x,1/2-\varepsilon)\right)\\ &= \frac{1}{a} r'\left(\tfrac{x-1/2}{a}\right) \lim_{\varepsilon\to 0}\left(p_R(x,1/2+\varepsilon) - p_R(x,1/2-\varepsilon)\right)\\ &= \frac{1}{a} r'\left(\frac{x-1/2}{a}\right) \int_{\partial\mathcal{M}} \beta,\end{aligned}$$

where we substituted (7), used the facts that p_L is continuous at $y = 1/2$ and that the discontinuity of p_R at $y = 1/2$ equals the circulation of β,

To conclude, ν_a represents a layering form on $\mathcal{M}$ having an edge-dislocation concentrated on the hyper-surface Γ_a. The locus of the dislocation is revealed by the boundary of the differential current induced by ν_a. Note that $\mathcal{M}\setminus\Gamma_a$ is defect-free only to the extent detectable by ν_a. Generally, $\mathcal{M}\setminus\Gamma_a$ may contain defects detected by other layering forms.

4 Homogenization of distributed edge-dislocations

In this section we show how a non-closed layering from (representing continuously-distributed dislocations) can be approximated, in the sense of currents, by an n-by-n array of singular edge-dislocations, each of magnitude of order $1/n^2$. We construct the approximation by "gluing" properly rescaled copies of the layering form ν_a constructed in Proposition 3.1.

For $(x_0, y_0) \in \mathbb{R}^2$, denote by $\tau_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ the translation operator

$$\tau_{(x_0,y_0)}(x, y) = (x + x_0, y + y_0).$$

Likewise, for $\lambda > 0$, denote by $S_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ the scaling operator

$$S_\lambda(x, y) = (\lambda x, \lambda y).$$

Let $n \in \mathbb{N}$ be given; for every $0 \le k, j < n$, let

$$\mathcal{M}_{n;kj} = S_{1/n} \circ \tau_{(k,j)}(\mathcal{M})$$

be translated and rescaled copies of $\mathcal{M}$, forming an n-by-n tiling of $\mathcal{M}$. By construction,

$$\iota_{n;kj} = S_{1/n} \circ \tau_{(k,j)} : \mathcal{M} \to \mathcal{M}_{n;kj} \tag{9}$$

are diffeomorphisms (see Figure 7). Similarly, let

$$\Gamma_{n;kj} = \iota_{n;kj}(\Gamma_{a/n})$$

be segments of lengths a/n^2 located at the centers of each square. Finally, denote by

$$\Gamma_n = \bigcup_{k,j=0}^{n-1} \Gamma_{n;kj}$$

the union of those segments.

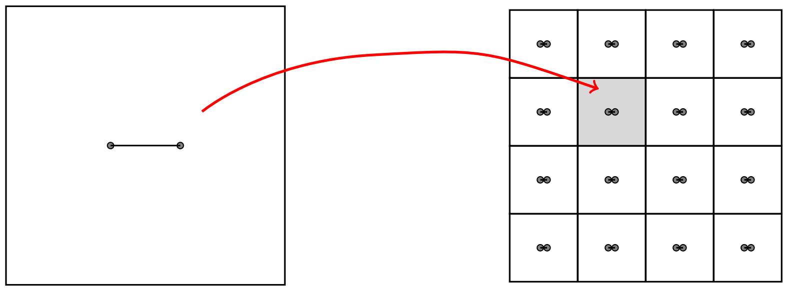


Fig. 7 The diffeomorphism $\iota_{n;kj}$ for $n = 4$, $k = 1$ and $j = 2$

Let $\beta \in \Omega^1(\mathcal{M})$ be a layering form. Let

$$\beta_{n;kj} = (\iota_{n;kj})^\star \beta|_{\mathcal{M}_{n;kj}} \in \Omega^1(\mathcal{M}), \tag{10}$$

be the pullback[2] of β (restricted to $\mathcal{M}_{n;kj}$) to $\mathcal{M}$ and let $\mu_{n;kj} \in \Omega^1(\mathcal{M} \setminus \Gamma_{a/n})$ be the singular layering form defined in Proposition 3.1, with $\beta_{n;kj}$ playing the role of β and the parameter a is scaled by a factor of $1/n$. We approximate β by a sequence of singular layering forms,

$$\nu_n \in \Omega^1(\mathcal{M} \setminus \Gamma_n),$$

by pushing forward $\mu_{n;kj}$ into $\mathcal{M}_{n;kj}$,

$$\nu_n|_{\mathcal{M}_{n;kj}} = (\iota_{n;kj})_\star \mu_{n;kj}. \tag{11}$$

Proposition 4.1. *Equation* (11) *for* $0 \le k, j < n$ *defines a layering form* ν_n *on* $\mathcal{M} \setminus \Gamma_n$, *satisfying*

(a) ν_n *is* C^1*-bounded.*

(b) ν_n *is closed.*

(c) ν_n *has the same circulation as* β *in each sub-domain: for every* $0 \le k, j \le n-1$,

---

[2] For a smooth map $f : \mathcal{M} \to \mathcal{N}$ between two manifolds and a k-form $\beta \in \Omega^k(\mathcal{N})$, we denote by $f^*\beta \in \Omega^k(\mathcal{M})$ its pullback,

$$(f^*\beta)_p(v_1, \dots, v_k) = \beta_{f(p)}(df_p(v_1), \dots, df_p(v_k)).$$

If f is a diffeomorphism, then k-forms can also be pushed forward.

$$\int_{\partial \mathcal{M}_{n;kj}} \nu_n = \int_{\partial \mathcal{M}_{n;kj}} \beta.$$

(d) ν_n *coincides with* β *on the vertical segments* $L_k = \{k/n\} \times [0,1]$ *for* $0 \le k \le n$*.*

Proof. We first show that ν_n is well-defined and satisfies Property (a). Since the $\mu_{n;kj}$ are smooth and C^1-bounded, ν_n is smooth and C^1-bounded in the interior of each $\mathcal{M}_{n;kj} \setminus \Gamma_{n;kj}$. It remains to prove that it is continuously-differentiable on the "skeleton" $\cup_{k,j} \partial \mathcal{M}_{n;kj}$. Note that

$$\partial \mathcal{M}_{n;kj} = \iota_{n;kj}(\mathcal{M}_{\text{left}}) \cup \iota_{n;kj}(\mathcal{M}_{\text{right}}) \cup \iota_{n;kj}(\mathcal{M}_{\text{top}}) \cup \iota_{n;kj}(\mathcal{M}_{\text{bottom}}).$$

By (10), since the diffeomorphism $\iota_{n;kj}$ is a combination of a translation and a scaling,

$$\beta_{n;kj}(\partial_x) = \frac{1}{n}\beta(\partial_x) \circ \iota_{n;kj} \qquad \text{and} \qquad \beta_{n;kj}(\partial_y) = \frac{1}{n}\beta(\partial_y) \circ \iota_{n;kj},$$

which are equalities between functions on $\mathcal{M}$. In particular, since $\iota_{n;k\,j+1}(x,0) = \iota_{n;kj}(x,1)$ and $\iota_{n;k\,j-1}(x,1) = \iota_{n;kj}(x,0)$, it follows that for every $x, y \in [0,1]$, and $v \in \{\partial_x, \partial_y\}$

$$\begin{aligned} \beta_{n;k\,j+1}(v)(x,0) &= \beta_{n;kj}(v)(x,1) \\ \beta_{n;k+1\,j}(v)(0,y) &= \beta_{n;kj}(v)(1,y) \end{aligned}$$

By the same argument, for $w \in \{\partial_x, \partial_y\}$

$$\begin{aligned} \mathcal{L}_w \beta_{n;k\,j+1}(v)(x,0) &= \mathcal{L}_w \beta_{n;kj}(v)(x,1) \\ \mathcal{L}_w \beta_{n;k+1\,j}(v)(0,y) &= \mathcal{L}_w \beta_{n;kj}(v)(1,y), \end{aligned}$$

where $\mathcal{L}_w$ is the Lie derivative along w. By (5), (6) and (7), the construction of $\mu_{n;kj}$ only depends on $\beta_{n;kj}$ (and the smooth function r). Moreover, $\mu_{n;kj}$ and its derivative on every side of $\partial \mathcal{M}$ depend only on $\beta_{n;kj}$ and its derivatives on that side. As a result, for every $x, y \in [0,1]$, and $v, w = \{\partial_x, \partial_y\}$,

$$\begin{aligned} \mu_{n;k\,j+1}(v)(x,0) &= \mu_{n;kj}(v)(x,1) \\ \mu_{n;k+1\,j}(v)(0,y) &= \mu_{n;kj}(v)(1,y) \\ \mathcal{L}_w \mu^n_{k\,j+1}(v)(x,0) &= \mathcal{L}_w \mu_{n;kj}(v)(x,1) \\ \mathcal{L}_w \mu^n_{k+1\,j}(v)(0,y) &= \mathcal{L}_w \mu_{n;kj}(v)(1,y). \end{aligned}$$

Since the relation between $\mu_{n;kj}$ and ν_n is once again a pullback under a combination of scaling and translation, we obtain that ν_n is continuously-differentiable along the skeleton.

We proceed to prove Property (d): by Property (c) of Proposition 3.1:

$$\begin{aligned}
\nu_n|_{\iota_{n;kj}(\mathcal{M}_{\text{left}})} &= (\iota_{n;kj})_\star \mu_{n;kj}|_{\iota_{n;kj}(\mathcal{M}_{\text{left}})} \\
&= (\iota_{n;kj})_\star \beta_{n;kj}|_{\iota_{n;kj}(\mathcal{M}_{\text{left}})} \\
&= (\iota_{n;kj})_\star (\iota_{n;kj})^\star \beta|_{\iota_{n;kj}(\mathcal{M}_{\text{left}})} \\
&= \beta|_{\iota_{n;kj}(\mathcal{M}_{\text{left}})},
\end{aligned}$$

i.e., ν_n coincides with β on the vertical components of the skeleton.

Property (b) is immediate as $\mu_{n;kj}$ are closed and closedness is invariant under pullback. Finally, Property (c) follows from Property (d) in Proposition 3.1: using the change of variables formula and the fact that $\mu_{n;kj}$ and $\beta_{n;kj}$ have the same circulation along $\partial\mathcal{M}$,

$$\begin{aligned}
\int_{\partial\mathcal{M}_{n;kj}} \nu_n &= \int_{\iota_{n;kj}(\partial\mathcal{M})} ((\iota_{n;kj})^{-1})^\star \mu_{n;kj} \\
&= \int_{\partial\mathcal{M}} \mu_{n;kj} = \int_{\partial\mathcal{M}} \beta_{n;kj} = \int_{\partial\mathcal{M}_{n;kj}} \beta.
\end{aligned}$$

□

As in the case of a single dislocation, we define for each n the 1-current induced by ν_n:

$$T_{\nu_n}(\alpha) = \int_{\mathcal{M}} \nu_n \wedge \alpha, \qquad \alpha \in \Omega^1_c(\mathcal{M}).$$

Its boundary ∂T_{ν_n} is a 0-current given by

$$\partial T_{\nu_n}(f) = \sum_{k,j=1}^{n-1} \int_{\Gamma_{n;kj}} f[\nu_n]_{\Gamma_{n;kj}}, \qquad f \in C^\infty_c(\mathcal{M}),$$

where $[\nu_n]_{\Gamma_{n;kj}}$ is the discontinuity jump of ν_n along $\Gamma_{n;kj}$, given by

$$[\nu_n]_{\Gamma_{n;kj}}(x, (j+1/2)/n) = \frac{n}{a} r' \left(\frac{nx - k - 1/2}{a} \right) \int_{\partial\mathcal{M}_{n;kj}} \beta.$$

We view ν_n as a layering form on $\mathcal{M}$ having n^2 edge-dislocations concentrated on Γ_n. The loci of the dislocations are revealed by the boundary of the differential current induced by ν_n. Once again, $\mathcal{M} \setminus \Gamma_n$ is defect-free only to the extent detectable by ν_n.

Theorem 4.1 (Homogenization). *The sequence ν_n of layering forms converges to β in the sense of currents: for every $\alpha \in \Omega^1_c(\mathcal{M})$,*

$$\lim_{n\to\infty} \int_{\mathcal{M}} \nu_n \wedge \alpha = \int_{\mathcal{M}} \beta \wedge \alpha,$$

or equivalently,

$$\lim_{n\to\infty} T_{\nu_n-\beta}(\alpha) = \lim_{n\to\infty} \int_{\mathcal{M}} (\nu_n - \beta)\wedge\alpha = 0. \tag{12}$$

Proof. Choose any metric on $\mathcal{M}$; for concreteness we take the Euclidean metric associated with the parametrization.

If $\beta = \beta_1\,dx + \beta_2\,dy$, then

$$\|\beta_{(x,y)}\|^2 = \beta_1^2(x,y) + \beta_2^2(x,y).$$

For every $\alpha \in \Omega^1_c(\mathcal{M})$,

$$\begin{aligned} T_{\nu_n-\beta}(\alpha) &= \sum_{k,j=0}^{n-1} \int_{\mathcal{M}_{n;kj}} (\nu_n-\beta)\wedge\alpha \\ &= \sum_{k,j=0}^{n-1} \int_{\iota_{n;kj}(\mathcal{M})} ((\iota_{n;kj})^{-1})^{\star}(\mu_{n;kj}-\beta_{n;kj})\wedge\alpha \\ &= \sum_{k,j=0}^{n-1} \int_{\mathcal{M}} (\mu_{n;kj}-\beta_{n;kj})\wedge(\iota_{n;kj})^{\star}\alpha, \end{aligned} \tag{13}$$

where the second equality follows from the definitions of ν_n and $\beta_{n;kj}$, and the third equality follows from the change of variables formula. Fix $0\le k,j\le n-1$. Since $\iota_{n;kj}$ involves a contraction by a factor of n,

$$\left\|(\iota_{n;kj})^{\star}\alpha|_{\mathcal{M}_{n;kj}}\right\|_\infty \le \frac{1}{n}\|\alpha\|_\infty.$$

It follows that

$$\begin{aligned} \left|\int_{\mathcal{M}} (\mu_{n;kj}-\beta_{n;kj})\wedge(\iota_{n;kj})^{\star}\,\alpha\right| &\le \frac{1}{n}\|\alpha\|_\infty \sup_{\|\xi\|_\infty=1}\left|\int_{\mathcal{M}}(\mu_{n;kj}-\beta_{n;kj})\wedge\xi\right| \\ &\le \frac{1}{n}\,\|\alpha\|_\infty \int_{\mathcal{M}} |\mu_{n;kj}-\beta_{n;kj}|\,dx\wedge dy. \end{aligned}$$

Combining with (13),

$$|T_{\nu_n-\beta}(\alpha)| \le n\,\|\alpha\|_\infty \sup_{0\le k,j<n} \int_{\mathcal{M}} |\mu_{n;kj}-\beta_{n;kj}|\,dx\wedge dy.$$

Next, writing $\beta_{n;kj}$ explicitly,

$$(\beta_{n;kj})_{(x,y)} = \frac{1}{n}\beta_1\left(\frac{x+k}{n},\frac{y+j}{n}\right)\,dx + \frac{1}{n}\beta_2\left(\frac{x+k}{n},\frac{y+j}{n}\right)\,dy.$$

By (6), for $x < 1/2 - a/2n$,

$$(\mu_{n;kj})_{(x,y)} = \frac{1}{n}\beta_1\left(\frac{x+k}{n}, \frac{y+j}{n}\right) dx + \left(\frac{1}{n}\beta_2\left(\frac{k}{n}, \frac{y+j}{n}\right) + \int_0^x \frac{1}{n^2}\frac{\partial\beta_1}{\partial y}\left(\frac{x'+k}{n}, \frac{y+j}{n}\right) dx'\right) dy,$$

so that

$$\begin{aligned} n\,|\mu_{n;kj} - \beta_{n;kj}|(x,y) &\le \left|\beta_2\left(\frac{x+k}{n}, \frac{y+j}{n}\right) - \beta_2\left(\frac{k}{n}, \frac{y+j}{n}\right)\right| \\ &\quad + \frac{1}{n}\int_0^x \left|\frac{\partial\beta_1}{\partial y}\left(\frac{x'+k}{n}, \frac{y+j}{n}\right)\right| dx' \\ &\le \frac{1}{n}\left(\left\|\frac{\partial\beta_2}{\partial x}\right\|_\infty + \left\|\frac{\partial\beta_1}{\partial y}\right\|_\infty\right). \end{aligned}$$

The same bound is obtained for $x > 1/2 + a/2n$. Finally, for $|x - 1/2| < a/2n$, using (7), and noting that p_L and p_R are $O(1/n)$, we obtain that

$$n\,|\mu_{n;kj} - \beta_{n;kj}|(x,y) \le \frac{C}{a}\|r'(x)\|_\infty,$$

where $C > 0$ is some constant. Putting it all together,

$$\begin{aligned} |T_{\nu_n-\beta}(\alpha)| &\le n\,\|\alpha\|_\infty \sup_{0\le k,j<n} \int_{\mathcal{M}\setminus\mathcal{M}_{a/n}} |\mu_{n;kj} - \beta_{n;kj}|\, dx\wedge dy \\ &\quad + n\,\|\alpha\|_\infty \sup_{0\le k,j<n} \int_{\mathcal{M}_{a/n}} |\mu_{n;kj} - \beta_{n;kj}|\, dx\wedge dy \\ &\le \frac{\|\alpha\|_\infty}{n}\left(\left\|\frac{\partial\beta_2}{\partial x}\right\|_\infty + \left\|\frac{\partial\beta_1}{\partial y}\right\|_\infty + \tilde{C}\,\|r'(x)\|_\infty\right), \end{aligned}$$

where in the estimation of the third term we used the fact that the volume of $\mathcal{M}_{a/n}$ is $O(1/n)$. Letting $n \to \infty$ we obtain the desired result. □

5 Singular torsion and its homogenization

Thus far, we analyzed a lattice structure through a single layering form, representing a single family of Bravais surfaces. In d dimensions, a lattice structure is fully determined by a set of d linearly-independent layering forms, i.e., by a coframe $\{\vartheta^i\}$. Denote by $\{e_i\}$ the frame field dual to $\{\vartheta^i\}$.

A frame-coframe structure induces a path-independent parallel transport,

$$\Pi_p^q : T_p\mathcal{M} \to T_q\mathcal{M} \qquad \text{given by} \qquad \Pi_p^q = e_i|_q \otimes \vartheta^i|_p. \tag{14}$$

The latter induces a connection ∇ having trivial holonomy, which locally implies zero curvature. By construction, the frame field $\{e_i\}$ and its dual $\{\vartheta^i\}$ are ∇-parallel sections,

$$\nabla e_i = 0 \qquad \text{and} \qquad \nabla \vartheta^i = 0.$$

The torsion tensor associated with ∇ is a $T\mathcal{M}$-valued 2-form τ, given by

$$\tau(e_i, e_j) = \nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] = [e_j, e_i].$$

Since for every $1 \le i, j, k \le d$,

$$\begin{aligned} d\vartheta^i(e_j, e_k) &= e_j(\vartheta^i(e_k)) - e_k(\vartheta^i(e_j)) - \vartheta^i([e_j, e_k]) \\ &= \vartheta^i([e_k, e_j]) \\ &= \vartheta^i(\tau(e_j, e_k)), \end{aligned}$$

we conclude that $d\vartheta^i = \vartheta^i \circ \tau$, or equivalently,

$$\tau = e_i \otimes d\vartheta^i, \tag{15}$$

where we adopt henceforth Einstein's summation rule, whereby repeated upper and lower indexes imply a summation. In particular, torsion vanishes if and only if $d\vartheta^i = 0$ for all $1 \le i \le d$, or equivalently, if $[e_i, e_j] = 0$ for all $1 \le i, j \le d$.

The question we are addressing henceforth is in what sense is the smooth torsion τ given by (15) a limit of singular torsions associated with singular dislocations. For example, let $\mathcal{M}$, β and ν_n be defined as in the previous section, and suppose that

$$\vartheta_n^1 = \nu_n \qquad \text{and} \qquad \vartheta_n^2 = dx$$

is a sequence of coframe fields (namely, ν_n are dx are linearly-independent everywhere in $\mathcal{M}$). By the analysis of the previous section (and trivially for ϑ^2),

$$\lim_{n\to\infty} T_{\vartheta_n^1} = T_\beta \qquad \text{and} \qquad \lim_{n\to\infty} T_{\vartheta_n^2} = T_{dx},$$

i.e.,

$$\lim_{n\to\infty} \{\vartheta_n^1, \vartheta_n^2\} = \{\beta, dx\}$$

in the sense of weak convergence of currents.

Since the coframe field $\{\vartheta_n^1, \vartheta_n^2\}$ consists of closed layering forms, the induced torsion on $\mathcal{M} \setminus \Gamma_n$ vanishes identically for every n,

$$\tau_n = e_i^n \otimes d\vartheta_n^i = 0,$$

which, if $d\beta \neq 0$, does not converge to the torsion

$$\tau = \frac{1}{\beta_2}\partial_y \otimes d\beta$$

associated with the limiting coframe field in any classical sense (we used here the fact that the frame dual to $\{\beta, dx\}$ is $\{\partial_y/\beta_2, \partial_x - \beta_1/\beta_2\,\partial_y\}$).

The question is how to cast a weak convergence of torsion in the framework of de-Rham currents. Torsion is a tangent bundle-valued 2-form. While it is possible to define currents associated with tangent bundle-valued forms, see e.g. [19], this approach doesn't seem applicable here. A simple heuristic argument shows that if we try to interpret torsion as a distribution for a discontinuous coframe field, we obtain the product of a discontinuous section e_i and the derivative $d\vartheta^i$ of a discontinuous section (loosely speaking, a product of a Heaviside function and a delta-function), which is not well-defined.

A hint toward a correct interpretation of singular torsion is obtained by considering Burgers circuits: Let C be a simple, oriented, regular closed curve in $\mathcal{M}$. The Burgers vector associated with the curve C is a parallel vector field B [20], whose value at a reference point p is given by

$$B_p = \oint_C \Pi^p_\gamma(d\gamma),$$

where Π^p is the parallel-transport to p, which by (14) is given by

$$\Pi^p = e_i|_p \otimes \vartheta^i,$$

and γ is a parametrization for C. Interpreting Π^p as a $T_p\mathcal{M}$-valued 1-form, we rewrite the Burgers vector B_p in a more succinct form,

$$B_p = \oint_C \Pi^p.$$

Applying Stokes' theorem,

$$B_p = \int_\Sigma d\Pi^p,$$

where $\partial\Sigma = C$. Hence,

$$B_p = e_i|_p \int_\Sigma d\vartheta^i.$$

Thus, having chosen a reference point p, the Burgers vector for a loop C is an integral over the area enclosed by this loop of a Burgers vector density

$$e_i|_p \otimes d\vartheta^i,$$

which is a $T_p\mathcal{M}$-valued 2-form; it is nothing but the torsion τ, whose output, once acting on a bivector, is parallel-transported to the reference point p. We henceforth denote by

$$\tau_p = \Pi^p \circ \tau = e_i|_p \otimes d\vartheta^i$$

the torsion transported to p. The notion of singular torsion may now be easily defined as the distributional counterpart of τ_p by replacing $d\vartheta^i$ with the boundary current ∂T_{ϑ^i}. However, we first need to define the notion of a singular frame. Rather than choosing the most general possible framework, we adopt a possibly restrictive but yet sufficiently rich and physically-motivated approach:

Definition 5.1. Let $\mathcal{M}$ be a compact d-dimensional manifold. A collection $\{\vartheta^i\}_{i=1}^d$ of 1-forms is called a **singular coframe** for $\mathcal{M}$ if for every $1 \le i \le d$, there exists a compact $(d-1)$-dimensional submanifold $\Gamma_i \subset \mathcal{M}$, such that

1. Each ϑ^i is a C^1-bounded 1-form on $\mathcal{M} \setminus \Gamma_i$.
2. $\{\vartheta^i_p\}$ is a basis for $T^*_p\mathcal{M}$ for every $p \in \mathcal{M} \setminus \Gamma$ where $\Gamma = \cup_i \Gamma_i$.
3. $\mathcal{M} \setminus \Gamma$ is path connected and $\partial\mathcal{M} \cap \Gamma = \emptyset$.

A **closed singular coframe** is a singular coframe $\{\vartheta^i\}$ satisfying $d\vartheta^i = 0$ on $\mathcal{M} \setminus \Gamma_i$ for every $1 \le i \le d$.

Recall that if a layering form $\omega \in \Omega^1(\mathcal{M})$ is closed, its induced layering structure (foliation) is defect free. A closed singular coframe therefore corresponds to isolated defects which are concentrated on a set of measure zero.

Definition 5.2. Let $\{\vartheta^i\}$ be a singular coframe field on $\mathcal{M}$ and let $p \in \mathcal{M} \setminus \Gamma$ be an arbitrary reference point. The **torsion current**, is a $T_p\mathcal{M}$-valued $(d-2)$-current given by,

$$\mathcal{T} = e_i|_p \, \partial T_{\vartheta^i}.$$

For a smooth coframe $\{\vartheta^i\}$, the torsion current is given by

$$\mathcal{T}(\alpha) = e_i|_p \, \partial T_{\vartheta^i}(\alpha) = e_i|_p T_{d\vartheta^i}(\alpha) = T_{\tau_p}(\alpha), \qquad \alpha \in \Omega_c^{d-2}(\mathcal{M}). \tag{16}$$

In other words, in the smooth case, the torsion current $\mathcal{T}$ is the $T_p\mathcal{M}$-valued $(d-2)$-current induced by the smooth $T_p\mathcal{M}$-valued 2-form τ_p.

In the case of a closed singular coframe (isolated defects), the singular torsion is supported on the singular hyper-surfaces $\{\Gamma_i\}$ and is given explicitly by

$$\mathcal{T}[p](\eta) = \sum_{i=1}^d \left(\int_{\Gamma_i} [\vartheta^i]_{\Gamma_i} \wedge \eta \right) e_i(p), \tag{17}$$

where $[\vartheta^i]_{\Gamma_i}$ is the discontinuity jump of ϑ^i along Γ_i and $\eta \in \Omega_c^{d-2}(\mathcal{M})$. For a general (non-closed) singular frame $\{\vartheta^i\}$, the torsion current naturally decomposes into a smooth component as in (16) and a singular component as in (17).

We have thus obtained the following corollary:

Corollary 5.1 (Homogenization of torsion). *Let $\{\vartheta_n^i\}$ be a sequence of (possibly) singular coframes and $p \in \mathcal{M}$ a reference point, satisfying:*

1. *There exists a (possibly) singular frame $\{\vartheta^i\}$ such that $\{\vartheta_n^i\}$ converges to $\{\vartheta^i\}$ in the sense of currents. That is*
$$T_{\vartheta_n^i} \to T_{\vartheta^i} \quad \text{as } n \to \infty, \qquad \forall\, 1 \le i \le d.$$
2. *The point p is outside the singularity sets of $\{\vartheta_n^i\}$ and $\{\vartheta^i\}$ and $(\vartheta_n^i)_p \to \vartheta_p^i$ (pointwise) for every $1 \le i \le d$.*

Let
$$\mathcal{T}_n = e_i^n|_p\, \partial T_{\vartheta_n^i} \qquad \text{and} \qquad \mathcal{T} = e_i|_p\, \partial T_{\vartheta^i}$$
be the corresponding $T_p\mathcal{M}$-valued $(d-2)$-torsion currents. Then, $\mathcal{T}_n \to \mathcal{T}$ in the sense of currents.

In particular, if $\{\vartheta_n^i\}$ are singular closed frames for every n and the limiting frame $\{\vartheta^i\}$ is smooth, then $\mathcal{T}_n$ and $\mathcal{T}$ are given by (17) and (16) respectively. The limiting smooth torsion is thus obtained as a limit of singular torsion currents supported on singular sets of measure zero.

For example, given a smooth coframe $\{\vartheta^1, \vartheta^2\}$ for the unit square $\mathcal{M} = [0,1]^2$, we have by Theorem 4.1 a sequence of closed singular frames $\{\vartheta_n^1, \vartheta_n^2\}$ corresponding to an array of dislocations which converge to the coframe $\{\vartheta^1, \vartheta^2\}$ in the sense of currents. The corresponding torsion currents $\mathcal{T}_n$ act on functions by integration along the dislocation segments of the $n \times n$ dislocation array corresponding to ϑ_n^1, and converge to a smooth current $\mathcal{T}$ acting on functions by integration over the whole of $\mathcal{M}$.

6 Homogenization for general surfaces

In this section, we extend the homogenization Theorem 4.1 to arbitrary compact, orientable, smooth two-dimensional manifold with boundary. We restrict our attention to manifold without corners. The results in this section rely on the gluing constructions for 1-forms developed in Appendix A.

Theorem 6.1. *Let $\mathcal{M}$ be a compact, orientable two-dimensional manifold, possibly with boundary. Let $\omega \in \Omega^1(\mathcal{M})$ be a (generally non-closed) layering form on $\mathcal{M}$. Then there exists sequences ω_n and Γ_n such that*

1. *Γ_n is a finite disjoint union of simple non-closed curves in $\mathcal{M}$ and is bounded away from $\partial\mathcal{M}$.*

2. ω_n *are closed* C^1*-bounded* 1*-forms on* $\mathcal{M} \setminus \Gamma_n$*.*

3. ω_n *converge to* ω *in the sense of currents. That is,* $T_{\omega_n} \to T_\omega$ *as* $n \to \infty$*.*

4. $\omega_n|_{\partial\mathcal{M}} = \omega|_{\partial\mathcal{M}}$*.*

We say that a manifold $\mathcal{M}$ satisfies the **homogenization property** if Theorem 6.1 holds for $\mathcal{M}$ and in addition $\mathcal{L}_X\omega_n|_{\partial\mathcal{M}} = \mathcal{L}_X\omega|_{\partial\mathcal{M}}$ for every vector-field $X \in \Gamma(\mathcal{M})$. The latter condition is technical and is required below for gluing together manifolds with boundaries.

Remark 6.1. Note that in Theorem 6.1, the layering forms ω_n coincide with ω along the entire boundary of $\mathcal{M}$, whereas in the case of a rectangle (Section 4), the layering forms coincide only on part of the boundary. In general, if $\mathcal{M}$ has a corner, then the tangent to the boundary at the corner spans the entire tangent space; thus, if ω_n coincides with ω in a neighborhood of the corner, then its derivatives are fully determined by those of ω and it might not be closed as required by the construction.

A key observation is the following:

Lemma 6.1. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be compact diffeomorphic two-dimensional manifolds with boundaries. Then, Theorem 6.1 holds for* $\mathcal{M}_1$ *if and only if it holds for* $\mathcal{M}_2$*.*

Proof. Suppose that Theorem 6.1 holds for $\mathcal{M}_1$. Let $f : \mathcal{M}_2 \to \mathcal{M}_1$ be a diffeomorphism and let $\omega_2 \in \Omega^1(\mathcal{M}_2)$. Applying Theorem 6.1 for $\omega_1 = f_\star\omega_2 \in \Omega^1(\mathcal{M}_1)$, we obtain a sequence $\omega_{1,n} \in \Omega_1(\mathcal{M}_1 \setminus \Gamma_1^n)$ satisfying properties $(1-3)$. Define

$$\omega_{2,n} = f^\star\omega_{1,n} \in \Omega(\mathcal{M}_2 \setminus \Gamma_2^n), \qquad (\Gamma_2^n = f(\Gamma_1^n)).$$

Since f is a diffeomorphism, Γ_2^n is a finite disjoint union of segments bounded away from $\partial\mathcal{M}_2$ (Property 1). Property 2 follows from the fact that pullback and exterior differentiation commute,

$$d\omega_{2,n} = d(f^\star\omega_{1,n}) = f^\star(d\omega_{1,n}) = 0.$$

By the change of variable formula (for forms), for every $\eta \in \Omega_c^1(\mathcal{M}_2)$,

$$T_{\omega_{2,n}}(\eta) = \int_{\mathcal{M}_2} \omega_{2,n} \wedge \eta = \int_{\mathcal{M}_1} \omega_{1,n} \wedge f_\star\eta = T_{\omega_{1,n}}(f_\star\eta).$$

Since $T_{\omega_{1,n}}(f_\star\eta) \to T_{\omega_1}(f_\star\eta)$,

$$\lim_{n\to\infty} T_{\omega_{2,n}}(\eta) = \lim_{n\to\infty} T_{\omega_{1,n}}(f_\star\eta) = T_{\omega_1}(f_\star\eta) = T_{f_\star\omega_2}(f_\star\eta) = T_{\omega_2}(\eta),$$

proving Property 3. Finally, Property 4 follows from the fact that for every n,

$$\omega_{2,n}|_{\partial\mathcal{M}_2} = f^*(\omega_{1,n}|_{\partial\mathcal{M}_1}) = f^*(\omega_1|_{\partial\mathcal{M}_1}) = \omega_2|_{\partial\mathcal{M}_2}.$$

□

The well-known classification theorem for orientable compact surfaces states that every closed, compact, orientable, connected surface is diffeomorphic to either the sphere S^2 or the n-fold torus T_n (a sphere with n handles). Likewise, any compact, orientable, connected surface $\mathcal{M}$ with boundary is diffeomorphic to either S^2 or T_n, with k holes, namely,

$$\mathcal{M} = S^2 \setminus \amalg_{i=1}^k U_i \qquad \text{or} \qquad \mathcal{M} = T_n \setminus \amalg_{i=1}^k U_i,$$

where U_i are disjoint open sets diffeomorphic to a disc; see e.g. [21] for a proof using Morse theory. Moreover, each of those surfaces can be constructed by gluing together a finite number of two building blocks: a closed disc, and a "pair-of-pants".

To prove a homogenization for compact, orientable surfaces with or without boundary we adopt the following strategy: We first prove the homogenization property for the two above-mentioned building blocks. Then, using a gluing lemma (Lemma A.2), we deduce the homogenization property for S^2 and T_n with k holes. We finally obtain the general case by combining Lemma 6.1 and the classification theorem of surfaces.

We start by constructing a layering form containing a single dislocation on the unit disk

$$D = \left\{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \le 1\right\}.$$

Lemma 6.2. *Let $\omega \in \Omega^1(D)$ and let $\Gamma_a = [-a,a] \times \{0\}$, where $0 < a < 1/2$. Then there exists a closed, C^1-bounded 1-form $\omega_a \in \Omega^1(D \setminus \Gamma_a)$ satisfying*

(a) $\int_{\partial D} \omega = \int_{\partial D} \omega_a$.

(b) $\omega|_{\partial D} = \omega_a|_{\partial D}$.

(c) $(\mathcal{L}_{\partial_r}\omega_a)\,|_{\partial D} = (\mathcal{L}_{\partial_r}\omega)\,|_{\partial D}$.

Proof (sketch). The proof follows the same lines as the proof of Proposition 3.1. We construct ω_a as the differential of a discontinuous function $f : D \to \mathbb{R}$. First fix $q_0 = (0,1) \in \partial D$ and define $f_0 : \partial D \to \mathbb{R}$ by

$$f_0(q) = \int_{q_0}^{q} \omega,$$

where the integration is counter-clockwise along ∂D. As in the case of a square, f_0 is discontinuous at q_0 but its differential is well-defined. Next, define $f : D \setminus \Gamma_a \to \mathbb{R}$ as follows: for every $q = (q_1, q_2) \in \partial D$, let l_q be the segment connecting q to $(aq_1, 0) \in \Gamma_a$ (see Figure 8). Then every $p \in D$ lies on a unique segment l_q, and we may define

$$f(p) = f_0(q) + \int_{[q,p]} \omega, \quad p \in l_q.$$

A straightforward computation as in the proof of Proposition 3.1 shows that $\omega_a = df$ satisfies the desired properties. □

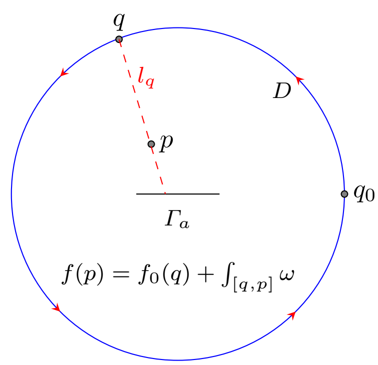


Fig. 8 Construction of a layering form on a disk containing a single dislocation.

We next prove the homogenization property for the closed disk.

Lemma 6.3. *The homogenization property holds for the disk*

$$D = \left\{ (x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 \leq 1 \right\}.$$

Proof. Let $\omega \in \Omega^1(D)$. For every $n \in \mathbb{N}$, let $D_n = \frac{1}{n}D$ and let $B_{i,n}$, $1 \leq i \leq 4$, be the sectors given by

$$B_{i,n} = \{(r\,\cos\theta, r\,\sin\theta) \;:\; 1/n \leq r \leq 1, \quad i\pi/4 \leq \theta \leq (i+1)\pi/4\}.$$

Then $B_{i,n} \simeq [0,1]^2$ and $D \simeq D_n \cup_{i=1}^4 B_{i,n}$ (see Figure 9). Let $\phi : B_{1,n} \to [0,1]^2$ be a diffeomorpism which preserves the left/right and upper/lower edges/arcs. Then its rotations $\phi_i = \phi \circ R_{(i-1)\pi/4} : B_{i,n} \to [0,1]^2$ $(i = 2,3,4)$ are diffeomorphisms as well. Using Proposition 4.1, we may construct singular closed layering forms $\omega_{i,n}$ on $B_{i,n}$ which combine together into a singular layering form $\tilde{\omega}_n$ on $D \setminus D_n$, whose singularity set of $\tilde{\omega}_n$ is a union of segments and it coincides with ω on $\partial D \setminus \frac{1}{n}D$. Finally, by Lemma 6.2, we may complete $\tilde{\omega}_n$ into a singular layering form ω_n on D. That

$$T_{\omega_n} \to T_\omega$$

follows from Theorem 4.1 (applied separately for each sector). □

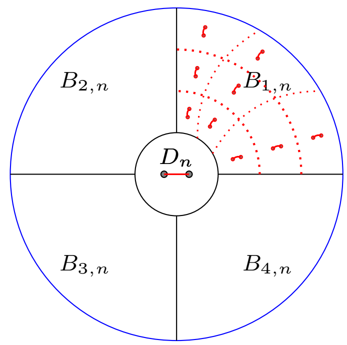


Fig. 9 The disk D is decomposed four sectors (diffeomorphic to a square) and a small disk. A layering form containing an "array of dislocations" is constructed in each sector, and glued together to obtain ω_n.

We next prove the homogenization property for a pair-of-pants which is diffeomorphic to the three-holed sphere (see Figure 10).

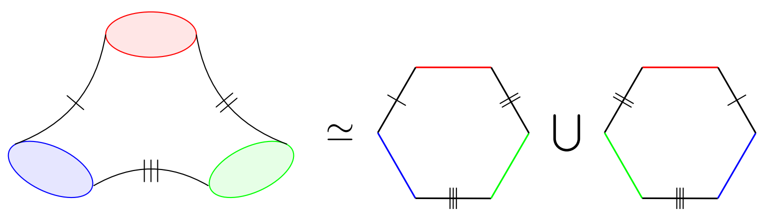


Fig. 10 A pair-of-pants. It can be obtained by gluing hexagons along three pairs of edges. The remaining pairs of (colored) edges are glued at their ends thus forming three boundary circles.

Lemma 6.4. *The homogenization property holds for a pair-of-pants.*

Proof. First, note that the hexagon, denoted by $\mathcal{O}$, also satisfies the homogenization property as well as the gluing conditions as in Lemma 6.3. The proof is almost identical to the proof for the disk (taking 6 rather than 4 sectors). Let $\mathcal{M}$ be a pair-of-pants. It can be obtained by identifying three pairs of edges of two hexagons $\mathcal{O}_1$ and $\mathcal{O}_2$ (see Figure 10). Hence, a layering form $\omega \in \Omega^1(\mathcal{M})$ induces layering forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$ on $\mathcal{O}_{1,2}$ satisfying (trivially) the gluing conditions of Lemma A.2. Since the homogenization property holds for each hexagon, there exist approximating sequences $\tilde{\omega}_{i,n}$ for $\tilde{\omega}_i$ $(i = 1, 2)$ which satisfy the gluing conditions and therefore form together an approximating sequence ω_n for ω. □

We next prove the following gluing argument.

Proposition 6.1. *Suppose that the homogenization property holds for compact orientable surfaces* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*. Let* $A_i \subset \partial\mathcal{M}_i$ *be connected components of the boundaries and* $h : A_1 \to A_2$ *a diffeomorphism. Finally, let* $\iota_{A_i} : [0,1) \times A_i \to \mathcal{M}_i$ *be collar neighborhoods for* A_i*. Then the glued manifold*

$$\mathcal{M} = \mathcal{M}_1 \amalg_h \mathcal{M}_2$$

satisfies the homogenization property (see the appendix for the definition of collar neighborhoods and gluing constructions).

Proof (sketch). Let $\omega \in \Omega^1(\mathcal{M})$. Then ω induces layering forms $\omega^i \in \Omega^1(\mathcal{M}_i)$ satisfying the gluing conditions, so that the restriction of ω and its first derivatives to $\mathcal{M}_i$ coincides with those of ω_i. Apply the homogenization property to obtain sequences of closed singular layering forms $\omega_{i,n}$, so that $T_{\omega_{i,n}}$ converges weakly to T_{ω_i}. We may choose the $\omega_{i,n}$ such that their values and their Lie derivatives coincide with those of ω_i at A_i. By the gluing Lemma A.2, $\omega_{1,n}$ and $\omega_{2,n}$ induce a closed and singular C^1-bounded layering form ω_n on $\mathcal{M} = \mathcal{M}_1 \amalg_h \mathcal{M}_2$. It follows directly from the construction that the sequence ω_n satisfies the required properties. □

Remark 6.2. We are mostly interested in the case where $\mathcal{M}_1$ and $\mathcal{M}_2$ are submanifolds of $\mathcal{M}$ with a common boundary (circle) component $A = A_1 = A_2 \simeq S^1$. In such a case, one can take a collar neighborhood induced by a vector field $X \in \Gamma(T\mathcal{M})$ which is transversal to A. Taking, $h = Id_A : A_1 \to A_2$ one obtains $\mathcal{M} \simeq \mathcal{M}_1 \amalg_h \mathcal{M}_2$. In other words, it is not necessary in this case to specify the collar neighborhoods and the boundary identifications, and the conditions for the gluing lemma to apply are satisfied automatically.

By applying Proposition 6.1 inductively we may finally prove Theorem 6.1:

Proof (of Theorem 6.1). Let $\mathcal{M}$ be a compact, orientable surface (possibly with boundary). By the classification of surfaces, we may decompose $\mathcal{M}$ into a finite number of pairs of pants and disks (glued along circles). By Lemmas 6.3 and 6.4, the homogenization property holds for the disk and for the pair-of-pants. Hence, given $\omega \in \Omega^1(\mathcal{M})$, we may inductively apply Proposition 6.1 (on larger and larger components of $\mathcal{M}$) to obtain the desired sequence ω_n. The convergence $T_{\omega_n} \to T_\omega$ follows immediately from the construction and the compactness of $\mathcal{M}$. □

Acknowledgments

We would like to thank Marcelo Epstein, Pavel Giterman, Cy Maor, Reuven Segev and Amitay Yuval for many helpful discussions. This research was partially funded by the Israel Science Foundation (Grant No. 1035/17), and by a grant from the Ministry of Science, Technology and Space, Israel and the Russian Foundation for Basic Research, the Russian Federation.

A Gluing constructions

The homogenization procedure presented in Sections 4 and 6 relies on gluing diffeomorphic copies of single isolated dislocations and their structure forms. To this end, we review some basic definitions and facts, following [10, Chap. 9] and prove a gluing lemma for 1-forms.

Let $\mathcal{M}$ be a smooth manifold with boundary. A neighborhood of $\partial\mathcal{M}$ is called a **collar neighborhood** if it is the image of a smooth embedding $\iota : [0,1) \times \partial\mathcal{M} \hookrightarrow \mathcal{M}$ sending (identically) $\{0\} \times \partial\mathcal{M}$ to $\partial\mathcal{M}$. It follows from the theory of flows that every smooth manifold with boundary admits a collar neighborhood; see [10, Theorem 9.25].

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be smooth manifolds with boundary of the same dimension, and let $A \subset \partial\mathcal{M}_1$, and $B \subset \partial\mathcal{M}_2$ be nonempty connected (possibly closed) submanifolds. Suppose that $h : B \to A$ is a diffeomorphism. h defines an equivalence relation on the disjoint union $\mathcal{M}_1 \amalg \mathcal{M}_2$ whereby $p \sim_h q$ if and only if $p = h(q)$. Let

$$\mathcal{M}_1 \amalg_h \mathcal{M}_2 := \{[p]_h \,|\, p \in \mathcal{M}_1 \amalg \mathcal{M}_2\},$$

where $[p]_h$ is the $\sim_h$-equivalence class of p. Then, $\mathcal{M}_1 \amalg_h \mathcal{M}_2$ is a topological manifold (possibly with boundary and corners); it admits a smooth structure such that the natural embeddings

$$\mathcal{M}_1 \hookrightarrow \mathcal{M}_1 \amalg_h \mathcal{M}_2, \qquad \mathcal{M}_2 \hookrightarrow \mathcal{M}_1 \amalg_h \mathcal{M}_2,$$

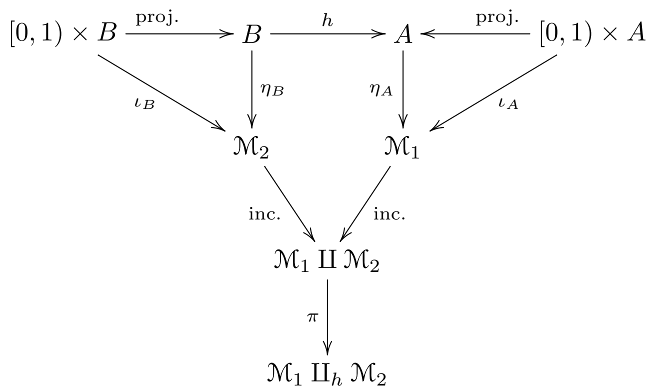
are smooth, $[\mathcal{M}_1]_h \cup [\mathcal{M}_2]_h = \mathcal{M}_1 \amalg_h \mathcal{M}_2$ and $[\mathcal{M}_1]_h \cap [\mathcal{M}_2]_h = [A]_h = [B]_h$. We will denote by

$$\pi : \mathcal{M}_1 \amalg \mathcal{M}_2 \to \mathcal{M}_1 \amalg_h \mathcal{M}_2$$

the projection map sending every point $p \in \mathcal{M}_1 \amalg \mathcal{M}_2$ to its equivalence class $[p]_h \in \mathcal{M}_1 \amalg_h \mathcal{M}_2$.

The construction of the smooth structure relies on gluing collar neighborhoods of A and B along h. In particular the smooth structure depends on the chosen collar neighborhoods; see [10, Theorem 9.29] for details.

Let $\iota_A : [0,1) \times A \to \mathcal{M}_1$ and $\iota_B : [0,1) \times B \to \mathcal{M}_2$ be collar neighborhoods for A and B; define also the inclusions $\eta_A : A \hookrightarrow \mathcal{M}_1$ and $\eta_B : B \hookrightarrow \mathcal{M}_1$ by $\eta_A(p) = \iota_A(0,p)$ and $\eta_B(p) = \iota_B(0,p)$; see diagram below.



For later use, we note that

$$\pi \circ \eta_B = \pi \circ \eta_A \circ h,$$

hence, differentiating, for $p \in B$,

$$d\pi_{\eta_B(p)} \circ (d\eta_B)_p = d\pi_{\eta_A(h(p))} \circ (d\eta_A)_{h(p)} \circ dh_p. \tag{18}$$

The collar neighborhoods define a decomposition of $T\mathcal{M}_1$ and $T\mathcal{M}_2$ at A and B: for example,

$$T\mathcal{M}_1|_{\eta(A)} = T\mathcal{M}_1^{\|} \oplus T\mathcal{M}_1^{\perp},$$

where

$$T\mathcal{M}_1^{\|} = (\eta_A)_\star TA,$$

and

$$T\mathcal{M}_1^{\perp} = \operatorname{span}(n_A),$$

where

$$n_A = (\iota_A)_\star(\partial_t)|_{A\times\{0\}} \tag{19}$$

is a vector field normal to $T\mathcal{M}_1^{\|}$ with respect to the collar neighborhood ι_A. Similar definitions apply for the tangent bundle of $\mathcal{M}_2$ at B.

We turn to characterize tangent vectors on the quotient space $\mathcal{M}_1 \amalg_h \mathcal{M}_2$. Suppose first that $p \in \mathcal{M}_1 \amalg \mathcal{M}_2 \setminus (A \cup B)$. Then, π is a local diffeomorphism in a neighborhood of p, hence $d\pi_p$ is a linear isomorphism. In other words, tangent vectors at $[p]_h$ can be identified with tangent vectors at p.

In contrast, let $p \in B$, i.e.,

$$\pi^{-1}(\pi(p)) = \{h(p), p\},$$

and let $v \in T_{\pi(p)}(\mathcal{M}_1 \amalg_h \mathcal{M}_2)$. Then, $d\pi^{-1}(v) = \{v_1, v_2\}$, where $v_1 \in T_{h(p)}\mathcal{M}_1$ and $v_2 \in T_p\mathcal{M}_2$. Each of the two vectors can be written in the form

$$v_1 = (\eta_A)_\star(v_1^{\|}) + v_1^{\perp}\, n_A \qquad \text{and} \qquad v_2 = (\eta_B)_\star(v_2^{\|}) + v_2^{\perp}\, n_B,$$

where $v_1^{\|} \in TA$, $v_2^{\|} \in TB$ and $v_1^{\perp}, v_2^{\perp} \in \mathbb{R}$.

We state without a proof:

Lemma A.1. *The following relations hold:*

$$v_1^{\|} = h_\star(v_2^{\|}), \tag{20}$$

and

$$v_1^{\perp} = -v_2^{\perp}. \tag{21}$$

Moreover,

$$\pi_\star(n_A) = -\pi_\star(n_B). \tag{22}$$

Our next goal is to glue together 1-forms along $A \subset \partial\mathcal{M}_1$ and $B \subset \partial\mathcal{M}_2$:

Lemma A.2 (Gluing of forms).

Let $\omega_1 \in \Omega^1(\mathcal{M}_1)$ and $\omega_2 \in \Omega^1(\mathcal{M}_2)$ satisfy the following conditions:

(i) Equality of tangential component:

$$h^\star(\eta_A^\star \omega_1) = \eta_B^\star \omega_2 \tag{23}$$

(this is an equality of 1-forms on B).

(ii) Matching of normal component:

$$\omega_1(n_A) \circ h = -\omega_2(n_B) \tag{24}$$

(this is an equality of functions on B).

(iii) Matching of normal derivative:

$$(\mathcal{L}_{n_A}\omega_1(n_A)) \circ h = -\mathcal{L}_{n_B}\omega_2(n_B),$$

and

$$h^\star\,(\eta_A^\star(\mathcal{L}_{n_A}\omega_1)) = -\eta_B^\star(\mathcal{L}_{n_B}\omega_2),$$

where $\mathcal{L}$ is the Lie derivative and n_A and n_B are extended to neigborhoods of $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ via (19).

Then, there exists a 1-form ω on $\mathcal{M}_1 \amalg_h \mathcal{M}_2$ which is C^1 with respect to the smooth structure induced by ι_A and ι_B, such that the restrictions of ω to $\mathcal{M}_1$ and $\mathcal{M}_2$ coincide with ω_1 and ω_2.

Proof. Let $\omega_1 \amalg \omega_2 \in \Omega^1(\mathcal{M}_1 \amalg \mathcal{M}_2)$ be the induced form on the disjoint union. We first show that Conditions (i) and (ii) imply that $\omega_1 \amalg \omega_2$ projects to a well defined 1-form ω on $\mathcal{M}_1 \amalg_h \mathcal{M}_2$.

Consider first $p \in \mathcal{M}_1 \amalg \mathcal{M}_2 \setminus (A \cup B)$, and let $v \in T_{\pi(p)}(\mathcal{M}_1 \amalg_h \mathcal{M}_2)$. Since $\pi^{-1}(\pi(p)) = \{p\}$ and $d\pi_p$ is an isomorphism, we may define

$$\omega_{\pi(p)}(v) = (\omega_1 \amalg \omega_2)_p(d\pi^{-1}(v)).$$

Next, let $p \in B$ and let $v \in T_{\pi(p)}(\mathcal{M}_1 \amalg_h \mathcal{M}_2)$. Now $\pi^{-1}(\pi(p)) = \{h(p), p\}$ and $d\pi^{-1}(v) = \{v_1, v_2\}$, where $v_1 \in T_{h(p)}\mathcal{M}_1$ and $v_2 \in T_p\mathcal{M}_2$. In order to define $\omega_{\pi(p)}(v)$ unambiguously, it suffices to show that $\omega_1(v_1) = \omega_2(v_2)$.

Write as above

$$\begin{aligned} v_1 &= d\eta_A(v_1^{\|}) + v_1^{\perp}\,(n_A)_{h(p)} \\ v_2 &= d\eta_B(v_2^{\|}) + v_2^{\perp}\,(n_B)_p, \end{aligned}$$

where $v_1^{\|} \in T_{h(p)}A$ and $v_2^{\|} \in T_pB$. Then,

$$\begin{aligned} \omega_1(d\eta_A(v_1^{\|})) = \eta_A^\star\omega_1(v_1^{\|}) &\overset{(20)}{=} \eta_A^\star\omega_1(dh(v_2^{\|})) = h^\star\eta_A^\star\omega_1(v_2^{\|}) \\ &\overset{(23)}{=} \eta_B^\star\omega_2(v_2^{\|}) = \omega_2(d\eta_B(v_2^{\|})), \end{aligned}$$

and

$$\begin{aligned} \omega_1(v_1^{\perp}\,(n_A)_{h(p)}) = v_1^{\perp}\,\omega_1(n_A)_{h(p)} &\overset{(21)}{=} -v_2^{\perp}\,\omega_1(n_A)_{h(p)} \\ &\overset{(24)}{=} v_2^{\perp}\,\omega_2(n_B)_p = \omega_2(v_2^{\perp}\,n_B)_p. \end{aligned}$$

We have thus proved that ω is well defined. It remains to show that ω (or equivalently $\Phi_\star\omega$) is continuously differentiable. For $(t,p) \in (-1,1) \times A$ and $\alpha\partial_t \oplus v \in T((-1,1) \times A) \simeq T_t(-1,1) \oplus T_pA$,

$$(\Phi_\star\omega)\,|_{(t,p)}(\alpha\partial_t \oplus v) = \begin{cases} \omega_1|_{\iota_A(-t,p)}(-\alpha n_A + v) & t < 0 \\ \omega_2|_{\iota_B(t,p)}(\alpha n_B + dh(v)) & t \geq 0. \end{cases}$$

Conditions (i) then implies that the tangential (to A) derivatives of $\Phi_\star\omega$ are continuous and Condition (iii) shows (by a similar calculation) that it is continuously differentiable in the "t" direction (one-sided limits coincide). This completes the proof. □

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