Asymptotic rigidity of Riemannian manifolds

Raz Kupferman
Institute of Mathematics
The Hebrew University

Cy Maor
Department of Mathematics
University of Toronto

Asaf Shachar
Institute of Mathematics
The Hebrew University

January 30, 2017

Contents

Abstract

Let $f : \mathcal{M} \to \mathcal{N}$ be a Lipschitz map between two oriented Riemannian manifolds, whose differential is almost everywhere a linear isometry. Gromov (1986) showed that if f is not assumed to be smooth, then it is not necessarily an isometric immersion; moreover, $\mathcal{M}$ may not even be isometrically immersible in $\mathcal{N}$. In this paper we prove that if f is additionally orientation-preserving (almost everywhere), then it is an isometric immersion. Moreover, we prove that if there exists a sequence of mapping $f_n : (\mathcal{M}, \mathfrak{h}) \to (\mathcal{N}, \mathfrak{h})$, whose differentials converge in L^p to the set of orientation-preserving isometries, then there exists a subsequence converging to an isometric immersion. These results are generalizations of celebrated rigidity theorems by Liouville (1850) and Reshetnyak (1967) from Euclidean to Riemannian settings. We describe an application of the generalized rigidity theorem to convergence notions of manifolds.

1 Introduction

In 1850, Liouville proved a celebrated rigidity theorem for conformal mappings [Lio50]. An important corollary of Liouville's theorem is that a sufficiently smooth mapping $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ that is everywhere a local isometry must be a global isometry. Specifically, if $f \in C^1(\Omega; \mathbb{R}^d)$ satisfies $df \in \mathrm{SO}(d)$ everywhere, then f is an affine function, i.e., an isometric embedding of Ω into $\mathbb{R}^d$.

While from modern perspective it seems rather trivial, Liouville's rigidity theorem was generalized in various highly non-trivial directions. One such direction is concerned with the regularity requirements on f. As it turns out, it suffices to require that f be Lipschitz continuous with $df \in \mathrm{SO}(d)$ almost everywhere (by Rademacher's theorem, Lipschitz continuous functions are a.e. differentiable). Indeed, if f is a local orientation-preserving isometry a.e., then it is weakly-harmonic, and by Weyl's lemma, it is smooth [Res67a]; for a more complete survey on regularity see [Lor13].

Another type of generalization is due to Reshetnyak [Res67b]. It is concerned with sequences of mappings $f_n : \Omega \to \mathbb{R}^d$ that are asymptotically locally rigid in an average sense. Specifically,

> *Let $\Omega \subset \mathbb{R}^d$ be an open, connected, bounded domain, and let $1 \le p < \infty$. If $f_n \in W^{1,p}(\Omega; \mathbb{R}^d)$ satisfy $\int_\Omega f_n \, dx = 0$ and*
>
> $$\lim_{n\to\infty} \int_\Omega \mathrm{dist}^p(df_n, \mathrm{SO}(d)) \, dx = 0,$$
>
> *then f_n has a subsequence converging in the strong $W^{1,p}(\Omega; \mathbb{R}^d)$ topology to an affine mapping.*

Here, $\mathrm{dist}(df, \mathrm{SO}(d)) : \Omega \to \mathbb{R}$ is a measure of local distortion of f. Liouville's theorem (for Lipschitz mappings) states that if this local distortion vanishes almost everywhere, then f is an isometric embedding. Reshetnyak's theorem states that a sequence of mappings, for which the L^p-norm of the local distortion tends to zero, converges (modulo a subsequence) to an isometric embedding.

Geometric formulation and main results This paper is concerned with generalizations of Liouville's and Reshetnyak's theorems to the realm of mappings between Riemannian manifolds. Throughout this paper, let $(\mathcal{M}, \mathfrak{g})$ and $(\mathcal{N}, \mathfrak{h})$ be compact, connected, oriented d-dimensional Riemannian manifolds (possibly with a C^1 boundary). Liouville's theorem for smooth mappings has a well-known generalization for manifolds:

> *Let $f \in C^1(\mathcal{M};\mathcal{N})$ satisfy $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ everywhere in $\mathcal{M}$. Then, f is smooth and rigid in the sense that every $x \in \mathcal{M}$ has a neighborhood U_x in which*
>
> $$f = \exp^{\mathcal{N}}_{f(x)} \circ df_x \circ (\exp^{\mathcal{M}}_x)^{-1}.$$

Here, $\exp^M$ and $\exp^{\mathcal{N}}$ are the respective exponential maps in $\mathcal{M}$ and $\mathcal{N}$; for $x \in \mathcal{M}$, $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})_x$ is the set of orientation-preserving isometries $T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$. The generalization of Liouville's theorem for the smooth case states that a Riemannian isometry can be (locally) factorized via the mapping and its derivative at a single point.

A first natural question is whether this generalization of Liouville's theorem holds if f is assumed less regular. A second natural question is whether a generalization of Reshetnyak's theorem can be established for mappings between manifolds: suppose that $\mathcal{M}$ can be mapped into $\mathcal{N}$ with arbitrarily small mean local distortion. Can one deduce that $\mathcal{M}$ is isometrically immersible into $\mathcal{N}$? Moreover, suppose that those mappings are diffeomorphisms. Can one deduce that $\mathcal{M}$ and $\mathcal{N}$ are isometric? This paper provides positive answers to all these questions.

Specifically, we prove the following generalization of Reshetnyak's theorem (Theorem 4.1 below):

> *Let $(\mathcal{M}, \mathfrak{g})$ and $(\mathcal{N}, \mathfrak{h})$ be compact, oriented, d-dimensional Riemannian manifolds with C^1 boundary. Let $1 \le p < \infty$. Let $f_n \in W^{1,p}(\mathcal{M};\mathcal{N})$ be a sequence of mappings satisfying*
>
> $$\operatorname{dist}_{(\mathfrak{g}, f_n^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})) \to 0 \qquad \text{in } L^p(\mathcal{M}),$$
>
> *where $\operatorname{dist}_{(\mathfrak{g}, f_n^*\mathfrak{h})}$ is the distance in $T^*\mathcal{M} \otimes f_n^*T\mathcal{N}$ induced by $\mathfrak{g}$ and $f_n^*\mathfrak{h}$. Then, $\mathcal{M}$ can is isometrically immersible into $\mathcal{N}$, and there exists a subsequence of f_n converging in $W^{1,p}(\mathcal{M};\mathcal{N})$ to a smooth isometric immersion $f : \mathcal{M} \to \mathcal{N}$.*
>
> *Moreover, if $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $Vol_{\mathfrak{g}}\mathcal{M} = Vol_{\mathfrak{h}}\mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are isometric and f is an isometry. In particular, these additional conditions hold if f_n are diffeomorphisms.*

The role of orientation Liouville's theorem for smooth mappings holds if $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ is replaced with $\mathrm{O}(\mathfrak{g}, f^*\mathfrak{h})$: indeed, a $C^1(\mathcal{M};\mathcal{N})$ mapping is either orientation-preserving or orientation-reversing (globally), which reduces the setting to the case of the differential being in $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$. However, both for Lipschitz mappings, and asymptotically-rigid mappings, Liouville's and Reshtnyak's theorems do not hold if $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ is replaced with $\mathrm{O}(\mathfrak{g}, f^*\mathfrak{h})$ (even in Euclidean settings).

The reason for the breakdown of both rigidity theorems is the following: maps whose differentials switch between the two connected components of $\mathrm{O}(\mathfrak{g}, f^*\mathfrak{h})$ can be highly irregular. For example, Goromov proved, using methods of convex integration, that given an arbitrary metric $\mathfrak{g}$ on the d-dimensional closed disc $\mathcal{D}^d$, there exists a mapping $f \in W^{1,\infty}(\mathcal{D}^d, \mathbb{R}^d)$, such that $f^\star\mathfrak{e} = \mathfrak{g}$ a.e., (i.e., $df \in \mathrm{O}(\mathfrak{g}, \mathfrak{e})$ a.e.); here $\mathfrak{e}$ denotes the Euclidean metric on $\mathbb{R}^d$ (see also [LP11, Remark 2.1])). It follows that a functional such as

$$f \mapsto \int_{\mathcal{M}} |\mathfrak{g} - f^*\mathfrak{h}|^p \,\mathrm{Vol}_{\mathfrak{g}}, \tag{1.1}$$

which does not account for orientation, is not a good measure of distortion, even though at first sight, it might seem more natural than

$$f \mapsto \int_{\mathcal{M}} \mathrm{dist}^p_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))\,\mathrm{Vol}_{\mathfrak{g}}. \tag{1.2}$$

These difficulties only arise when mappings can switch orientations; the results of this paper hold if the distance from orientation-preserving isometries is replaced by the distance from orientation-reversing isometries (which amounts to choosing a different orientation on either $\mathcal{M}$ or $\mathcal{N}$), or if $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ in (1.2) is replaced with $\mathrm{O}(\mathfrak{g}, f^*\mathfrak{h})$ or with (1.1), but the mappings are restricted to (local) diffeomorphisms.

Sketch of proof We now present a rough sketch of the proof, emphasizing its main ideas; applications of the theorem are discussed afterwards.

As a starting point, note the following well-known linear algebraic fact: $A \in \mathrm{SO}(d)$ if and only if $\det A = 1$ and $\operatorname{cof} A = A$, where $\operatorname{cof} A$ is the cofactor matrix of A, i.e., the transpose of the adjugate matrix of A. This fact can be reformulated for mappings between abstract inner-product spaces: $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ if and only if $\mathrm{Det}\, df = 1$ and $\mathrm{Cof}\, df = df$, where Det and Cof are the intrinsic determinant and cofactor operators (see Section 2.3 for details).

The assumptions on the sequence (f_n) imply that it is precompact in the weak $W^{1,p}$-topology. However, the direct methods of the calculus of variations cannot be used directly to deduce that a limit function f is an isometric immersionn, since the functional (1.2) is not lower-semicontinuous in the weak $W^{1,p}$-topology. Instead, we follow the ideas behind the proof of [JK90] to Reshetnyak's (Euclidean) rigidity theorem; we use Young measures to show that any weak limit f of (f_n) must satisfy $\operatorname{Det} df = 1$ and $\operatorname{Cof} df = df$ a.e., hence $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ a.e.

The generalization of [JK90] is not straightforward. The fundamental theorem of Young measures is formulated for sequences of vector-valued functions. A generalization of this theory to sections of a *fixed* vector bundle is relatively straightforward (see Section 2.2 for details). In our case, however, df_n is a section of $T^*\mathcal{M} \otimes f_n^*T\mathcal{N}$, i.e., every df_n is a section of a *different* vector bundle. Trying to overcome this difficulty by the standard procedure of embedding $\mathcal{N}$ isometrically into a high-dimensional Euclidean space $\mathbb{R}^D$ (so that all df_n become sections of the same vector bundle $T^*\mathcal{M} \otimes \mathbb{R}^D$) does not solve the problem, because information about orientation is lost (as discussed above, the theorem does not hold if $\mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})$ is replaced by $\mathrm{O}(\mathfrak{g}, f_n^*\mathfrak{h})$). This difficulty is overcome by a combination of extrinsic (embedded) and intrinsic (local) treatments of $\mathcal{N}$ in different parts of the argument.

In addition, this generalization of [JK90] only works for $p > d$ (otherwise $\operatorname{Det} df_n \not\rightharpoonup \operatorname{Det} df$, and even worse, the use of local coordinates for the intrinsic part of the analysis is impossible). To encompass the case $1 \leq p \leq d$, we use a truncation argument from [FJM02, LP11], adapted to our setting.

Having established that $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ almost everywhere, the problem is reduced to proving the following version of Liouville theorem for Lipschitz mappings (Theorem 3.1):

> *Let $f \in W^{1,\infty}(\mathcal{M};\mathcal{N})$ satisfy $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ almost everywhere. Then f is a smooth isometric immersion.*

While being well-known for $C^1(\mathcal{M};\mathcal{N})$ mappings, this is non-trivial for Lipschitz mappings. As discussed above, if one removes, for example, the orientation requirement, i.e., assume $df \in \mathrm{O}(\mathfrak{g}, f^*\mathfrak{h})$ a.e. (or equivalently $f^\star\mathfrak{h} = \mathfrak{g}$ a.e.), then this statement is false.

For $\mathcal{N} = \mathbb{R}^d$, Liouville's theorem for Lipschitz mappings follows from the observation that for a smooth $\varphi : \mathcal{M} \to \mathbb{R}^d$, $\operatorname{div}_\mathfrak{g} \operatorname{Cof} d\varphi = 0$ ($\operatorname{Cof} d\varphi$ is a d-tuple of sections of $T^*\mathcal{M}$, each of which divergence-free); see e.g. [Eva98, Chapter 8.1.4.b.] and [LP11, Lemma 3.1]. Since $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{e})$ implies that $df = \operatorname{Cof} df$, it follows that if $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{e})$ a.e., then $\operatorname{div}_\mathfrak{g} df = 0$ in a weak sense, i.e. f is weakly-harmonic. By Weyl's lemma, f is smooth, and therefore it is an isometric immersion.

To generalize this analysis to the case where $\mathcal{N}$ is not Euclidean, we adopt once again a combination of intrinsic and an extrinsic approaches. First, using an intrinsic approach, we generalize the "Div-Cof-Grad=0" identity: we prove that $\delta_\nabla \operatorname{Cof} d\varphi = 0$ for every $\varphi \in C^2(\mathcal{M};\mathcal{N})$, where δ_∇ is the co-differential induced by the Riemannian connection on $\varphi^*T\mathcal{N}$. Second, embedding $\mathcal{N}$ isometrically into a Euclidean space, $\iota : \mathcal{N} \to \mathbb{R}^D$, we write this identity in a weak form, from which we deduce the weak harmonicity of f, hence its smoothness.

The combination of intrinsic and extrinsic approaches seems necessary: on the one hand the "Div-Cof-Grad=0" identity cannot be formulated for mappings $\mathcal{M}^d \to \mathbb{R}^D$, $d < D$, at least in a way that will enable us to deduce the harmonicity of f. On the other hand, it is not clear how to formulate a weak form of this identity without an embedding, which naturally embeds $T^*\mathcal{M} \otimes f^*T\mathcal{N}$ into a vector bundle $T^*\mathcal{M} \otimes \mathbb{R}^D$ independent of f.

The proof of Theorem 4.1 is completed by showing that $f_n \to f$ in the strong (rather than weak) $W^{1,p}(\mathcal{M};\mathcal{N})$ topology (using again Young measures). This stronger convergence, along with the conditions that $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $\operatorname{Vol}_\mathfrak{g}\mathcal{M} = \operatorname{Vol}_\mathfrak{h}\mathcal{N}$, imply that f is an isometry. Note that this last part has no equivalent in the Euclidean version of Reshetnyak's theorem.

Applications Two applications of Theorem 4.1 are presented in Section 5. First, an immediate corollary (Corollary 5.1) of Theorem 4.1 is that if $\mathcal{M}$ is not isometrically immersible in $\mathcal{N}$, then

$$\inf_{f\in W^{1,p}(\mathcal{M};\mathcal{N})} \int_\mathcal{M} \operatorname{dist}^p_{(\mathfrak{g},f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))\, d\mathrm{Vol}_\mathfrak{g} > 0.$$

This corollary is relevant to the field of non-Euclidean elasticity [LP11, KS12, ESK13]. It was previously established only for $\mathcal{N} = \mathbb{R}^d$; see [LP11, Theorem 2.2].

Second, weak notions of converging manifolds were investigated in a recent series of works on manifolds with singularities [KM15, KM15, KM16a]. A sequence of Riemannian manifolds $(\mathcal{M}_n, \mathfrak{g}_n)$ was defined to converge to a Riemannian manifold $(\mathcal{M}, \mathfrak{g})$ if (up to some additional assumptions) there exist diffeomorphisms $F_n : \mathcal{M} \to \mathcal{M}_n$, such that

$$\operatorname{dist}^p(dF_n, \mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n)) \to 0 \qquad \text{in } L^p(\mathcal{M}),$$

and similarly for F_n^{-1}. Theorem 4.1 implies that such a notion of convergence is well-defined for p large enough, in the sense that the limit is unique (independent of the choice of F_n). Moreover, we present some examples, showing that this notion of metric convergence can be substantially different from Gromov–Hausdorff convergence.

Open questions

1. A discussion of generalizations of Liouville's rigidity theorem cannot be complete without mentioning the far-reaching result of [FJM02], which is a quantitative version of Reshetnyak's theorem:

   Let $\Omega \subset \mathbb{R}^d$ be an open, connected Lipschitz domain, and let $1 < p < \infty$. Then, there exists a constant $C > 0$ such that for every $f \in W^{1,p}(\Omega;\mathbb{R}^d)$ there exists an affine map $\tilde{f}$ such that

   $$\|f - \tilde{f}\|^p_{W^{1,p}(\Omega;\mathbb{R}^d)} \le C \int_\Omega \operatorname{dist}^p(df, \mathrm{SO}(d))\, dx.$$

   This result has been generalized in various ways, see e.g. [Lor16, CM16] and the references therein. All these generalizations are in Euclidean settings. A natural question is whether this theorem can be generalized to mappings between Riemannian manifolds.

2. While the results of this paper imply that for $\mathcal{M}$ not immersible into $\mathcal{N}$,

   $$\inf_{f \in W^{1,p}(\mathcal{M};\mathcal{N})} \int_\mathcal{M} \operatorname{dist}^p_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))\, d\mathrm{Vol}_\mathfrak{g} > 0,$$

   they do not provide an estimate on how large this infimum is. Since the local obstruction of isometric immersibility can be related to a mismatch of curvatures, one would expect curvature-dependent lower bounds. some results in this direction exist for $\mathcal{N} = \mathbb{R}^d$ [KS12], however the general picture is still widely open.

3. In this paper, we assume, for the sake of simplicity, that the Riemannian metrics $\mathfrak{g}$ and $\mathfrak{h}$ are smooth; all the results hold for metrics of class $C^{1,\alpha}$. It is of interest whether our results can be extended to less regular metrics including singularities. In the context of the convergence of manifolds presented in Section 5, an important example is the convergence to smooth surfaces of locally-flat surfaces with conic singularities [KM16b, KM16a]. The uniqueness of the limit in such cases is yet to be established.

Structure of this paper In Section 2, we present definitions, along with a brief survey of results, which are used in various parts of this paper; specifically, we consider Sobolev spaces, Young measures and the cofactor of a linear transformation, all in a general Riemannian setting. Most of those results are known; some less known results (or ones that we did not find a good reference for) are proved in the appendices for completeness. Sections 3–4 are the core of the paper. In Section 3 we prove Liouville's rigidity theorem for Lipschitz mappings between Riemannian manifolds. In Section 4, we prove the generalization of Reshetnyak's asymptotic rigidity theorem for mappings between Riemannian manifolds. In Section 5, we present the two above-mentioned applications.

2 Preliminaries

2.1 Sobolev spaces between manifolds

The following definitions and results are well-known; see [Haj09, Weh04] for proofs and for further references.

Let $\mathcal{M}, \mathcal{N}$ be compact Riemannian manifolds, and let D be large enough such that there exists an isometric embedding $\iota : \mathcal{N} \to \mathbb{R}^D$ (Nash's theorem). For $p \in [1, \infty)$, we define the Sobolev space $W^{1,p}(\mathcal{M}; \mathcal{N})$ by

$$W^{1,p}(\mathcal{M}; \mathcal{N}) = \left\{ u : \mathcal{M} \to \mathcal{N} \,:\, \iota \circ u \in W^{1,p}(\mathcal{M}; \mathbb{R}^D) \right\}.$$

This space inherits the strong and weak topologies of $W^{1,p}(\mathcal{M}; \mathbb{R}^D)$, which are independent of the embedding ι.

Generally, these spaces are larger than the closure of $C^\infty(\mathcal{M}; \mathcal{N})$ in the strong/weak $W^{1,p}(\mathcal{M}; \mathbb{R}^D)$ topology. However, when $p \geq d = \dim \mathcal{M}$, $W^{1,p}(\mathcal{M}; \mathcal{N})$ is the strong closure of $C^\infty(\mathcal{M}; \mathcal{N})$ in the strong topology [Haj09, Theorem 2.1].

By the standard Sobolev embedding theorems, it follows that for $p > d$, $W^{1,p}(\mathcal{M}; \mathcal{N})$ consists of continuous functions whose image is in $\mathcal{N}$ everywhere. Moreover, $W^{1,p}(\mathcal{M}; \mathcal{N})$ convergence implies uniform convergence for $p > d$. Therefore, when $p > d$, $W^{1,p}(\mathcal{M}; \mathcal{N})$ can be defined "locally", namely

$$\begin{aligned} W^{1,p}(\mathcal{M}; \mathcal{N}) = \Big\{ u \in C(\mathcal{M}; \mathcal{N}) : \phi \circ u \in W^{1,p}(u^{-1}(U), \mathbb{R}^d), \\ \text{for every local chart } \phi : U \subset \mathcal{N} \to \mathbb{R}^d \Big\}. \end{aligned}$$

In addition, $u_n \to u$ in $W^{1,p}(\mathcal{M};\mathcal{N})$ if and only if $u_n \to u$ uniformly, and $\phi \circ u_n \to \phi \circ u$ in $W^{1,p}(u^{-1}(U), \mathbb{R}^d)$ in every coordinate patch [Weh04, Lemmas B.5 and B.7].

Moreover, it follows from [Hei05, Theorem 4.9] that for $p > d$, $u \in W^{1,p}(\mathcal{M};\mathcal{N})$ is differentiable almost everywhere and that its strong and weak derivatives coincide almost everywhere.

Finally, note that for every $p \geq 1$ (including $p \leq d$), there is a notion of weak derivative du of $u \in W^{1,p}(\mathcal{M};\mathcal{N})$ (and not only of $\iota \circ u$), which is measurable as a function $T\mathcal{M} \to T\mathcal{N}$ [CS16].

2.2 Young measures on vector bundles

Young measures play a central role in the calculus of variations. In this paper, we will make use of the following theorem [Bal89]:

Theorem 2.1 Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable. Let $u_n : \Omega \to \mathbb{R}^m$ be a sequence of Lebesgue measurable functions. Suppose that (u_n) satisfies the boundedness condition

$$\lim_{M\to\infty} \sup_n \operatorname{meas}\{x \in \Omega \cap B(0,R) : |u_n(x)| \geq M\} = 0, \tag{2.1}$$

for every $R > 0$. Then, there exists a (not relabeled) subsequence u_n and a family $(\nu_x)_{x\in\Omega}$ of Radon probability measures on $\mathbb{R}^m$, depending measurably on x, such that,

$$\psi \circ u_n \rightharpoonup \left\{ x \mapsto \int_{\mathbb{R}^m} \psi(\lambda)\, d\nu_x(\lambda) \right\} \quad \text{in } L^1(\Omega)$$

for every continuous function $\psi : \mathbb{R}^m \to \mathbb{R}$, for which $(\psi \circ u_n)$ is sequentially weakly relatively compact in $L^1(\Omega)$. That is, for every $h \in L^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \psi(u_n(x))h(x)\, dx = \int_\Omega \left(\int_{\mathbb{R}^m} \psi(\lambda)\, d\nu_x(\lambda) \right) h(x)\, dx.$$

Remarks:

1. Condition (2.1) is equivalent to the existence, for every $R > 0$, of a function $g_R : [0,\infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} g_R(t) = \infty$ and

$$\sup_n \int_{\Omega\cap B(0,R)} g_R(|u_n(x)|)\, dx < \infty.$$

   In particular, it holds whenever the sequence u_n is uniformly bounded in $L^1(\Omega; \mathbb{R}^m)$.

2. If Ω is bounded, then the criterion that $(\psi \circ u_n)$ be sequentially weakly relatively compact in $L^1(\Omega)$ is equivalent to

$$\sup_n \int_\Omega \varphi(|\psi \circ u_n|)\, dx < \infty \tag{2.2}$$

for some continuous function $\varphi : [0, \infty) \to \mathbb{R}$, such that $\lim_{t\to\infty} \varphi(t)/t = \infty$. This is known as de la Vallée Poussin's criterion [Bal89, Remark 3].

The fundamental theorem of Young measures has a natural generalization in the case where Ω is replaced by a manifold and u_n are sections of a vector bundle equipped with a norm, namely, sections of a Finsler bundle:

Theorem 2.2 *Let $(\mathcal{M}, \mathfrak{g})$ be a compact Riemannian manifold. Let $E \to \mathcal{M}$ be a Finsler vector bundle. Let (ξ_n) be a sequence of measurable sections of E, bounded in $L^1(\mathcal{M}; E)$. Then, there exists a subsequence (ξ_n) and a family $(\nu_x)_{x\in\mathcal{M}}$ of Radon probability measures on E_x, depending measurably on x, such that*

$$\psi \circ \xi_n \rightharpoonup \left\{ x \mapsto \int_{E_x} \psi_x(\lambda)\, d\nu_x(\lambda) \right\} \quad \text{in } L^1(\mathcal{M}; W), \tag{2.3}$$

for every Finsler vector bundle $W \to \mathcal{M}$ and every continuous bundle map $\psi : E \to W$, satisfying that $(\psi \circ \xi_n)$ is sequentially weakly relatively compact in $L^1(\mathcal{M}; W)$.

The above theorem makes use of the following definitions. Its proof follows the proof of the Euclidean case (as in [Bal89]), with natural adaptations.

Definition 2.3 *Let $(\mathcal{M}, \mathfrak{g})$ be a compact Riemannian manifold. Let $W, E \to \mathcal{M}$ be Finsler vector bundles.*

1. *The space $C_0(E; W)$ is the space of continuous bundle maps (not necessarily linear) $E \to W$ that are decaying fiberwise. That is, if $h \in C_0(E; W)$, then for every $x \in \mathcal{M}$,*

$$\lim_{E_x \ni e \to \infty} |h_x(e)|_{W_x} = 0.$$

2. *$M(E)$ is the bundle of bounded Radon measures on E. A section μ of $M(E)$ is measurable (more accurately weak-$*$-measurable) if for every bundle map $f \in C_0(E; \mathbb{R})$, the real-valued function*

$$\left\{ x \mapsto \int_{E_x} f_x(e)\, d\mu_x(e) \right\} : \mathcal{M} \to \mathbb{R}$$

is measurable; note that this implies the measurability of

$$\left\{x \mapsto \int_{E_x} f_x(e)\, d\mu_x(e)\right\} : \mathcal{M} \to W$$

for every $f \in C_0(E; W)$.

3. $h_n \to h$ *weakly in* $L^1(\mathcal{M}; W)$ *if for every* $\phi \in L^\infty(W^*)$,

$$\int_{\mathcal{M}} \phi \circ h_n \; dVol_{\mathfrak{g}} \to \int_{\mathcal{M}} \phi \circ h \; dVol_{\mathfrak{g}}$$

where $L^\infty(W^*)$ *is the space of essentially bounded measurable vector bundle morphisms* $\phi : W \to \mathcal{M} \times \mathbb{R}$. *Note that while, generally, the composition of measurable functions is not measurable, in this case the fiberwise linearity of* ϕ *implies that the composition amounts to a scalar multiplication of vectors, which is measurable.*

2.3 Intrinsic determinant and cofactor

The determinant and the cofactor of a matrix are encountered in every elementary course in linear algebra. While the determinant has a well-known generalization for linear maps between inner-product spaces, the notion of a cofactor of a linear map is less common. In this section, we present an intrinsic coordinate-free definition of both determinant and cofactor of a linear map. These definitions are used further below in a natural, coordinate-free analysis of mappings between Riemannian manifolds. For completeness, expressions in local coordinates are derived in Appendix A.

Definition 2.4 (determinant) Let V and W be d-dimensional, oriented, inner-product spaces. Let $\star_V^k : \Lambda_k(V) \to \Lambda_{d-k}(V)$ *and* $\star_W^k : \Lambda_k(W) \to \Lambda_{d-k}(W)$ *be their respective Hodge-dual operators. Let* $A \in \operatorname{Hom}(V, W)$. *The* ***determinant*** *of* A, $\operatorname{Det} A \in \mathbb{R}$, *is defined by*

$$\operatorname{Det} A := \star_W^d \circ \bigwedge^d A \circ \star_V^0,$$

where $\bigwedge^d A = A \wedge \ldots \wedge A$, *d times, and we identify* $\bigwedge^0 V \simeq \bigwedge^0 W \simeq \mathbb{R}$.

This definition of the determinant coincides with the determinant of the matrix representing A with respect to any orthonormal bases of V and W. In particular, let $(\mathcal{M}, \mathfrak{g})$ and $(\mathcal{N}, \mathfrak{h})$ be oriented d-dimensional Riemannian manifolds. We denote by

$\star_{\mathcal{M}}^k : \Lambda_k(T\mathcal{M}) \to \Lambda_{d-k}(T\mathcal{M})$ and $\star_{\mathcal{N}}^k : \Lambda_k(T\mathcal{N}) \to \Lambda_{d-k}(T\mathcal{N})$ the Hodge-dual operators of the tangent bundles (note that the Hodge-dual in Riemannian settings usually applies to the exterior algebra of the cotangent bundle). Let $f : \mathcal{M} \to \mathcal{N}$ a differentiable mapping. Then

$$\operatorname{Det} df = \star_{\mathcal{N}}^d \circ \bigwedge^d df \circ \star_{\mathcal{M}}^0.$$

Some properties of the intrinsic determinant are proved in Appendix A. In particular the determinant of the differential is associated with a scalar scaling of the volume forms, namely,

$$\operatorname{Det} df = \frac{f^\star d\mathrm{Vol}_{\mathfrak{h}}}{d\mathrm{Vol}_{\mathfrak{g}}}.$$

Definition 2.5 (cofactor operator) Let V and W be d-dimensional, oriented, inner-product spaces. Let $A \in \operatorname{Hom}(V,W)$. The cofactor of A, $\operatorname{Cof} A \in \operatorname{Hom}(V,W)$, is defined by

$$\operatorname{Cof} A := (-1)^{d-1} \star_W^{d-1} \circ \bigwedge^{d-1} A \circ \star_V^1,$$

where we identify $\bigwedge^1 V \simeq V$ and $\bigwedge^1 W \simeq W$.

Properties of the cofactor $\operatorname{Cof} A$ are presented in Appendix A. In particular, we prove that the following identities, which are intrinsic versions of the well-known properties of the matrix-cofactor,

$$\operatorname{Det} A \ \mathrm{Id}_V = A^T \circ \operatorname{Cof} A = (\operatorname{Cof} A)^T \circ A,$$

and

$$\operatorname{Det} A \ \mathrm{Id}_W = A \circ (\operatorname{Cof} A)^T = \operatorname{Cof} A \circ A^T.$$

An immediate corollary is:

Corollary 2.6 Let V and W be d-dimensional, oriented, inner-product spaces. Let $A \in \operatorname{Hom}(V,W)$. Then $A \in \mathrm{SO}(V,W)$ if and only if $\operatorname{Det} A = 1$ and $\operatorname{Cof} A = A$.

In the context of differentiable mappings, $f : \mathcal{M} \to \mathcal{N}$, this corollary implies that

Corollary 2.7 Let $(\mathcal{M}, \mathfrak{g})$ and $(\mathcal{N}, \mathfrak{h})$ be oriented d-dimensional manifolds. Then,

$$df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$$

if and only if

$$\operatorname{Det} df = 1 \quad \text{and} \quad \operatorname{Cof} df = df.$$

Remarks:

1. For $d > 2$, it can easily be checked that $A \in \mathrm{SO}(V, W)$ if and only if $\operatorname{Cof} A = A \neq 0$ (the condition on the determinant is satisfied automatically). For $d = 2$, $\operatorname{Cof} : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ is a linear operator; the set

$$\{A \in \mathrm{Hom}(V, W) \ : \ \operatorname{Cof} A = A\}$$

is a linear subspace, consisting of all conformal maps.

2. The characterization of isometries through cofactors and the role of dimension can be illuminated by the following simple heuristic argument: $\operatorname{Cof} A$ defines the action of A on $(d-1)$-dimensional parallelepipeds, i.e., it determines volume changes of $(d-1)$-dimensional shapes, whereas, A determines volume changes of 1-dimensional shapes (lengths). When $d - 1 \neq 1$, the condition $\operatorname{Cof} A = A$ implies that metric changes in two different dimensions are fully correlated, which is a rigidity constraint, forcing A to be either trivial, or an isometry (cf. $r^{d-1} = r$ if and only if $r = 0$ or $r = 1$).

2.4 Vector-valued forms

The differential df of a map $f : \mathcal{M} \to \mathcal{N}$ is a section of the vector bundle $T\mathcal{M} \otimes f^*T\mathcal{N}$ over $\mathcal{M}$; alternatively, it can be viewed as an $f^*T\mathcal{N}$-valued 1-form on $\mathcal{M}$. In this section, we present the properties of vector-valued forms that are used in the proof of Liouville's theorem. Proofs are provided in Appendix B.

Definition 2.8 (Hodge-dual of vector-valued forms) Let V be a d-dimensional, oriented inner-product space, and let W be a finite dimensional vector space. We define the Hodge-dual operator

$$\star^k_{V,W} : \Lambda_k(V) \otimes W \to \Lambda_{d-k}(V) \otimes W,$$

by the linear extension of

$$\star^k_{V,W} (v \otimes w) = (\star^k_V v) \otimes w, \qquad v \in \Lambda_k(V), \ w \in W, \tag{2.4}$$

where $\star^k_V$ is the standard Hodge-dual on $\Lambda_k(V)$.

The following property of $\star^k_{V,W}$ is an immediate consequence of its definition, along with the well-known property for scalar-valued forms:

Lemma 2.9 *[Duality of the Hodge-dual] Let V and W be as in Definition 2.8. The Hodge-dual operators $\star^k_{V,W}$ are isomorphisms satisfying*

$$\star^{d-k}_{V,W}\star^k_{V,W} = (-1)^{k(d-k)}\,\mathrm{Id}\,.$$

The following lemma, provides a useful characterization of the Hodge-dual for vector-valued forms, when the target space W is endowed with an inner-product:

Lemma 2.10 (Characterization of the Hodge-dual) The Hodge-dual $\star^k_{V,W}$ is the unique linear operator

$$\Lambda_k(V)\otimes W \to \Lambda_{d-k}(V)\otimes W,$$

satisfying

$$\mathrm{tr}_W\left(\alpha\wedge\star^k_{V,W}\beta\right) = (\alpha,\beta)_{V,W}\,(\star^0_V 1) \tag{2.5}$$

for every $\alpha,\beta\in\Lambda_k(V)\otimes W$; note that this holds independently of the inner-product on W.

Note that if V is replaced with V^*, then by definition, $\star^0_{V^*}1 = \mathrm{Vol}_V$, hence

$$\mathrm{tr}_W\left(\alpha\wedge\star^k_{V^*,W}\beta\right) = (\alpha,\beta)_{V^*,W}\,\mathrm{Vol}_V \tag{2.6}$$

for every $\alpha,\beta\in\Lambda_k(V^*)\otimes W$.

The following lemma shows that linear operators commute with the Hodge-dual operator:

Lemma 2.11 *Let V,W be as in Definition 2.8 Let $A\in\mathrm{Hom}(V,W)\simeq\Lambda_1(V^*)\otimes W$. Then,*

$$(\star^1_{V^*,W}A)(v_1,\dots,v_{d-1}) = (-1)^{d-1}A\left(\star^{d-1}_V(v_1\wedge\cdots\wedge v_{d-1})\right)$$

for every $v_1,\dots,v_{d-1}\in V$.

Thus far, we considered linear maps between arbitrary inner-product space. We next turn to consider vector-valued forms on Riemannian manifolds. Let $(\mathcal{M},\mathfrak{g})$ be a d-dimensional Riemannian manifold. Let E be a vector bundle over $\mathcal{M}$ (of arbitrary finite rank n), endowed with a Riemannian metric $\mathfrak{h}$ and a metric affine connection ∇^E.

We start by establishing the commutation between the Hodge-dual operator and covariant differentiation. Note that ∇^E induces a connection on $\Lambda_k(E)$ (also denoted by ∇^E); this induced connection is compatible with the metric induced on $\Lambda_k(E)$ by $\mathfrak{h}$.

Lemma 2.12 (Hodge-dual commutes with covariant derivative) Let $(E, \mathcal{M})$ be defined as above. Denote by $\star^k_E$ the fiber-wise Hodge-dual $\Lambda_k(E) \to \Lambda_{n-k}(E)$. Then,

$$\star^k_E(\nabla^E_X \beta) = \nabla^E_X(\star^k_E \beta)$$

for every $\beta \in \Gamma(\Lambda_k(E))$ and $X \in \Gamma(T\mathcal{M})$.

We denote by

$$\Omega^k(\mathcal{M}; E) = \Gamma(\Lambda_k(T^*\mathcal{M}) \otimes E)$$

the space of k-forms on $\mathcal{M}$ with values in E. The metrics on $\mathcal{M}$ and E induce a metric on $\Omega^k(\mathcal{M}; E)$, denoted $\langle \cdot, \cdot \rangle_{\mathfrak{g},\mathfrak{h}}$. The Hodge-dual operator

$$\star^k_{T^*\mathcal{M},E} : \Omega^k(\mathcal{M}; E) \to \Omega^{d-k}(\mathcal{M}; E)$$

is defined fiber-wise.

Definition 2.13 (Covariant exterior derivative) The covariant exterior derivative,

$$d_\nabla : \Omega^k(\mathcal{M}; E) \to \Omega^{k+1}(\mathcal{M}; E),$$

is defined by the linear extension of

$$d_\nabla(\omega \otimes \eta) = d\omega \otimes \eta + (-1)^k \omega \wedge \nabla^E \eta, \qquad \omega \in \Omega^k(\mathcal{M}),\ \eta \in \Gamma(E).$$

It is well-known that d_∇ is given by the formula

$$\begin{aligned} d_\nabla \sigma(X_1, \cdots, X_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} \nabla^E_{X_i} \left(\sigma(X_1, \ldots, \widehat{X}_i \ldots, X_{k+1}) \right) \\ & + \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}), \end{aligned} \tag{2.7}$$

where $\sigma \in \Omega^k(\mathcal{M}; E)$. Note that the covariant exterior derivative only depends on the connection on E and not on the connection on $\mathcal{M}$.

We conclude this section by introducing the covariant coderivative:

Definition 2.14 (Covariant coderivative) The covariant coderivative,

$$\delta_{\nabla^E} : \Omega^k(\mathcal{M}; E) \to \Omega^{k-1}(\mathcal{M}; E),$$

is defined by the relation,

$$\int_{\mathcal{M}} \langle \sigma, \delta_{\nabla^E} \rho \rangle_{\mathfrak{g},\mathfrak{h}} \; Vol_{\mathfrak{g}} = \int_{\mathcal{M}} \langle d_{\nabla^E} \sigma, \rho \rangle_{\mathfrak{g},\mathfrak{h}} \; Vol_{\mathfrak{g}},$$

for all $\rho \in \Omega^k(\mathcal{M}; E)$ *and compactly-supported* $\sigma \in \Omega^{k-1}(\mathcal{M}; E)$.

Lemma 2.15 The covariant coderivative is given by

$$\delta_{\nabla^E} = (-1)^{dk+d+1} \star^{d-k+1}_{T^*\mathcal{M},E} d_{\nabla^E} \star^{k}_{T^*\mathcal{M},E} .$$

3 Liouville's theorem for Lipschitz mappings

In this section we prove the following generalization of Liouville's theorem for Lipschitz maps between manifolds:

Theorem 3.1 (Liouville's rigidity for Lipschitz functions) Let $f \in W^{1,\infty}(\mathcal{M}; \mathcal{N})$ *satisfy* $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ *almost everywhere. Then* f *is a smooth isometric immersion.*

Theorem 3.1 is basically a statement about regularity, since for smooth mappings, $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ almost everywhere implies $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ everywhere. The smoothness of f is the consequence of the following key proposition:

Proposition 3.2 (A.e. local isometries are harmonic) Let $f \in W^{1,\infty}(\mathcal{M}; \mathcal{N})$ *satisfy* $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ *almost everywhere. Then,* f *is weakly-harmonic in the sense of [Hél02]:*

$$\int_{\mathcal{M}} \left\langle d(\iota \circ f), \nabla^{\mathcal{M}\times\mathbb{R}^D} \xi \right\rangle_{\mathfrak{g},\mathfrak{e}} \; Vol_{\mathfrak{g}} = \int_{\mathcal{M}} \langle \mathrm{tr}_{\mathfrak{g}} \, f^* A(df, df), \xi \rangle_{\mathfrak{e}} \; Vol_{\mathfrak{g}}, \tag{3.1}$$

for all $\xi \in W^{1,2}_0(\mathcal{M}; \mathbb{R}^D) \cap L^\infty(\mathcal{M}; \mathbb{R}^D)$*, where* $\iota : \mathcal{N} \to \mathbb{R}^D$ *is an isometric embedding,* A *is the second fundamental form induced by* ι*, and* $\mathfrak{e}$ *denotes the Euclidean metric on* $\mathbb{R}^D$.
In particular, f *is smooth.*

The fact that f is smooth follows from a well known regularity theorem for continuous, weakly-harmonic mappings [Hél02, Theorem 1.5.1]. Since $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$, implies $\mathrm{Cof}\, df = df$ (Corollary 2.7), the fact that f is weakly-harmonic follows from the following generalization of the Euclidean "Div-Cof-Grad=0" identity:

Proposition 3.3 (Div-Cof-Grad=0, weak formulation) Let $f \in W^{1,p}(\mathcal{M};\mathcal{N})$ where $p \geq 2(d-1)$ ($p > 2$ if $d = 2$). Let $\iota : \mathcal{N} \to \mathbb{R}^D$ denote an isometric embedding of $\mathcal{N}$ in $\mathbb{R}^D$ with second fundamental form A. Then,

$$\int_{\mathcal{M}} \left\langle f^* d\iota \circ \operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D} \xi \right\rangle_{\mathfrak{g},\mathfrak{e}} Vol_{\mathfrak{g}} = \int_{\mathcal{M}} \langle \operatorname{tr}_{\mathfrak{g}} f^* A(\operatorname{Cof} df, df), \xi \rangle_{\mathfrak{e}} \, Vol_{\mathfrak{g}}, \tag{3.2}$$

for all $\xi \in W_0^{1,2}(\mathcal{M};\mathbb{R}^D) \cap L^\infty(\mathcal{M};\mathbb{R}^D)$.

Comments:

1. Eq. (3.2) reads as follows in local coordinates: Let the indices i, j denote coordinates on $\mathcal{M}$, let the indices α, β denote coordinates on $\mathcal{N}$ and let the indices a, b denote coordinates on $\mathbb{R}^D$. We denote by $\mathfrak{g}_{ij}$ the entries of the metric $\mathfrak{g}$, by $\mathfrak{g}^{ij}$ the entries of the matrix inverse to $\mathfrak{g}_{ij}$, $|\mathfrak{g}| = \det \mathfrak{g}_{ij}$, and A^a are the entries of the second fundamental form induced by ι. The differential df has entries $\partial_i f^\alpha$. We also denote by $\partial_i f$ the vectors in $T\mathcal{N}$, whose entries are $\partial_i f^\alpha$; the same holds for $\operatorname{Cof} df = (\operatorname{Cof} df)_i^\alpha$. Then,(3.2) reads

$$\int_{\mathcal{M}} \mathfrak{g}^{ij} \, \partial_\alpha \iota^a (\operatorname{Cof} df)_i^\alpha \, \delta_{ab} \, \partial_j \xi^b \, \sqrt{|\mathfrak{g}|} \, dx = \int_{\mathcal{M}} \mathfrak{g}^{ij} \, A^a((\operatorname{Cof} df)_i, \partial_j f) \, \delta_{ab} \, \xi^b \, \sqrt{|\mathfrak{g}|} \, dx,$$

   where ι and A are evaluated at $f(x)$.

2. In view of the last remark in Section 2.3, in the two-dimensional case, Proposition 3.3 is stronger than Proposition 3.2: Let $f \in W^{1,p}(\mathcal{M};\mathcal{N})$, $\dim \mathcal{M} = \dim \mathcal{N} = 2$ and let $p > 2$. If df is either 0 or a conformal map a.e., then f is weakly-harmonic, and in particular smooth. This was already known for $f \in C^2$ (see [HW08, Section 2.2, Example 11]), and Proposition 3.3 generalizes it to less regular maps.

The next subsection proves Proposition 3.3 and related identities.

3.1 Generalization of the "Div-Cof-Grad=0" relation

We begin this section by proving a strong formulation of the "Div-Cof-Grad=0" relation for mappings between manifolds. This relation serves as the main step in proving Proposition 3.3, but it is also interesting in its own. Note that the formulation of the relation does not require embedding the target space into a larger Euclidean space. In this sense, it is intrinsic. Note also that we do not require here the manifolds to be compact, unlike in the rest of this paper.

Proposition 3.4 (Div-Cof-Grad=0, intrinsic formulation) Let $(\mathcal{M}, \mathfrak{g})$ and $(\mathcal{N}, \mathfrak{h})$ be oriented, d-dimensional Riemannian manifolds. Let $f \in C^\infty(\mathcal{M}, \mathcal{N})$. Then, for every compactly supported $\chi \in \Gamma(f^*T\mathcal{N})$,

$$\int_{\mathcal{M}} \langle \operatorname{Cof} df, d_\nabla \chi \rangle_{\mathfrak{g},\mathfrak{h}} \, Vol_{\mathfrak{g}} = 0. \tag{3.3}$$

Equivalently,

$$\delta_{\nabla^{f^*T\mathcal{N}}} \operatorname{Cof} df = 0.$$

Using the same index conventions as above, (3.3) reads in local coordinates:

$$\int_{\mathcal{M}} (\operatorname{Cof} df)^\alpha_i \, \mathfrak{g}^{ij} \mathfrak{h}_{\alpha\beta} \left(\partial_j \xi^\beta + \partial_j f^\gamma \Gamma^\beta_{\gamma\delta} \xi^\delta \right) \sqrt{|\mathfrak{g}|} \, dx = 0,$$

where $\mathfrak{h}_{\alpha\beta}$ are the entries of the metric $\mathfrak{h}$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $\nabla^{\mathcal{N}}$ and both $\mathfrak{h}$ and Γ are evaluated at $f(x)$.

Proof: By Lemma 2.15 (with $E = f^*T\mathcal{N}$), we need to prove that

$$\star^d_{T^*\mathcal{M}, f^*T\mathcal{N}} \, d_\nabla \star^1_{T^*\mathcal{M}, f^*T\mathcal{N}} \operatorname{Cof} df = 0.$$

Since the Hodge-dual operator is an isomorphism, this amount to proving that

$$d_\nabla \star^1_{T^*\mathcal{M}, f^*T\mathcal{N}} \operatorname{Cof} df = 0.$$

By Definition 2.5 of the cofactor operator,

$$\operatorname{Cof} df(u) = (-1)^{d-1} \star^{d-1}_{f^*T\mathcal{N}} \left(\wedge^{d-1} df \right) \star^1_{T\mathcal{M}} u.$$

Appying Lemma 2.11, for every sequence $X_1, \dots, X_{d-1} \in \Gamma(T\mathcal{M})$ of vector fields,

$$\begin{aligned} \left(\star^1_{T^*\mathcal{M}, f^*T\mathcal{N}} \operatorname{Cof} df \right) (X_1, \dots, X_{d-1}) &= (-1)^{d-1} \operatorname{Cof} df (\star^{d-1}_{T\mathcal{M}} (X_1 \wedge \cdots \wedge X_{d-1})) \\ &= \star^{d-1}_{f^*T\mathcal{N}} \left(\wedge^{d-1} df \right) \star^1_{T\mathcal{M}} \star^{d-1}_{T\mathcal{M}} (X_1 \wedge \cdots \wedge X_{d-1}) \\ &= (-1)^{d-1} \star^{d-1}_{f^*T\mathcal{N}} \left(df(X_1) \wedge \cdots \wedge df(X_{d-1}) \right), \end{aligned} \tag{3.4}$$

where we used the dual property of $\star^k_{T^*\mathcal{M}}$ (Lemma 2.9). Since our goal is to prove that $d_\nabla \star^1_{T^*\mathcal{M}, f^*T\mathcal{N}} \operatorname{Cof} df = 0$, we will henceforth omit the $(-1)^{d-1}$ factor.

By (2.7) and (3.4),

$$
\begin{aligned}
&(d_\nabla \star^1_{T^*\mathcal{M},f^*T\mathcal{N}} \operatorname{Cof} df)(X_1,\dots,X_d) \\
&\qquad = \sum_{j=1}^d (-1)^{j-1} \nabla^{f^*T\mathcal{N}}_{X_j} \star^{d-1}_{f^*T\mathcal{N}} \Big(df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad\quad + \sum_{i<j} (-1)^{i+j} \star^{d-1}_{f^*T\mathcal{N}} \Big(df([X_i,X_j]) \wedge df(X_1)\wedge\cdots\wedge \widehat{df(X_i)} \wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big)
\end{aligned}
$$

Note that for every j,

$$
\begin{aligned}
&\nabla^{f^*T\mathcal{N}}_{X_j} \star^{d-1}_{f^*T\mathcal{N}} \Big(df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad = \star^{d-1}_{f^*T\mathcal{N}} \nabla^{f^*T\mathcal{N}}_{X_j} \Big(df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad = \sum_{i\neq j} \star^{d-1}_{f^*T\mathcal{N}} \Big(df(X_1)\wedge\cdots\wedge \nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) \wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad = \sum_{i<j} (-1)^{i-1} \star^{d-1}_{f^*T\mathcal{N}} \Big(\nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) \wedge df(X_1)\wedge\cdots\wedge \widehat{df(X_i)} \wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad\quad + \sum_{i>j} (-1)^{i} \star^{d-1}_{f^*T\mathcal{N}} \Big(\nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) \wedge df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge \widehat{df(X_i)} \wedge\cdots\wedge df(X_d)\Big),
\end{aligned}
$$

where in the first equality we used the commutation of the covariant derivative and the Hodge-dual operator (Lemma 2.12). Note that here j is fixed, so in the last equality (and only there) the symbol $\sum_{i<j}$ represent summation only over i, and not over both i and j. It follows that

$$
\begin{aligned}
&\sum_{j=1}^d (-1)^{j-1} \nabla^{f^*T\mathcal{N}}_{X_j} \star^{d-1}_{f^*T\mathcal{N}} \Big(df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad = \sum_{i<j} (-1)^{i+j} \star^{d-1}_{f^*T\mathcal{N}} \Big(\nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) \wedge df(X_1)\wedge\cdots\wedge \widehat{df(X_i)} \wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge df(X_d)\Big) \\
&\qquad\quad - \sum_{i>j} (-1)^{i+j} \star^{d-1}_{f^*T\mathcal{N}} \Big(\nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) \wedge df(X_1)\wedge\cdots\wedge \widehat{df(X_j)} \wedge\cdots\wedge \widehat{df(X_i)} \wedge\cdots\wedge df(X_d)\Big).
\end{aligned}
$$

From the symmetry of the Levi-Civita connection of $T\mathcal{N}$, it follows that

$$
\nabla^{f^*T\mathcal{N}}_{X_i}(df(X_j)) - \nabla^{f^*T\mathcal{N}}_{X_j}(df(X_i)) = df([X_i,X_j]),
$$

therefore,

$$\sum_{j=1}^{d}(-1)^{j-1}\nabla^{f^*T\mathcal{N}}_{X_j}\star^{d-1}_{f^*T\mathcal{N}}\left(df(X_1)\wedge\cdots\wedge\widehat{df(X_j)}\wedge\cdots\wedge df(X_d)\right)$$
$$=\sum_{i<j}(-1)^{i+j}\star^{d-1}_{f^*T\mathcal{N}}\left(df([X_j,X_i])\wedge df(X_1)\wedge\cdots\wedge\widehat{df(X_i)}\wedge\cdots\wedge\widehat{df(X_j)}\wedge\cdots\wedge df(X_d)\right)$$

Thus, we finally obtain

$$\begin{aligned}(d_\nabla\star^1_{T^*\mathcal{M},f^*T\mathcal{N}}\operatorname{Cof}df)(X_1,\dots,X_d)\\ &=\sum_{i<j}(-1)^{i+j}\star^{d-1}_{f^*T\mathcal{N}}\left(df([X_j,X_i])\wedge df(X_1)\wedge\cdots\wedge\widehat{df(X_i)}\wedge\cdots\wedge\widehat{df(X_j)}\wedge\cdots\wedge df(X_d)\right)\\ &\qquad+(-1)^{i+j}\star^{d-1}_{f^*T\mathcal{N}}\left(df([X_i,X_j])\wedge df(X_1)\wedge\cdots\wedge\widehat{df(X_i)}\wedge\cdots\wedge\widehat{df(X_j)}\wedge\cdots\wedge df(X_d)\right)\\ &=0,\end{aligned}$$

which completes the proof.

■

An immediate corollary of Proposition 3.4 and Corollary 2.7 is the well-known fact that smooth isometric immersions between manifolds of the same dimensions are harmonic:

Corollary 3.5 *Let* $(\mathcal{M},\mathfrak{g})$ *and* $(\mathcal{N},\mathfrak{h})$ *be oriented, d-dimensional Riemannian manifolds, and let* $f\in C^\infty(\mathcal{M};\mathcal{N})$. *If*

$$df\in\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h}),$$

then for every compactly-supported $\chi\in\Gamma(f^*T\mathcal{N})$,

$$\int_{\mathcal{M}}\left\langle df,\nabla^{f^*T\mathcal{N}}\chi\right\rangle_{\mathfrak{g},\mathfrak{h}}\,Vol_{\mathfrak{g}}=0. \tag{3.5}$$

Proof of Proposition 3.3: As a first stage, we show that (3.2) holds for every $f\in C^\infty(\mathcal{M},\mathcal{N})$. Given an isometric embedding $\iota:(\mathcal{N},\mathfrak{h})\to(\mathbb{R}^D,\mathfrak{e})$,

$$d\iota:T\mathcal{N}\to\mathcal{N}\times\mathbb{R}^D\qquad\text{and}\qquad f^*d\iota:f^*T\mathcal{N}\to\mathcal{M}\times\mathbb{R}^D.$$

Then, Proposition 3.4 can be rewritten as

$$\int_{\mathcal{M}}\left\langle f^*d\iota\circ\operatorname{Cof}df,f^*d\iota\circ\nabla^{f^*T\mathcal{N}}\chi\right\rangle_{\mathfrak{g},\mathfrak{e}}\mathrm{Vol}_{\mathfrak{g}}=0 \tag{3.6}$$

for all $\chi \in \Gamma_0(f^*T\mathcal{N})$.

Denote by $N\mathcal{N}$ the normal bundle of $\iota(\mathcal{N})$ in $\mathbb{R}^D$, that is, $N\mathcal{N} \subset \mathcal{N}\times\mathbb{R}^D$ is the orthogonal complement of $d\iota(T\mathcal{N})$ in $(\mathcal{N}\times\mathbb{R}^D, \mathfrak{e})$. Denote by P and $P^\perp$ the orthogonal projections of $\mathcal{N}\times\mathbb{R}^D$ into $d\iota(T\mathcal{N})$ and $N\mathcal{N}$. For a section $\zeta \in \Gamma(T\mathcal{N})$, the Levi-Civita connection on $T\mathcal{N}$ is induced by the Levi-Civita connection on the trivial bundle, $\mathcal{N}\times\mathbb{R}^D$, by the classical relation

$$d\iota \circ \nabla^{T\mathcal{N}}\zeta = P\left(\nabla^{\mathcal{N}\times\mathbb{R}^D}(d\iota\circ\zeta)\right).$$

Let $\zeta \in \Gamma(T\mathcal{N})$ have compact support in $f(\mathcal{M})$. Then, $f^*\zeta \in \Gamma_0(f^*T\mathcal{N})$, and

$$\begin{aligned} f^*d\iota \circ \nabla^{f^*T\mathcal{N}} f^*\zeta &= f^*\left(d\iota\circ\nabla^{T\mathcal{N}}\zeta\right) \\ &= f^*\left(P\left(\nabla^{\mathcal{N}\times\mathbb{R}^D}(d\iota\circ\zeta)\right)\right) \\ &= (f^*P)\left(\nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*d\iota\circ f^*\zeta)\right), \end{aligned}$$

where in the last step we used the fact that $f^*\nabla^{\mathcal{N}\times\mathbb{R}^D} = \nabla^{\mathcal{M}\times\mathbb{R}^D}$. Since sections of the form $f^*\zeta$ span $\Gamma(f^*T\mathcal{N})$ locally, it follows that

$$f^*d\iota \circ \nabla^{f^*T\mathcal{N}}\chi = (f^*P)\left(\nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*d\iota\circ\chi)\right). \tag{3.7}$$

Next, we note that

$$f^*d\iota\circ\chi \in \Gamma_0(f^*d\iota(T\mathcal{N})) \subset \Gamma_0(M\times\mathbb{R}^D).$$

Sections in $\Gamma_0(f^*d\iota(T\mathcal{N}))$ can be represented by sections in $\Gamma_0(M\times\mathbb{R}^D)$ projected onto $f^*d\iota(T\mathcal{N})$. That is, setting $f^*d\iota\circ\chi = (f^*P)(\xi)$, and combining (3.6), (3.7) we get

$$\int_{\mathcal{M}} \left\langle f^*d\iota\circ\operatorname{Cof} df, (f^*P)\left(\nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*P)(\xi)\right)\right\rangle_{\mathfrak{g},\mathfrak{e}} \operatorname{Vol}_{\mathfrak{g}} = 0$$

for all $\xi \in \Gamma_0(\mathcal{M}\times\mathbb{R}^D)$. Since $f^*d\iota\circ\operatorname{Cof} df \in \Gamma(f^*d\iota(T\mathcal{N}))$, the outer projection can be omitted, yielding,

$$\int_{\mathcal{M}} \left\langle f^*d\iota\circ\operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*P)(\xi)\right\rangle_{\mathfrak{g},\mathfrak{e}} \operatorname{Vol}_{\mathfrak{g}} = 0.$$

Next, set $(f^*P)(\xi) = \xi - (f^*P^\perp)(\xi)$. Then, for all $\xi \in \Gamma_0(\mathcal{M}\times\mathbb{R}^D)$,

$$\begin{aligned} &\int_{\mathcal{M}} \left\langle f^*d\iota\circ\operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D}\xi\right\rangle_{\mathfrak{g},\mathfrak{e}} \operatorname{Vol}_{\mathfrak{g}} \\ &\qquad = \int_{\mathcal{M}} \operatorname{tr}_{\mathfrak{g}} \left\langle f^*d\iota\circ\operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*P^\perp)(\xi)\right\rangle_{\mathfrak{e}} \operatorname{Vol}_{\mathfrak{g}}, \end{aligned} \tag{3.8}$$

where on the right-hand side, we have separated the inner-product on $T^*\mathcal{M}\otimes\mathbb{R}^D$ into, first, an inner-product over $\mathbb{R}^D$, followed by a trace over $T^*\mathcal{M}$.

Let $A : T\mathcal{N}\times T\mathcal{N} \to N\mathcal{N}$ be the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^D$. That is,

$$\langle A(u,v),\eta\rangle_{\mathfrak{e}} = \left\langle d\iota\circ u, \nabla_v^{\mathcal{N}\times\mathbb{R}^D}\eta\right\rangle_{\mathfrak{e}},$$

for $u,v\in\Gamma(T\mathcal{N})$ and $\eta\in\Gamma(N\mathcal{N})$. Pulling back with f,

$$\langle f^*A(u,df(X)),\eta\rangle_{\mathfrak{e}} = \left\langle f^*d\iota\circ u, \nabla_X^{\mathcal{M}\times\mathbb{R}^D}\eta\right\rangle_{\mathfrak{e}},$$

for $u\in\Gamma(f^*T\mathcal{N})$, $X\in\Gamma(T\mathcal{M})$ and $\eta\in\Gamma(f^*N\mathcal{N})$. Setting $\eta=(f^*P^\perp)(\xi)$ and $u=\operatorname{Cof} df(X)$,

$$\langle f^*A(\operatorname{Cof} df(X),df(X)),(f^*P^\perp)(\xi)\rangle_{\mathfrak{e}} = \left\langle f^*d\iota\circ \operatorname{Cof} df(X), \nabla_X^{\mathcal{M}\times\mathbb{R}^D}(f^*P^\perp)(\xi)\right\rangle_{\mathfrak{e}}.$$

Since the range of A is the $N\mathcal{N}$, the projection $f^*P^\perp$ on the left-hand side can be omitted. Moreover, replacing the vector field X by the components X_i of an orthonormal frame field, and summing over i, we obtain

$$\langle \operatorname{tr}_{\mathfrak{g}} f^*A(\operatorname{Cof} df, df),\xi\rangle_{\mathfrak{e}} = \operatorname{tr}_{\mathfrak{g}}\left\langle f^*d\iota\circ\operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D}(f^*P^\perp)(\xi)\right\rangle_{\mathfrak{e}}.$$

Substituting this last identity into (3.8), we finally obtain

$$\int_{\mathcal{M}}\left\langle f^*d\iota\circ\operatorname{Cof} df, \nabla^{\mathcal{M}\times\mathbb{R}^D}\xi\right\rangle_{\mathfrak{g},\mathfrak{e}}\operatorname{Vol}_{\mathfrak{g}} = \int_{\mathcal{M}}\langle \operatorname{tr}_{\mathfrak{g}} f^*A(\operatorname{Cof} df, df),\xi\rangle_{\mathfrak{e}}\operatorname{Vol}_{\mathfrak{g}},$$

for all $f\in C^\infty(\mathcal{M},\mathcal{N})$ and all $\xi\in\Gamma_0(\mathcal{M}\times\mathbb{R}^D)$.

It remains to show that this identity holds for all $f\in W^{1,p}(\mathcal{M};\mathcal{N})$ and all $\xi\in W^{1,2}_0(\mathcal{M};\mathbb{R}^D)\cap L^\infty(\mathcal{M};\mathbb{R}^D)$. This follows by first approximating f by smooth functions in the $W^{1,p}$ topology (this is possible since $p\ge d$), and then approximating ξ with smooth sections of $\mathcal{M}\times\mathbb{R}^D$ in the $W^{1,2}$ topology. Since $p\ge 2(d-1)$, then $f^*d\iota\circ\operatorname{Cof} df\in L^2(\mathcal{M};T^*\mathcal{M}\otimes\mathbb{R}^D)$, hence the first integrand is well defined for $\xi\in W^{1,2}(\mathcal{M};\mathbb{R}^D)$. Since $p\ge d$, $\operatorname{tr}_{\mathfrak{g}} f^*A(df,\operatorname{Cof} df)\in L^1(\mathcal{M};\mathbb{R}^D)$, and the second integrand is well-defined for $\xi\in L^\infty(\mathcal{M};\mathbb{R}^D)$. The fact that $f_n^*d\iota\circ\operatorname{Cof} df_n\to f^*d\iota\circ\operatorname{Cof} df$ in L^2 and $f_n^*A(df_n,\operatorname{Cof} df_n)\to f^*A(df,\operatorname{Cof} df)$ in L^1 also hinges on the fact that $p>d$, hence the convergence $f_n\to f$ is uniform. The necessity of uniform convergence is also the reason for assuming $p>2$ for $d=2$, rather than $p\ge 2(d-1)=2$.

■

4 Reshetnyak's rigidity theorem for manifolds

In this section we prove the following theorem, which is a generalization of Reshetnyak's theorem for the case where both source and target spaces are Riemannian manifolds:

Theorem 4.1 *Let* $(\mathcal{M}, \mathfrak{g})$ *and* $(\mathcal{N}, \mathfrak{h})$ *be compact, oriented, d-dimensional Riemannian manifolds with* C^1 *boundary. Let* $1 \le p < \infty$ *and let* $f_n \in W^{1,p}(\mathcal{M}; \mathcal{N})$ *be a sequence of mappings satisfying*

$$\operatorname{dist}_{(\mathfrak{g}, f_n^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})) \to 0 \qquad \text{in } L^p(\mathcal{M}). \tag{4.1}$$

Then, $\mathcal{M}$ *can be immersed isometrically into* $\mathcal{N}$ *and there exists a subsequence of* f_n *converging in* $W^{1,p}(\mathcal{M}; \mathcal{N})$ *to a smooth isometric immersion* $f : \mathcal{M} \to \mathcal{N}$*.*

Moreover, if $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ *and* $Vol_{\mathfrak{g}}\mathcal{M} = Vol_{\mathfrak{h}}\mathcal{N}$*, then* $\mathcal{M}$ *and* $\mathcal{N}$ *are isometric and* f *is an isometry. In particular, these conditions hold if* f_n *are diffeomorphisms.*

Proof: Since the proof is long and technical, we divide it into six steps. In Steps I–III, we assume that $p > d$; in Step I, we show that f_n converges to f uniformly; in Step II, we show that f is an isometric immersion; in Step III, we show that the convergence of f_n to f takes also place in $W^{1,p}(\mathcal{M}; \mathcal{N})$. In Step IV we relax the $p > d$ assumption, and prove that the results of Steps I-III hold for $p \ge 1$. Finally, in Steps V-VI we prove that f is an isometry if the additional assumption on f_n and the equality of volumes are satisfied.

Step I: Prove that f_n *has a uniformly converging subsequence*

As described in Section 2.1, Sobolev maps between manifolds are conveniently defined by first embedding the target manifold isometrically into a higher-dimensional Euclidean space. The idea is to exploit the well-established theory of vector-valued Sobolev functions on Riemannian manifolds.

As before, let $\iota : (\mathcal{N}, \mathfrak{h}) \to (\mathbb{R}^D, \mathfrak{e})$ be a smooth isometric embedding of $\mathcal{N}$, where $\mathfrak{e}$ denotes the standard Euclidean metric on $\mathbb{R}^D$. Let

$$F_n = \iota \circ f_n : \mathcal{M} \to \mathbb{R}^D$$

be the "extrinsic representative" of f_n. For $x \in \mathcal{M}$, denote by $O(\mathfrak{g}_x, \mathfrak{e})$ the set of linear isometries $(T_x\mathcal{M}, \mathfrak{g}_x) \to (\mathbb{R}^D, \mathfrak{e})$. Note that when mapping a vector space into a vector space of higher dimension, there is no notion of preservation of orientation;

in particular, $\mathrm{SO}(\mathfrak{g},\mathfrak{e})$ is not defined. However, since $A \in \mathrm{O}(\mathfrak{g}_x, \mathfrak{h}_{f(x)})$ implies that $d\iota_{f(x)} \circ A \in \mathrm{O}(\mathfrak{g}_x, \mathfrak{e})$, it follows that

$$\begin{aligned}\operatorname{dist}_{(\mathfrak{g},\mathfrak{e})}(dF_n, \mathrm{O}(\mathfrak{g},\mathfrak{e})) &\leq \operatorname{dist}_{(\mathfrak{g},f_n^*\mathfrak{h})}(df_n, \mathrm{O}(\mathfrak{g}, f_n^*\mathfrak{h})) \\ &\leq \operatorname{dist}_{(\mathfrak{g},f_n^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})).\end{aligned}$$

In particular, (4.1) implies that

$$\operatorname{dist}(dF_n, \mathrm{O}(\mathfrak{g},\mathfrak{e})) \to 0 \qquad \text{in } L^p(\mathcal{M}). \tag{4.2}$$

Since, by the compactness of $\mathcal{N}$, $\mathrm{Image}(F_n) \subseteq \iota(\mathcal{N}) \subset \mathbb{R}^D$ is bounded, it follows from the Poincaré inequality that F_n are uniformly bounded in $W^{1,p}(\mathcal{M};\mathbb{R}^D)$. Hence, F_n has a subsequence converging weakly in $W^{1,p}(\mathcal{M};\mathbb{R}^D)$ to a limit $F \in W^{1,p}(\mathcal{M};\mathbb{R}^D)$.

Since $p > d$, it follows from the Rellich-Kondrachov theorem [AF03, Theorem 6.3] that $F_n \to F$ uniformly; since $\iota(\mathcal{N})$ is closed in $\mathbb{R}^D$, it follows that $F(\mathcal{M}) \subset \iota(\mathcal{N})$, i.e., $\iota^{-1} \circ F : \mathcal{M} \to \mathcal{N}$ is well-defined. The compactness of $\mathcal{N}$ implies that the intrinsic and the extrinsic distances on $\mathcal{N}$ are strongly equivalent (see [Coh]). Therefore, $f_n \to \iota^{-1} \circ F$ uniformly; we denote this limit by f; it is in $W^{1,p}(\mathcal{M};\mathcal{N})$ by the very definition of that space.

Step II: Prove that f is an isometric immersion

By Theorem 3.1, it is sufficient to prove that $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ a.e. Note that this is a local statement; thus, it suffices to show that every $x \in \mathcal{M}$ has an open neighborhood in which this property holds. Using local coordinate charts, this statement can be reformulated in terms of mappings between a manifold and a Euclidean space of the same dimension; as already discussed, the equality of dimension is critical for keeping track of orientation-preserving linear maps.

So let $x \in \mathcal{M}$ and let $\phi : U \subset \mathbb{R}^d \to \mathcal{N}$ be a coordinate chart around $f(x) \in \mathcal{N}$. Let $\mathcal{M}'$ be an open neighborhood of x such that $\overline{f(\mathcal{M}')} \subset \phi(U)$. Since $f_n \to f$ uniformly, and the distance between $f(\mathcal{M}')$ and the boundary of $\phi(U)$ is positive, $f_n(\mathcal{M}') \subset \phi(U)$ for n large enough.

In the rest of this step of the proof, we will view f_n and f as mappings $\mathcal{M}' \to U \subset \mathbb{R}^d$; for $y \in U$, $T_yU \simeq \mathbb{R}^d$ will be endowed with either the Euclidean metric $\mathfrak{e}$ or the pullback metric $\phi^\star\mathfrak{h}$, with entries $\mathfrak{h}_{ij}(y) = \mathfrak{h}(\partial_i, \partial_j)|_{\phi(y)}$. Since we can assume that $f_n(\mathcal{M}')$ are all contained in the same compact subset of $\phi(U)$, it follows that we can assume that all the entries $\mathfrak{h}_{ij}$ and $\mathfrak{h}^{ij}$ of the metric and its dual are uniformly continuous, and in particular uniformly bounded.

The uniform boundedness of $\mathfrak{h}_{ij}$ and $\mathfrak{h}^{ij}$ implies that the norms on $T\mathcal{M}' \otimes \mathbb{R}^d$ and $T^*\mathcal{M}' \times \mathbb{R}^d$ induced by (i) $\mathfrak{g}$ and $f_n^*\mathfrak{h}$, (ii) $\mathfrak{g}$ and $f^*\mathfrak{h}$, and (iii) $\mathfrak{g}$ and $\mathfrak{e}$ are all equivalent; moreover, the constants in these equivalences are independent of both n and x (see Lemma D.3). This implies that both weak and strong convergence in $L^q(\mathcal{M}'; T^*\mathcal{M} \otimes \mathbb{R}^d)$ are the same with respect to either of those norms.

As distances in $T\mathcal{M}' \otimes \mathbb{R}^d$ with respect to $(\mathfrak{g}, f_n^*\mathfrak{h})$ and $(\mathfrak{g}, f^*\mathfrak{h})$ are equivalent, (4.1) implies that

$$\operatorname{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})) \to 0 \qquad \text{in } L^p(\mathcal{M}').$$

The uniform boundedness of entries of $\mathfrak{h}$ along with the uniform convergence of f_n to f implies that

$$\operatorname{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(\mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h}), \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) \to 0$$

uniformly in $\mathcal{M}'$, where the distance here is the Hausdorff distance induced by $\operatorname{dist}_{(\mathfrak{g}, \mathfrak{e})}$. Hence

$$\operatorname{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) \to 0 \qquad \text{in } L^p(\mathcal{M}'). \tag{4.3}$$

Comparing (4.3) and (4.1), we replaced the n-dependent set $\mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})$ by the fixed set $\mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$ and the n-dependent metric induced by $\mathfrak{g}$ and $f_n^*\mathfrak{h}$ by the fixed metric induced by $\mathfrak{g}$ and $f^*\mathfrak{h}$. It follows from (4.3) that df_n is uniformly bounded in $L^p(\mathcal{M}'; T^*\mathcal{M}' \otimes \mathbb{R}^d)$. Since, moreover, $f_n(\mathcal{M}')$ is uniformly bounded in $\mathbb{R}^d$, it follows that f_n has a subsequence that weakly converges in $W^{1,p}(\mathcal{M}'; \mathbb{R}^d)$. Since weak convergence in $W^{1,p}(\mathcal{M}'; \mathbb{R}^d)$ implies uniform convergence, the limit coincides with f.

Henceforth, denote by E the vector bundle $T^*\mathcal{M}' \otimes \mathbb{R}^d$ with the metric induced by $\mathfrak{g}$ and $f^*\mathfrak{h}$. Note that we view all the mappings df_n as sections of the same vector bundle E, which is the key reason for using a local coordinate chart for $\mathcal{N}$.

The sequence df_n satisfies the conditions of Theorem 2.2, including the boundedness in L^1 (since df_n are bounded in L^p and $\mathrm{Vol}(\mathcal{M}) < \infty$). Hence, there exists a subsequence f_n, and a family of Radon probability measures $(\nu_x)_{x\in\mathcal{M}'}$ on E_x, such that

$$\psi \circ df_n \rightharpoonup \left(x \mapsto \int_{E_x} \psi_x(\lambda)\, d\nu_x(\lambda) \right) \quad \text{in } L^1(\mathcal{M}'; W) \tag{4.4}$$

for every Riemannian vector bundle $W \to \mathcal{M}'$ and every continuous bundle map $\psi : E \to W$, such that $\psi \circ df_n$ is sequentially weakly relatively compact in $L^1(\mathcal{M}'; W)$. The idea is to exploit the general relation (4.4) for various choices of W and ψ.

First, consider (4.4) for $W = \mathbb{R}$ and $\psi = \operatorname{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(\cdot, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))$. The compactness condition is satisfied since $\psi \circ df_n$ is bounded in $L^p(\mathcal{M}'; \mathbb{R})$ and $p > 1$ [LL01, p. 68]. We obtain that

$$\operatorname{dist}_{(\mathfrak{g},f^*\mathfrak{h})}(df_n,\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h}))\rightharpoonup\left(x\mapsto\int_{E_x}\operatorname{dist}_{(\mathfrak{g},f^*\mathfrak{h})}(\lambda,\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h}))|_x\,d\nu_x(\lambda)\right) \tag{4.5}$$

in $L^1(\mathcal{M}';\mathbb{R})$. Multiplying by the test function $1\in L^\infty(\mathcal{M};\mathbb{R})$ and integrating over $\mathcal{M}'$ we obtain

$$\begin{aligned}0&=\lim_n\int_{\mathcal{M}'}\operatorname{dist}_{(\mathfrak{g},f^*\mathfrak{h})}(df_n,\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h}))\,d\mathrm{Vol}_{\mathfrak{g}}\\&=\int_{\mathcal{M}'}\left(\int_{E_x}\operatorname{dist}_{(\mathfrak{g},f^*\mathfrak{h})}(\lambda,\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h}))|_x\,d\nu_x(\lambda)\right)d\mathrm{Vol}_{\mathfrak{g}}|_x.\end{aligned}$$

This implies that ν_x is supported on $\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})_x$ for almost every $x\in\mathcal{M}'$.

Next, consider (4.4) for the following choices of W and ψ,

$$\begin{array}{ll}W=E & \psi=Id\\ W=\mathbb{R} & \psi=\mathrm{Det}\\ W=E & \psi=\mathrm{Cof},\end{array}$$

where the determinant and the cofactor are defined with respect to the metric induced by $\mathfrak{g}$ and $f^*\mathfrak{h}$ (see Section 2.3 for intrinsic definitions of the determinant and the cofactor). Since $p>d$, all three choices of ψ imply that $\psi\circ df_n$ satisfy the L^1-weakly sequential compactness condition.

Therefore,

$$\begin{aligned}df_n&\rightharpoonup\left(x\mapsto\textstyle\int_{E_x}\lambda\,d\nu_x(\lambda)\right) && \text{in } L^1(\mathcal{M};E)\\ \mathrm{Det}(df_n)&\rightharpoonup\left(x\mapsto\textstyle\int_{E_x}\mathrm{Det}(\lambda)|_x\,d\nu_x(\lambda)\right) && \text{in } L^1(\mathcal{M}';\mathbb{R})\\ \mathrm{Cof}(df_n)&\rightharpoonup\left(x\mapsto\textstyle\int_{E_x}\mathrm{Cof}(\lambda)|_x\,d\nu_x(\lambda)\right) && \text{in } L^1(\mathcal{M}';E),\end{aligned} \tag{4.6}$$

where the dependence of $\mathrm{Det}(\lambda)$ and $\mathrm{Cof}(\lambda)$ on x is via the metrics $\mathfrak{g}$ and $f^*\mathfrak{h}$. Since ν_x is supported on $\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})$, and $\mathrm{Det}(\lambda)=1$ and $\mathrm{Cof}(\lambda)=\lambda$ for $\lambda\in\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})$, (4.6) reduces to

$$\begin{aligned}df_n&\rightharpoonup\left(x\mapsto\textstyle\int_{\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})_x}\lambda\,d\nu_x(\lambda)\right) && \text{in } L^1(\mathcal{M}';E)\\ \mathrm{Det}(df_n)&\rightharpoonup 1 && \text{in } L^1(\mathcal{M};\mathbb{R})\\ \mathrm{Cof}(df_n)&\rightharpoonup\left(x\mapsto\textstyle\int_{\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})_x}\lambda\,d\nu_x(\lambda)\right) && \text{in } L^1(\mathcal{M}';E).\end{aligned} \tag{4.7}$$

On the other hand, by the weak continuity of determinants and cofactors (see Proposition A.9),

$$\begin{aligned} df_n &\rightharpoonup df && \text{in } L^p(\mathcal{M}';E) \\ \operatorname{Det}(df_n) &\rightharpoonup \operatorname{Det}(df) && \text{in } L^{p/d}(\mathcal{M}';\mathbb{R}) \\ \operatorname{Cof}(df_n) &\rightharpoonup \operatorname{Cof}(df) && \text{in } L^{p/(d-1)}(\mathcal{M}';E) \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), it follows from the uniqueness of the limit in L^1 that

$$\begin{aligned} df(x) &= \int_{\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})_x} \lambda\, d\nu_x(\lambda) \\ \operatorname{Det}(df(x)) &= 1 && a.e. \\ \operatorname{Cof}(df(x)) &= \int_{\mathrm{SO}(\mathfrak{g},f^*\mathfrak{h})_x} \lambda\, d\nu_x(\lambda) = df(x). \end{aligned} \tag{4.9}$$

Since $\operatorname{Cof}(df) = df$ and $\operatorname{Det}(df) = 1$ a.e., it follows from Corollary 2.7 that

$$df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}) \qquad \text{a.e.,} \tag{4.10}$$

which concludes the second step of the proof. Note that while the analysis above refers to weak (distributional) derivatives, (4.10) holds for the standard notion of the differential df, since $p > d$, as discussed in Section 2.1. By Proposition 3.2 $f : \mathcal{M} \to \mathcal{N}$ is smooth as a map between manifolds with boundary.

Step III: Prove that $f_n \to f$ in $W^{1,p}(\mathcal{M};\mathcal{N})$

We have thus far obtained that $f_n \to f$ uniformly, that f is a.e. differentiable and $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$. We proceed to show that $f_n \to f$ (strongly) in $W^{1,p}(\mathcal{M};\mathcal{N})$.

As in Step I, let $\iota : \mathcal{N} \to \mathbb{R}^D$ be a smooth isometric embedding and let $F_n = \iota \circ f_n$ and $F = \iota \circ f$. By definition, $f_n \to f$ in $W^{1,p}(\mathcal{M};\mathcal{N})$ if $F_n \to F$ in $W^{1,p}(\mathcal{M};\mathbb{R}^d)$ (Section 2.1).

We repeat a similar analysis as in Step II for the sections dF_n of $T^*\mathcal{M} \otimes \mathbb{R}^D$. We obtain a family of Young probability measures $(\mu_x)_{x\in\mathcal{M}}$ on $T^*_x\mathcal{M} \otimes \mathbb{R}^D$ that correspond to dF_n. That is,

$$\psi \circ dF_n \rightharpoonup \left(x \mapsto \int_{T^*_x\mathcal{M}\otimes\mathbb{R}^D} \psi_x(\lambda)\, d\mu_x(\lambda) \right) \quad \text{in } L^1(\mathcal{M};W) \tag{4.11}$$

for every Riemannian vector bundle $W \to \mathcal{M}$ and every continuous bundle map $\psi : T^*\mathcal{M} \otimes \mathbb{R}^D \to W$, such that $\psi \circ dF_n$ is sequentially weakly relatively compact in $L^1(\mathcal{M};W)$.

Since $\|\operatorname{dist}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e}))\|_p \to 0$ (see (4.2)), we obtain, by an analysis similar to that leading to (4.5), that μ_x is supported on $\mathrm{O}(\mathfrak{g}_x, \mathfrak{e})$ for almost every $x \in \mathcal{M}$. As in (4.9), we obtain

$$dF_x = \int_{\mathrm{O}(\mathfrak{g},\mathfrak{e})_x} \lambda \, d\mu_x(\lambda), \qquad \text{a.e.}$$

We also know that f is an isometric immersion, hence $dF \in \mathrm{O}(\mathfrak{g}, \mathfrak{e})$. Since μ_x is a probability measure, we have just obtained that an element in $\mathrm{O}(\mathfrak{g}, \mathfrak{e})_x$ is equal to a convex combination of elements in $\mathrm{O}(\mathfrak{g}, \mathfrak{e})_x$. However, $\mathrm{O}(\mathfrak{g}, \mathfrak{e})_x$ is a subset of the sphere of radius $\sqrt{d}$ around the origin in $T^*_x\mathcal{M} \otimes \mathbb{R}^D$, and therefore, it is strictly convex. It follows that the convex combination must be trivial, namely,

$$\mu_x = \delta_{dF_x} \quad \text{a.e. in } \mathcal{M},$$

which together with (4.11) implies that

$$\psi \circ dF_n \rightharpoonup \psi \circ dF \quad \text{in } L^1(\mathcal{M}; W).$$

If we could take for $W = \mathbb{R}$ and $\xi \in T^*\mathcal{M} \otimes \mathbb{R}^D$,

$$\psi(\xi) = |\xi - dF|^p,$$

then we would be done, however, this function does not satisfy the sequential weak relative compactness condition. Hence, let

$$\psi(\xi) = |\xi - dF|^p \, \varphi\left(\frac{|\xi|}{3\sqrt{d}}\right),$$

where $\varphi : [0, \infty) \to [0, 1]$ is continuous, compactly-supported and satisfies $\varphi(t) = 1$ for $t \le 1$ and $\varphi(t) < 1$ for $t > 1$. This choice of ψ satisfies the de la Vallée Poussin criterion (2.2) and therefore

$$\psi \circ dF_n \rightharpoonup 0$$

in $L^1(\mathcal{M})$. In particular, taking the test function $1 \in L^\infty(\mathcal{M})$,

$$\lim_n \int_{\mathcal{M}} \psi \circ dF_n \, d\mathrm{Vol}_{\mathfrak{g}} = 0. \tag{4.12}$$

We now split the integral in (4.12) into integrals over two disjoint sets, $\mathcal{M}_n$ and $\mathcal{M}_n^c$, where

$$\mathcal{M}_n = \{x \in \mathcal{M} \;:\; |(dF_n)_x| \le 3\sqrt{d}\}.$$

By the definition of φ,
$$\psi \circ dF_n = |dF_n - dF|^p \qquad \text{in } \mathcal{M}_n.$$
On the other hand, in $\mathcal{M}_n^c$,
$$|dF_n - dF| \le |dF_n| + |dF| = |dF_n| + \sqrt{d} \le 2(|dF_n| - \sqrt{d}) \le 2\, \operatorname{dist}_{\mathfrak{g},\mathfrak{e}}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e})), \tag{4.13}$$
where the last inequality follows from the reverse triangle inequality.
Combining (4.12) and (4.13),
$$\begin{aligned}
\limsup_{n\to\infty} &\int_{\mathcal{M}} |dF_n - dF|^p \, d\mathrm{Vol}_{\mathfrak{g}} \\
&= \limsup_{n\to\infty} \left(\int_{\mathcal{M}_n} |dF_n - dF|^p \, d\mathrm{Vol}_{\mathfrak{g}} + \int_{\mathcal{M}_n^c} |dF_n - dF|^p \, d\mathrm{Vol}_{\mathfrak{g}} \right) \\
&\le \limsup_{n\to\infty} \int_{\mathcal{M}} \psi \circ dF_n \, d\mathrm{Vol}_{\mathfrak{g}} + \limsup_{n\to\infty} \int_{\mathcal{M}_n^c} |dF_n - dF|^p \, d\mathrm{Vol}_{\mathfrak{g}} \\
&= \limsup_{n\to\infty} \int_{\mathcal{M}_n^c} |dF_n - dF|^p \, d\mathrm{Vol}_{\mathfrak{g}} \\
&\le \limsup_{n\to\infty} 2^p \int_{\mathcal{M}_n^c} \operatorname{dist}^p_{\mathfrak{g},\mathfrak{e}}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e})) \, d\mathrm{Vol}_{\mathfrak{g}} \\
&\le \limsup_{n\to\infty} 2^p \int_{\mathcal{M}} \operatorname{dist}^p_{\mathfrak{g},\mathfrak{e}}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e})) \, d\mathrm{Vol}_{\mathfrak{g}} = 0,
\end{aligned}$$
where the last equality follows from (4.2). Therefore, $dF_n \to dF$ in $L^p(\mathcal{M}; T^*\mathcal{M} \otimes \mathbb{R}^D)$. Since F_n converges uniformly to F, we get that $F_n \to F$ in $W^{1,p}(\mathcal{M}; \mathbb{R}^D)$, and, by definition, $f_n \to f$ in $W^{1,p}(\mathcal{M}; \mathcal{N})$.

Step IV: Extension to $1 \le p \le d$

Suppose now that $p \ge 1$. The idea is to replace the functions f_n by functions f_n' that are more regular (specifically, uniformly Lipschitz), and then apply Steps I–III to the approximate mappings f_n'.

As in Step I of the proof, we choose a smooth isometric embedding $\iota : (\mathcal{N}, \mathfrak{h}) \to (\mathbb{R}^D, \mathfrak{e})$, and set $F_n = \iota \circ f_n : \mathcal{M} \to \mathbb{R}^D$. Our assumptions on f_n imply that $F_n \in W^{1,p}(\mathcal{M}; \mathbb{R}^D)$ (in fact, this is how $W^{1,p}(\mathcal{M}; \mathcal{N})$ is defined), and
$$\operatorname{dist}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e})) \to 0 \qquad \text{in } L^p(\mathcal{M}).$$

As in Step I, it follows that dF_n has a weakly converging subsequence, and together with the Poinrcaré inequality, implies that F_n has a subsequence weakly converging in $W^{1,p}(\mathcal{M};\mathbb{R}^D)$. However, since $p < d$, convergence is not uniform, and the limit does not necessarily lie in the image of ι.

To overcome this problem, we approximate the mappings F_n by another sequence $F'_n \in W^{1,\infty}(\mathcal{M};\mathcal{N})$, using the following truncation argument [FJM02, Proposition A.1]:

> *Let $p \geq 1$. There exists a constant C, depending only on p and $\mathfrak{g}$, such that for every $u \in W^{1,p}(\mathcal{M};\mathbb{R}^D)$ and every $\lambda > 0$, there exists $\tilde{u} \in W^{1,\infty}(\mathcal{M};\mathbb{R}^D)$ such that*
>
> $$\|d\tilde{u}\|_\infty \leq C\lambda,$$
> $$Vol_{\mathfrak{g}}\left(\{x \in \mathcal{M} : \tilde{u}(x) \neq u(x)\}\right) \leq \frac{C}{\lambda^p}\int_{\{|du(x)|>\lambda\}} |du|^p \, dVol_{\mathfrak{g}},$$
> $$\|d\tilde{u} - du\|_p^p \leq C\int_{\{|du(x)|>\lambda\}} |du|^p \, dVol_{\mathfrak{g}}.$$

The original proposition ([FJM02, Proposition A.1]) refers to a bounded Lipschitz domain in $\mathbb{R}^d$, but the partition of unity argument used to obtain the result for an arbitrary Lipschitz domain (Step 3 in the proof) applies to any compact Riemannian manifold with Lipschitz boundary (the constant C depends on the manifold, of course).

Let $\lambda > 2\sqrt{d}$, so that $|A| > \lambda$, for $A \in T^*\mathcal{M}_x \otimes \mathbb{R}^D$, implies that $|A| < 2\,\mathrm{dist}(A, \mathrm{O}(\mathfrak{g}_x, \mathfrak{e}))$ (compare with (4.13)). Apply the truncation argument to F_n, we obtain mappings $\tilde{F}_n \in W^{1,\infty}(\mathcal{M};\mathbb{R}^D)$, with a uniform Lipschitz constant C, such that

$$\begin{aligned}\mathrm{Vol}_{\mathfrak{g}}\left(\left\{x \in \mathcal{M} : \tilde{F}_n(x) \neq F_n(x)\right\}\right) &\leq C\int_{\{|dF_n\}(x)|>\lambda} |dF_n|^p \, d\mathrm{Vol}_{\mathfrak{g}} \\ &\leq C\int_{\{|dF_n(x)|>\lambda\}} 2^p \,\mathrm{dist}^p(dF_n, \mathrm{O}(\mathfrak{g},\mathfrak{e}))\, d\mathrm{Vol}_{\mathfrak{g}} \\ &\leq 2^p C\int_{\mathcal{M}} \mathrm{dist}^p(dF_n, \mathrm{O}(\mathfrak{g},\mathfrak{e}))\, d\mathrm{Vol}_{\mathfrak{g}},\end{aligned} \tag{4.14}$$

and

$$\|d\tilde{F}_n - dF_n\|_p^p \leq 2^p C\int_{\mathcal{M}} \mathrm{dist}^p(dF_n, \mathrm{O}(\mathfrak{g},\mathfrak{e}))\, d\mathrm{Vol}_{\mathfrak{g}}. \tag{4.15}$$

for some $C > 0$, independent of n. In particular,

$$\lim_{n\to\infty} \mathrm{Vol}_{\mathfrak{g}}\left(\{x \in \mathcal{M} : \tilde{F}_n(x) \neq F_n(x)\}\right) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|d\tilde{F}_n - dF_n\|_p^p = 0. \tag{4.16}$$

Since $\operatorname{dist}(dF_n, \mathrm{O}(\mathfrak{g}, \mathfrak{e})) \to 0$ in L^p, (4.14) implies that for every $\varepsilon > 0$, every n large enough and every ball $B \subset \mathcal{M}$ of radius ε, there exists a point $x \in B$ such that $\tilde{F}_n(x) = F_n(x) \in \iota(\mathcal{N})$ (there is a positive lower bound on the volumes of balls of radius ε in a compact Riemannian manifold). Since $\tilde{F}_n$ are uniformly Lipschitz, it follows that for large enough n, $\max_{x \in \mathcal{M}} \operatorname{dist}(\tilde{F}_n(x), \iota(\mathcal{N})) < C\varepsilon$, and therefore

$$\max_{x \in \mathcal{M}} \operatorname{dist}(\tilde{F}_n(x), \iota(\mathcal{N})) \to 0.$$

Thus, for n large enough, $\tilde{F}_n$ lies in a tubular neighborhood of $\iota(\mathcal{N})$, in which the orthogonal projection P onto $\iota(\mathcal{N})$ is well-defined and smooth (and in particular Lipschitz). We define $F'_n := P \circ \tilde{F}_n$. It immediately follows that $F'_n \in W^{1,\infty}(\mathcal{M}; R^D)$ are uniformly Lipschitz, and by definition, their image is in $\iota(\mathcal{N})$. Moreover, since

$$\{F'_n \neq \tilde{F}_n\} = \{\tilde{F}_n \notin \iota(\mathcal{N})\} \subset \{\tilde{F}_n \neq F_n\} \quad \text{and} \quad \{F'_n \neq F_n\} \subset \{\tilde{F}_n \neq F_n\},$$

(4.16) implies that

$$\lim_{n\to\infty} \operatorname{Vol}_{\mathfrak{g}}\left(\{F'_n \neq \tilde{F}_n\}\right) = \lim_{n\to\infty} \operatorname{Vol}_{\mathfrak{g}}\left(\{F'_n \neq F_n\}\right) = 0. \tag{4.17}$$

Since $\tilde{F}_n$ and F'_n are uniformly Lipschitz, it follows that

$$\|dF'_n - d\tilde{F}_n\|_p^p = \int_{\{F'_n \neq \tilde{F}_n\}} |dF'_n - d\tilde{F}_n|^p \, d\mathrm{Vol}_{\mathfrak{g}} \leq C \mathrm{Vol}_{\mathfrak{g}}\left(\{F'_n \neq \tilde{F}_n\}\right) \to 0,$$

where in the equality we used the fact that $dF'_n - d\tilde{F}_n = 0$ almost everywhere on the set $F'_n - \tilde{F}_n = 0$ (see [EG15, Theorem 4.4]), and the inequality follows by the fact that both F'_n and $\tilde{F}_n$ are uniformly Lipschitz.

Together with (4.16) we obtain that $\|dF'_n - dF_n\|_p \to 0$. Finally, since

$$\int_{\mathcal{M}} |F'_n(x) - F_n(x)|^p \, d\mathrm{Vol}_{\mathfrak{g}} \leq 2^p \max_{y \in \mathcal{N}} |\iota(y)|^p \operatorname{Vol}_{\mathfrak{g}}\left(\{F'_n \neq F_n\}\right) \to 0,$$

we conclude that

$$\lim_{n\to\infty} \|F'_n - F_n\|_{W^{1,p}(\mathcal{M};\mathbb{R}^D)} = 0. \tag{4.18}$$

Next, define $f'_n = \iota^{-1} \circ F'_n$. By definition $f'_n \in W^{1,\infty}(\mathcal{M}; \mathcal{N})$, and moreover, f'_n are uniformly Lipschitz (since intrinsic and extrinsic distances in $\iota(\mathcal{N})$ are equivalent). Since $dF'_n = dF_n$ almost everywhere in the set $\{F'_n = F_n\}$ (again, [EG15, Theorem 4.4]), we have that $df_n' = df_n$ almost everywhere in the set $\{f_n' = f_n\}$. Using the uniform bound on df_n', we obtain

$$\begin{aligned}
&\int_{\mathcal{M}} \operatorname{dist}^p_{(\mathfrak{g}, f'^*_n\mathfrak{h})}(df_n', \mathrm{SO}(\mathfrak{g}, f'^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} \\
&\qquad \le \int_{\{f_n'=f_n\}} \operatorname{dist}^p_{(\mathfrak{g}, f^*_n\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} + C\,\mathrm{Vol}_{\mathfrak{g}}\left(\{f_n' \ne f_n\}\right) \\
&\qquad \le \int_{\mathcal{M}} \operatorname{dist}^p_{(\mathfrak{g}, f^*_n\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} + C\,\mathrm{Vol}_{\mathfrak{g}}\left(\{f_n' \ne f_n\}\right) \to 0.
\end{aligned} \tag{4.19}$$

Moreover, for any $p < q < \infty$,

$$\begin{aligned}
&\int_{\mathcal{M}} \operatorname{dist}^q_{(\mathfrak{g}, f'^*_n\mathfrak{h})}(df_n', \mathrm{SO}(\mathfrak{g}, f'^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} \\
&\qquad \le \int_{\mathcal{M}} (|df_n'| + c)^{q-p} \operatorname{dist}^p_{(\mathfrak{g}, f'^*_n\mathfrak{h})}(df_n', \mathrm{SO}(\mathfrak{g}, f'^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} \\
&\qquad \le C \int_{\mathcal{M}} \operatorname{dist}^p_{(\mathfrak{g}, f'^*_n\mathfrak{h})}(df_n', \mathrm{SO}(\mathfrak{g}, f'^*_n\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} \to 0.
\end{aligned} \tag{4.20}$$

Next, we apply Steps I, II and III of the proof with f_n replaced by f_n' and any $q > d$. We obtain that f_n' converge in $W^{1,q}(\mathcal{M};\mathcal{N})$ to a smooth isometric immersion $f : \mathcal{M} \to \mathcal{N}$ (or equivalently $F_n' \to \iota \circ f$ in $W^{1,q}(\mathcal{M};\mathbb{R}^D)$). By (4.18), it follows that $F_n \to F$ in $W^{1,p}(\mathcal{M};\mathbb{R}^D)$, so by definition, $f_n \to f$ in $W^{1,p}(\mathcal{M};\mathcal{N})$.

Step V: Prove that f is an isometry under additional assumptions

Suppose that $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $\mathrm{Vol}_{\mathfrak{h}}\mathcal{N} = \mathrm{Vol}_{\mathfrak{g}}\mathcal{M}$. To show that f is an isometry, it suffices to show that $f|_{\mathcal{M}^\circ}$ is a surjective isometry $\mathcal{M}^\circ \to \mathcal{N}^\circ$. Indeed, if this is the case, then, since f is continuous and $\mathcal{M}$ is compact, $f(\mathcal{M})$ contains $\mathcal{N}^\circ$ and is closed in $\mathcal{N}$, i.e., $f(\mathcal{M}) = \mathcal{N}$. Finally, f is an isometry, because for every $x, y \in \mathcal{M}$, let $\mathcal{M}^\circ \ni x_n \to x$ and $\mathcal{M}^\circ \ni y_n \to y$; by the continuity of the metrics $d_{\mathcal{M}}$ and $d_{\mathcal{N}}$,

$$d_{\mathcal{N}}(f(x), f(y)) = \lim_{n\to\infty} d_{\mathcal{N}}(f(x_n), f(y_n)) = \lim_{n\to\infty} d_{\mathcal{M}}(x_n, y_n) = d_{\mathcal{M}}(x, y).$$

Note that the intrinsic distance function on $\mathcal{M}^\circ$ is the same as the extrinsic distance $d_{\mathcal{M}}$, and similarly for $\mathcal{N}$, so there is no ambiguity here regarding which metric we use.

We proceed to show that $f|_{\mathcal{M}^\circ}$ is a Riemannian isometry $\mathcal{M}^\circ \to \mathcal{N}^\circ$. Recall that $f : \mathcal{M} \to \mathcal{N}$ is smooth as a map between manifolds with boundary, and $df \in \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})$

is invertible at every point. Thus for any interior point $x \in \mathcal{M}^\circ$, the image $f(x)$ must be an interior point of $\mathcal{N}$, hence $f(\mathcal{M}^\circ) \subset \mathcal{N}^\circ$. Since (by the inverse function theorem) $f : \mathcal{M}^\circ \to \mathcal{N}^\circ$ is a local diffeomorphism and in particular an open map, $f(\mathcal{M}^\circ)$ is open in $\mathcal{N}^\circ$.

Since $f_n \to f$ in $W^{1,p}(\mathcal{M};\mathcal{N})$, it follows from the trace theorem (when viewing f_n as elements in $W^{1,p}(\mathcal{M};\mathbb{R}^D)$) that $f_n|_{\partial\mathcal{M}} \to f|_{\partial\mathcal{M}}$ in $L^p(\partial\mathcal{M};\mathbb{R}^D)$, and (after taking a subsequence) pointwise almost everywhere in $\partial\mathcal{M}$. Since $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$, and since $\partial\mathcal{N}$ is closed and f is continuous we conclude that $f(\partial\mathcal{M}) \subset \partial\mathcal{N}$. The reason for adopting an extrinsic viewpoint in the last argument is that the trace theorem relies upon the density of smooth functions in $W^{1,p}$. This density does not hold for mappings between manifolds for $p < d$. Using a truncation argument here would result in losing the condition that $f(\partial\mathcal{M}) \subset \partial\mathcal{N}$.

Let $f(x_n) \in \mathcal{N}^\circ$ converges to $y \in \mathcal{N}^\circ$. Since $\mathcal{M}$ is compact and f is continuous, we may assume, by taking a subsequence, that $x_n \to x \in \mathcal{M}$, and $y = f(x)$. Since $f(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $y \in \mathcal{N}^\circ$, it follows that $x \in \mathcal{M}^\circ$, i.e., $y \in f(\mathcal{M}^\circ)$, which implies that $f(\mathcal{M}^\circ)$ is closed in $\mathcal{N}^\circ$. We have thus obtained that $f(\mathcal{M}^\circ)$ is clopen in $\mathcal{N}^\circ$. Since $\mathcal{N}^\circ$ is connected, $f(\mathcal{M}^\circ) = \mathcal{N}^\circ$, i.e., $f|_{\mathcal{M}^\circ}$ is surjective.

It remains to prove that $f|_{\mathcal{M}^\circ}$ is injective; this is where we use a volume argument. The area formula for f implies that

$$\mathrm{Vol}_{\mathfrak{g}}\mathcal{M} = \int_{\mathcal{M}} |\operatorname{Det} df|\, d\mathrm{Vol}_{\mathfrak{g}} = \int_{\mathcal{N}} |f^{-1}(y)|\, d\mathrm{Vol}_{\mathfrak{h}}|_y \geq \mathrm{Vol}_{\mathfrak{h}}\mathcal{N}, \tag{4.21}$$

where $|f^{-1}(y)|$ denotes the cardinality of the inverse image of y, and the last inequality follows from the surjectivity of $f|_{\mathcal{M}^\circ}$. Since, by assumption, $\mathrm{Vol}_{\mathfrak{h}}\mathcal{N} = \mathrm{Vol}_{\mathfrak{g}}\mathcal{M}$, (4.21) is in fact an equality, hence

$$\mathrm{Vol}_{\mathfrak{h}}\left(\left\{q \in \mathcal{N} \;:\; |f^{-1}(q)| > 1\right\}\right) = 0.$$

It follows that f is injective on $\mathcal{M}^\circ$. Indeed, assume $f(p_1) = q = f(p_2)$, where $p_1 \neq p_2 \in \mathcal{M}^\circ$ and $q \in \mathcal{N}^\circ$. Since f is a local diffeomorphism, there exist disjoint open neighborhoods $U_i \ni p_i$ and $V \ni q$ such that $f(U_i) = V$, hence

$$\mathrm{Vol}_{\mathfrak{h}}\left(\{q \in \mathcal{N} : |f^{-1}(q)| > 1\}\right) \geq \mathrm{Vol}_{\mathfrak{h}}(V) > 0,$$

which is a contradiction. This completes the proof.

Step VI: If f_n are diffeomorphisms then $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $Vol_{\mathfrak{h}}\mathcal{N} = Vol_{\mathfrak{g}}\mathcal{M}$

If f_n are diffeomorphisms, then obviously $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$, and therefore (4.21) holds. It remains to show that $\mathrm{Vol}_{\mathfrak{h}}\mathcal{N} = \mathrm{Vol}_{\mathfrak{g}}\mathcal{M}$, and by (4.21), it is enough to show that $\mathrm{Vol}_{\mathfrak{g}}\mathcal{M} \leq \mathrm{Vol}_{\mathfrak{h}}\mathcal{N}$.

For $p \geq d$, the equality of volumes is straightforward: since $\mathcal{M}$ is connected, f_n are either globally orientation-preserving of globally orientation-reversing. Since $\mathrm{dist}(\mathrm{GL}_d^-, \mathrm{SO}(d)) = c(d) > 0$, an orientation-reversing diffeomorphism $\phi : \mathcal{M} \to \mathcal{N}$ satisfies

$$\int_{\mathcal{M}} \mathrm{dist}^p_{\mathfrak{g},\phi^*\mathfrak{h}}(d\phi, \mathrm{SO}(\mathfrak{g}, \phi^*\mathfrak{h}))\, d\mathrm{Vol}_{\mathfrak{g}} \geq c^p \mathrm{Vol}_{\mathfrak{g}}\mathcal{M}.$$

By (4.1), f_n are orientation-preserving for large enough n. If $p \geq d$, then $\mathrm{Vol}_{\mathfrak{g}}\mathcal{M} = \mathrm{Vol}_{\mathfrak{h}}\mathcal{N}$ follows from Lemma C.2 and (4.1).

For $p < d$, we can use the truncated mappings f_n' defined in step IV to show that $\mathrm{Vol}_{\mathfrak{g}}(\mathcal{M}) \leq \mathrm{Vol}_{\mathfrak{g}}(\mathcal{N})$. By (4.20), $\mathrm{dist}_{(\mathfrak{g},f'^*_n\mathfrak{h})}(df_n', \mathrm{SO}(\mathfrak{g}, f'^*_n\mathfrak{h})) \to 0$ in L^q for any $q \in [1,\infty)$, but f_n' are not diffeomorphisms, so we cannot use the above reasoning directly. However, (4.20) (with $q = d$) and Lemma C.1 imply that $|\mathrm{Det}\, df_n'| \to 1$ in $L^1(\mathcal{M})$. Therefore,

$$\begin{aligned}
\mathrm{Vol}_{\mathfrak{g}}(\mathcal{M}) &= \int_{\mathcal{M}} |\mathrm{Det}\, df_n'|\, d\mathrm{Vol}_{\mathfrak{g}} + o(1) \\
&= \int_{\{f_n = f_n'\}} |\mathrm{Det}\, df_n|\, d\mathrm{Vol}_{\mathfrak{g}} + \int_{\{f_n \neq f_n'\}} |\mathrm{Det}\, df_n'|\, d\mathrm{Vol}_{\mathfrak{g}} + o(1) \\
&\leq \int_{\{f_n = f_n'\}} |\mathrm{Det}\, df_n|\, d\mathrm{Vol}_{\mathfrak{g}} + C\mathrm{Vol}_{\mathfrak{g}}(\{f_n \neq f_n'\}) + o(1) \\
&\overset{(4.17)}{=} \int_{\{f_n = f_n'\}} |\mathrm{Det}\, df_n|\, d\mathrm{Vol}_{\mathfrak{g}} + o(1) \\
&\leq \int_{\mathcal{M}} |\mathrm{Det}\, df_n|\, d\mathrm{Vol}_{\mathfrak{g}} + o(1) = \mathrm{Vol}_{\mathfrak{h}}(\mathcal{N}) + o(1),
\end{aligned}$$

where in the first inequality we used the fact that f_n' are uniformly Lipschitz. Therefore $\mathrm{Vol}_{\mathfrak{g}}(\mathcal{M}) \leq \mathrm{Vol}_{\mathfrak{g}}(\mathcal{N})$, and together with (4.21) we obtain that $\mathrm{Vol}_{\mathfrak{g}}(\mathcal{M}) = \mathrm{Vol}_{\mathfrak{g}}(\mathcal{N})$.

■

We conclude this section with a number or remarks concerning the assumptions in Theorem 4.1.

1. Neither of the assumptions $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ and $\mathrm{Vol}_{\mathfrak{h}}\mathcal{N} = \mathrm{Vol}_{\mathfrak{g}}\mathcal{M}$, which were used to prove that f is an isometry, can be dropped. Take for example $\mathcal{M} = [-1,1]^d$,

and let $\mathcal{N} = \mathcal{M}/\sim$ be the flat d-torus with $\sim$ the standard equivalence relation. Then $f_n : \mathcal{M} \to \mathcal{N}$ given by $f_n(x) = (1 - 1/n)x$ are injective and satisfy (4.1), but converge uniformly to $\pi : \mathcal{M} \to \mathcal{N}$ the quotient map, which is obviously not an isometry but merely an isometric immersion. This example shows that the assumption $f_n(\partial\mathcal{M}) \subset \partial\mathcal{N}$ cannot be relaxed.

In order to see that the condition $\mathrm{Vol}_{\mathfrak{g}}\mathcal{M} = \mathrm{Vol}_{\mathfrak{h}}\mathcal{N}$ cannot be relaxed, recall that there is an isometric immersion from the circle of radius 2 in $\mathbb{R}^2$ into the circle of radius 1.

2. Yet another alternative condition implying that f is an isometry is the following "symmetric condition": there exist surjective mappings $f_n \in W^{1,p}(\mathcal{M};\mathcal{N})$ and $g_n \in W^{1,p}(\mathcal{N};\mathcal{M})$ such that

$$\mathrm{dist}_{(\mathfrak{g},f_n^*\mathfrak{h})}(df_n, \mathrm{SO}(\mathfrak{g}, f_n^*\mathfrak{h})) \to 0 \qquad \text{in } L^p(\mathcal{M}),$$

and

$$\mathrm{dist}_{(\mathfrak{h},g_n^*\mathfrak{g})}(dg_n, \mathrm{SO}(\mathfrak{h}, g_n^*\mathfrak{g})) \to 0 \qquad \text{in } L^p(\mathcal{N}).$$

The proof follows the same steps as Theorem 4.1 for both f_n and g_n, resulting in $f_n \to f$, $g_n \to g$, where f and g are surjective isometric immersions. It follows that $g \circ f : \mathcal{M} \to \mathcal{M}$ is a surjective isometric immersion, and therefore ([BBI01, Theorem 1.6.15]) it is a metric isometry. Then, $f : \mathcal{M} \to \mathcal{N}$ is a metric isometry, and by the Myers-Steenrod theorem, it is a Riemannian isometry.

3. Generally, the compactness of $\mathcal{N}$ is essential for the proof of Theorem 4.1. However, we used the compactness of $\mathcal{N}$ only in the following places: (i) in Step I of the proof, where we applied the Poincaré inequality for the global mappings F_n; (ii) in Step I again, for the equivalence of intrinsic and extrinsic distances when we isometrically embed $\mathcal{N} \subset \mathbb{R}^D$; and (iii) in Step V, for obtaining (4.18). Thus, the compactness of $\mathcal{N}$ can be replaced by alternative assumptions, as long as these three properties hold. In particular, the following holds:

Corollary 4.2 *Let* $(\mathcal{M}, \mathfrak{g})$ *be a compact d-dimensional manifold with* C^1 *boundary. Let* $p > 1$ *and let* $f_n \in W^{1,p}(\mathcal{M};\mathbb{R}^d)$ *be a sequence of mappings such that*

$$\mathrm{dist}_{(\mathfrak{g},\mathfrak{e})}(df_n, \mathrm{SO}(\mathfrak{g}, \mathfrak{e})) \to 0 \qquad \textit{in } L^p(\mathcal{M}),$$

and $\int_{\mathcal{M}} f_n d\mathrm{Vol}_{\mathfrak{g}} = 0$. *Then* f_n *has a subsequence converging in* $W^{1,p}(\mathcal{M};\mathbb{R}^d)$ *to a limit* f, *which is a smooth isometric immersion. In particular,* $\mathcal{M}$ *is flat.*

In this case, the proof is in fact much simpler, since the global and local stages can be merged, and there is no need to locally replace $\mathfrak{h}$ by $\mathfrak{e}$.

The conclusion that $\mathcal{M}$ is flat when $(\mathcal{N}, \mathfrak{h}) = (\mathbb{R}^d, \mathfrak{e})$ was already proved in [LP11, Theorem 2.2]; in fact, certain parts of their proof resemble analogous parts in the proof of Theorem 4.1.

5 Applications

5.1 Non-Euclidean elasticity

An immediate corollary of Theorem 4.1 is the following:

Corollary 5.1 Let $\mathcal{M}$ be a compact d-dimensional manifold with boundary, and let $\mathcal{N}$ be either $\mathbb{R}^d$, or a compact d-dimensional manifold with boundary. If $\mathcal{M}$ is not isometrically immersible in $\mathcal{N}$, then

$$\inf_{f \in W^{1,p}(\mathcal{M};\mathcal{N})} \int_{\mathcal{M}} \operatorname{dist}^p_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))\, dVol_{\mathfrak{g}} > 0$$

for every $p \geq 1$.

As stated above, for $\mathcal{N} = \mathbb{R}^d$, this corollary has already been proven in [LP11, Theorem 2.2].

Corollary 5.1 is relevant to the field of non-Euclidean elasticity [ESK09, LP11, KS12, ESK13, KM16a, KOS]. In non-Euclidean elasticity, the manifold $\mathcal{M}$ represents the body manifold of an elastic body whose intrinsic geometry is non-Euclidean, and $\mathcal{N}$ is the space manifold, which can either be Euclidean space, or have curved geometry. The function $\operatorname{dist}^p_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))$ is a typical lower bound for an energy density measuring the local energetic cost of a configuration $f : \mathcal{M} \to \mathcal{N}$ of the body.

In this context, Corollary 5.1 states that the configurations of an elastic body that cannot be immersed isometrically into the space manifold, and whose elastic energy density is bounded from below by $\operatorname{dist}^p(\cdot, \mathrm{SO}(\mathfrak{g}, \mathfrak{e}))$, have an energy bounded away from zero. In particular, this is the case for any non-flat manifold, when the target space is Euclidean. Therefore, non-Euclidean elastic bodies are always internally strained, justifying the often-used nomenclature of "pre-strained materials".

5.2 Convergence of manifolds

The following definition is motivated by a series of works on the homogenization of manifolds with distributed singularities [KM15, KM16a, KM16b]:

Definition 5.2 *Let* $(\mathcal{M}_n, \mathfrak{g}_n)_{n\in\mathbb{N}}$ *and* $(\mathcal{M}, \mathfrak{g})$ *be compact d-dimensional Riemannian manifolds (possibly with* C^1 *boundary). We say that the sequence* $\mathcal{M}_n$ *converges to* $\mathcal{M}$ *with exponents* p, q *if there exists a sequence of diffeomorphisms* $F_n : \mathcal{M} \to \mathcal{M}_n$ *such that*

$$\|\operatorname{dist}_{(\mathfrak{g},F_n^*\mathfrak{g}_n)}(dF_n, \mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n))\|_{L^p(\mathcal{M},\mathfrak{g})} \to 0, \tag{5.1}$$

$$\|\operatorname{dist}_{(\mathfrak{g}_n,(F_n^{-1})^*\mathfrak{g})}(dF_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (F_n^{-1})^*\mathfrak{g}))\|_{L^p(\mathcal{M}_n,\mathfrak{g}_n)} \to 0, \tag{5.2}$$

and the volume forms converge, that is

$$\|\operatorname{Det} F_n - 1\|_{L^q(\mathcal{M},\mathfrak{g})} \to 0, \qquad \|\operatorname{Det} F_n^{-1} - 1\|_{L^q(\mathcal{M}_n,\mathfrak{g}_n)} \to 0. \tag{5.3}$$

Theorem 5.3 *The convergence in Definition 5.2 is well-defined for* $q > 1$ *and* $p \ge 2+1/(q-1)$*: if* $(\mathcal{M}_n, \mathfrak{g}_n) \to (\mathcal{M}, \mathfrak{g})$ *and* $(\mathcal{M}_n, \mathfrak{g}_n) \to (\mathcal{N}, \mathfrak{h})$*, then* $(\mathcal{M}, \mathfrak{g})$ *and* $(\mathcal{N}, \mathfrak{h})$ *are isometric.*

Note that if $p \ge d$, then (5.1) and (5.2) imply (5.3) for $q = p/d$; this follows from Lemma C.1. Thus, the convergence in Definition 5.2 is well-defined for $p \ge d$ and $p \ge 2 + 1/(p/d - 1)$, which after a short calculation amounts to $p \ge \frac{1}{2}(d + 2 + \sqrt{d^2 + 4})$.

Proof: Assume that $\mathcal{M}_n \to \mathcal{M}$ with respect to $F_n : \mathcal{M} \to \mathcal{M}_n$, whereas $\mathcal{M}_n \to \mathcal{N}$ with respect to $G_n : \mathcal{N} \to \mathcal{M}_n$. By the same argument as in the first comment below the proof of Theorem 4.1, we may assume that both F_n and G_n are orientation-preserving for every n.

Eq. (5.3) for F_n^{-1} implies that

$$\lim_n \operatorname{Vol}_{\mathfrak{g}_n}\mathcal{M}_n = \lim_n \int_{\mathcal{M}} \operatorname{Det} F_n \operatorname{Vol}_{\mathfrak{g}} = \int_{\mathcal{M}} \operatorname{Vol}_{\mathfrak{g}} = \operatorname{Vol}_{\mathfrak{g}}\mathcal{M}.$$

By symmetry,

$$\operatorname{Vol}_{\mathfrak{g}}\mathcal{M} = \lim_n \operatorname{Vol}_{\mathfrak{g}_n}\mathcal{M}_n = \operatorname{Vol}_{\mathfrak{h}}\mathcal{N}. \tag{5.4}$$

Define the sequence of diffeomorphisms $H_n = G_n^{-1} \circ F_n : \mathcal{M} \to \mathcal{N}$. We will show that

$$\operatorname{dist}(dH_n, \mathrm{SO}(\mathfrak{g}, H_n^*\mathfrak{h})) \to 0 \qquad \text{in } L^r(\mathcal{M})$$

for some $r \ge 1$. By Theorem 4.1, it follows that $\mathcal{M}$ and $\mathcal{N}$ are isometric.

Denote by $q_n \in \Gamma(\mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n)) \subset \Gamma(T^*\mathcal{M} \otimes F_n^*T\mathcal{M}_n)$ the section satisfying

$$|dF_n - q_n| = \mathrm{dist}(dF_n, \mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n)),$$

and by $r_n \in \Gamma(\mathrm{SO}(\mathfrak{g}_n, (G_n^1)^*\mathfrak{h})) \subset \Gamma(T^*\mathcal{M}_n \otimes (G_n^{-1})^*T\mathcal{N})$ the section satisfying

$$|dG_n^{-1} - r_n| = \mathrm{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h})).$$

Then, since $dH_n = F_n^*dG_n^{-1} \circ dF_n$ and $F_n^*r_n \circ q_n \in \Gamma(T^*\mathcal{M} \otimes H_n^*T\mathcal{N})$,

$$\begin{aligned}
\mathrm{dist}(dH_n, \mathrm{SO}(\mathfrak{g}, H_n^*\mathfrak{h})) &\leq |F_n^*dG_n^{-1} \circ dF_n - F_n^*r_n \circ q_n| \\
&= |(dF_n^*dG_n^{-1} - F_n^*r_n) \circ dF_n + F_n^*r_n \circ (dF_n - q_n)| \\
&\leq |(dF_n^*dG_n^{-1} - F_n^*r_n) \circ dF_n| + |F_n^*r_n \circ (dF_n - q_n)| \\
&\leq F_n^*|dG_n^{-1} - r_n|\,|dF_n| + |dF_n - q_n| \\
&= F_n^*\,\mathrm{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\,|dF_n| + \mathrm{dist}(dF_n, \mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n)).
\end{aligned} \tag{5.5}$$

In the passage to the third line we used the triangle inequality; in the passage to the fourth line we used the fact that $F_n^*r_n$ is an isometry and the sub-multiplicativity of the Frobenius norm; in the passage to the fifth line we used the defining properties of r_n and q_n.

By the definition of p, q-convergence, the second term on the right-hand side of (5.5) tends to zero in $L^p(\mathcal{M})$. Thus, it suffices to prove that

$$F_n^*\,\mathrm{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\,|dF_n| \to 0 \quad \text{in } L^r(\mathcal{M}) \tag{5.6}$$

for some $1 \leq r < p$.

Since

$$|dF_n| \leq \mathrm{dist}_{(\mathfrak{g}, F_n^*\mathfrak{g}_n)}(dF_n, \mathrm{SO}(\mathfrak{g}, F_n^*\mathfrak{g}_n)) + C(d),$$

it follows that $\|dF_n\|_{L^p(\mathcal{M},\mathfrak{g})}$ is uniformly bounded; the same holds for $\|dF_n^{-1}\|_{L^p(\mathcal{M}_n,\mathfrak{g}_n)}$. Note that we used here the boundedness of $\mathrm{Vol}_{\mathfrak{g}_n}\mathcal{M}_n$ in order to control the norm of $C(d)$ uniformly.

Using these observations along with Hölder's inequality, we obtain

$$
\begin{aligned}
&\|F_n^* \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\, |dF_n|\|_{L^r(\mathcal{M},\mathfrak{g})} \\
&\quad \le \|F_n^* \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\|_{L^{rp/(p-r)}(\mathcal{M},\mathfrak{g})} \|dF_n\|_{L^p(\mathcal{M},\mathfrak{g})} \\
&\quad \le C \|F_n^* \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\|_{L^{rp/(p-r)}(\mathcal{M},\mathfrak{g})} \\
&\quad = C \left(\int_{\mathcal{M}_n} \operatorname{dist}^{pr/(p-r)}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h}))\, \frac{(F_n^{-1})^\star d\mathrm{Vol}_\mathfrak{g}}{d\mathrm{Vol}_{\mathfrak{g}_n}}\, d\mathrm{Vol}_{\mathfrak{g}_n} \right)^{(p-r)/rp} \qquad (5.7) \\
&\quad \le C \left\| \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h})) \right\|_{L^{qrp/(q-1)(p-r)}(\mathcal{M}_n,\mathfrak{g}_n)} \left\| \frac{(F_n^{-1})^\star d\mathrm{Vol}_\mathfrak{g}}{d\mathrm{Vol}_{\mathfrak{g}_n}} \right\|_{L^q(\mathcal{M}_n,\mathfrak{g}_n)}^{(p-r)/rp} \\
&\quad \le C \left\| \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, (G_n^{-1})^*\mathfrak{h})) \right\|_{L^{prq/(q-1)(p-r)}(\mathcal{M}_n,\mathfrak{g}_n)}
\end{aligned}
$$

where $C > 0$ is an appropriate constant varying from line to line. Now choose $r = p/(2 + 1/(q-1)) \ge 1$. Then $prq/(q-1)(p-r) \le p$, hence (5.7) reads

$$
\|F_n^* \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, \mathfrak{h}))\, |dF_n|\|_{L^r(\mathcal{M},\mathfrak{g})} \le C \left\| \operatorname{dist}(dG_n^{-1}, \mathrm{SO}(\mathfrak{g}_n, \mathfrak{h})) \right\|_{L^p(\mathcal{M}_n,\mathfrak{g}_n)} \to 0.
$$

Therefore (5.6) holds and the proof is complete. ■

Remark: Instead of (5.3), it is sufficient to assume that $\mathrm{Vol}_{\mathfrak{g}_n} \mathcal{M}_n$ and $\| \operatorname{Det} F_n^{-1} - 1\|_{L^q(\mathcal{M}_n,\mathfrak{g}_n)}$ are bounded. Equation (5.4) in no longer valid, however (4.1) still holds for H_n and some exponent $r \ge 1$, so $\mathcal{M}$ and $\mathcal{N}$ are isometric by Theorem 4.1.

Example: We now sketch two examples of convergence of manifolds according to Definition 5.2. In the first one, the limit coincides with the Gromov-Hausdorff limit; in the second the two limits are different.

Note that these two examples involve singular metrics; in order to use the uniqueness result (Theorem 5.3) one needs to consider a smoothed version of the sequence. This can easily be done without changing the limit. Note also that both examples are two-dimensional; this is for the sake of simplicity; both have higher dimensional generalizations.

1. Let $(\mathcal{M}, \mathfrak{g})$ be a two-dimensional Riemannian manifold. For each n, choose a geodesic triangulation of $\mathcal{M}$, such that all the edge lengths are in $[c/n, c'/n]$ for some $0 < c < c'$ independent of n (in particular, the angles in all the triangles are bounded away from zero, uniformly in n). Denote the triangles by $(T_{n,i})_{i=1}^{k_n}$.

Construct $(\mathcal{M}_n, \mathfrak{g}_n)$ by replacing each triangle $T_{n,i}$ with a Euclidean triangle $R_{n,i}$ having the same edge lengths.

Let $F_{n,i} : T_{n,i} \to R_{n,i}$ be a smooth diffeomorphism that preserves lengths along the edges of the triangles. Since $T_{n,i}$ are very small, with angles bounded away from zero, they are "almost Euclidean", so $F_{n,i}$ can be chosen such that

$$\operatorname{dist}(dF_{n,i}, \mathrm{SO}(\mathfrak{g}, F_{n,i}^*\mathfrak{g}_n)),\ \operatorname{dist}(dF_{n,i}^{-1}, \mathrm{SO}(\mathfrak{g}_n, (F_{n,i}^{-1})^*\mathfrak{g})) < \frac{C}{n}$$

for some $C > 0$ independent of n. $F_n : \mathcal{M} \to \mathcal{M}_n$ is then defined as the union of $F_{n,i}$. The above bound on the distortion implies that $\mathcal{M}_n \to \mathcal{M}$ according to Definition 5.2 for every choice of exponents p, q (including $p = q = \infty$). F_n are also maps of vanishing distortion in the metric-space sense, that is

$$\max_{x,y\in\mathcal{M}} \left| d_{\mathcal{M}}(x, y) - d_{\mathcal{M}_n}(F_n(x), F_n(y)) \right| \to 0,$$

and therefore $\mathcal{M}_n \to \mathcal{M}$ also in the Gromov-Hausdorff metric (see [KM16b] for a similar construction).

2. Let $\mathcal{M} = [0, 1]^2$ endowed with the standard Euclidean metric. Fix, say, $\varepsilon = 1/10$. For every $n \in \mathbb{N}$, define a discontinuous metric $\mathfrak{g}_n$ on $\mathcal{M}$ as follows:

$$\mathfrak{g}_n(x, y) = \begin{cases} \varepsilon\,\mathfrak{e} & (x, y) \in D \\ \mathfrak{e} & \text{otherwise}, \end{cases}$$

where $(x, y) \in D$ if and only if there exists a $j \in \{1, \ldots, n\}$ such that $|x - j/n| < 1/2n^2$ or $|y - j/n| < 1/2n^2$. That is, $\mathfrak{g}_n = \mathfrak{e}$ everywhere except on a set of n horizontal and n vertical strips of width $1/n^2$, in which it is shrunk isotropically by a factor ε. Let $F_n : (\mathcal{M}, \mathfrak{e}) \to (\mathcal{M}, \mathfrak{g}_n)$ be the map $x \mapsto x$. Then $dF_n \in \mathrm{SO}(\mathfrak{e}, \mathfrak{g}_n)$ everywhere except for a set of volume of order $1/n$; on that "defective" set, $\operatorname{dist}(dF_n, \mathrm{SO}(\varepsilon, \mathfrak{g}_n))$ is a constant independent of n. The same properties apply for F_n^{-1}. It follows that $(\mathcal{M}, \mathfrak{g}_n) \to (\mathcal{M}, \mathfrak{e})$ according to Definition 5.2 for every choice of $p, q < \infty$. On the other hand, $(\mathcal{M}, \mathfrak{g}_n)$ converges in the Gromov-Hausdorff sense to the "taxi-driver" ℓ^1 metric on $[0, \varepsilon]^2$.

Note that there exist $\alpha > 0$ and p, q such that if we take $\varepsilon_n = n^{-\alpha}$ rather than a fixed ε, $(\mathcal{M}, \mathfrak{g}_n) \to (\mathcal{M}, \mathfrak{e})$ according to Definition 5.2, whereas The Gromov-Hausdorff limit of this sequence is just the point.

▲▲▲

Acknowledgements We are grateful to Pavel Giterman for useful discussions and to Andreas Cap for providing us with a proof of Proposition B.3. This research was partially supported by the Israel Science Foundation (Grant No. 661/13), and by a grant from the Ministry of Science, Technology and Space, Israel and the Russian Foundation for Basic Research, the Russian Federation.

A Intrinsic determinant and cofactor

In this section we prove some useful properties of the determinant and cofactor operators defined in Section 2.3. Most of this section is linear algebra. We include it here for the sake of completeness. At the end of this section we include expressions for the determinant and cofactor in local coordinates.

Proposition A.1 Let V and W be d-dimensional oriented inner-product spaces. Let $\star_V$ and $\star_W$ be their Hodge-dual operators. The inner-products and the orientations induce volume forms on Vol_V and Vol_W (i.e., $Vol_V(e_1, \dots, e_d) = 1$ for every positively-oriented orthonormal basis of V). Let $T \in \operatorname{Hom}(V, W)$. Then,

$$\operatorname{Det} T = \frac{T^* Vol_W}{Vol_V}.$$

Proof: Let v_i, w_i be oriented orthonormal bases for V and W. Write $T(v_i) = a_i^j w_j$, and denote by A the matrix (a_i^j). Note that A^T is the matrix representing T in the bases v_i, w_i. On the one hand,

$$\begin{aligned} T^*\mathrm{Vol}_W(v_1, \dots, v_d) &= \mathrm{Vol}_W\left(T\left(v_1\right), \dots, T\left(v_d\right)\right) \\ &= \mathrm{Vol}_W\left(a_1^j w_j, \dots, a_d^j w_j\right) \\ &= (\det A) \cdot \mathrm{Vol}_W\left(w_1, \dots, w_d\right) \\ &= \det A. \end{aligned} \tag{A.1}$$

where the passage to the third line follows from the alternating property of Vol_W and the definition of the determinant of a matrix.

On the other hand,

$$\begin{aligned}
\operatorname{Det} T &= \operatorname{Det} T \cdot \mathrm{Vol}_V\,(v_1,\ldots,v_d) \\
&= \star_W \circ \bigwedge^d T \circ \star_V(1) \\
&= \star_W\left(\bigwedge^d T\,(v_1\wedge\ldots\wedge v_d)\right) \\
&= \star_W\,(T(v_1)\wedge\ldots\wedge T(v_d)) \\
&= \star_W\left(a_1^j w_j\wedge\ldots\wedge a_d^j w_j\right) \\
&= \star_W\,(\det A\,(w_1\wedge\ldots\wedge w_d)) \\
&= \det A,
\end{aligned} \tag{A.2}$$

where the passage to the sixth line follows from the alternating property of the wedge product and the definition of the determinant of a matrix (see [War71, Exercise 10, p. 78]).

Combining (A.1), (A.2),

$$(T^*\mathrm{Vol}_W)\,(v_1,\ldots,v_d) = \operatorname{Det} T\cdot \mathrm{Vol}_V\,(v_1,\ldots,v_d)\,.$$

Since the linear space of volume forms is one-dimensional, it is sufficient to check this equality on a single basis, hence $T^*\mathrm{Vol}_W = \operatorname{Det} T\,\mathrm{Vol}_V$ as required.

■

Proposition A.2 Let $f:\mathcal{M}\to\mathcal{N}$, *then*

$$\operatorname{Det} df = \frac{f^\star dVol_{\mathfrak{h}}}{dVol_{\mathfrak{g}}}$$

Proof: This is an immediate consequence of Proposition A.1 ■

Before we proceed to the definition of the intrinsic cofactor, we note a couple of simple facts on transposed maps. Recall that if V and W are d-dimensional oriented inner-product spaces and $A\in\mathrm{Hom}(V,W)$, then $A^T\in\mathrm{Hom}(W,V)$ is defined by

$$(Av,w)_W = (v,A^Tw)_V.$$

In particular, if $\{v_i\}$ and $\{w_i\}$ are orthonormal bases for V and W, then the matrices representing A and A^T are the mutual transposes of each other.

Corollary A.3 *Let V and W be d-dimensional oriented inner-product spaces. Let $A \in \mathrm{Hom}(V, W)$. Then,*

$$\mathrm{Det}\, A = \mathrm{Det}\, A^T.$$

Proof: This follows from (A.2) and the standard result for matrices. ■

Lemma A.4 *Let $S \in \mathrm{Hom}(V, W)$. Then ,*

$$\left(\bigwedge^p S\right)^T = \bigwedge^p S^T.$$

Proof:

$$\begin{aligned}
\left\langle \bigwedge^p S(v_1 \wedge \ldots \wedge v_p), w_1 \wedge \ldots \wedge w_p \right\rangle_{\Lambda_p(W)} &= \left\langle S(v_1) \wedge \ldots \wedge S(v_p), w_1 \wedge \ldots \wedge w_p \right\rangle_{\Lambda_p(W)} \\
&= \det\left(\left\langle S(v_i), w_j \right\rangle_W\right) \\
&= \det\left(\left\langle v_i, S^T(w_j) \right\rangle_V\right) \\
&= \left\langle v_1 \wedge \ldots \wedge v_p, S^T(w_1) \wedge \ldots \wedge S^T(w_p) \right\rangle_{\Lambda_p(V)} \\
&= \left\langle v_1 \wedge \ldots \wedge v_p, \bigwedge^p S^T\left(w_1 \wedge \ldots \wedge w_p\right) \right\rangle_{\Lambda_p(V)}.
\end{aligned}$$

The desired equality follows from the definition of the transpose map. ■

Lemma A.5 *The Hodge-dual operator satisfies:*

$$(\star_V^p)^T = (-1)^{p(d-p)} \star_V^{d-p}.$$

Proof: It is well known that $\star_V$ is an isometry, and that $\star_V^p \circ \star_V^{d-p} = (-1)^{p(d-p)} Id$. Thus

$$(\star_V^p)^T = (\star_V^p)^{-1} = (-1)^{p(d-p)} \star_V^{d-p}.$$

■

Corollary A.6 *Let $S \in \mathrm{Hom}(V, W)$. Then,*

$$\mathrm{Det}\, S = \mathrm{Det}\, S^T \qquad \text{and} \qquad (\mathrm{Cof}\, S)^T = \mathrm{Cof}\, S^T.$$

Proof: The claim about the determinant was proved above, but it can also be proved in the following way: since

$$\operatorname{Det} S = \star_W^d \circ \bigwedge^d S \circ \star_V^0,$$

it follows that

$$(\operatorname{Det} S)^T = (\star_V^0)^T \circ \left(\bigwedge^d S\right)^T \circ (\star_W^d)^T = \star_V^d \circ \left(\bigwedge^d S^T\right) \circ \star_W^0 = \operatorname{Det} S^T.$$

Since on $\mathbb{R}$, the transpose is the identity map,

$$\operatorname{Det} S = (\operatorname{Det} S)^T = \operatorname{Det} S^T.$$

Similarly, the cofactor is defined by

$$\operatorname{Cof} S = (-1)^{d-1} \star_W^{d-1} \circ \bigwedge^{d-1} S \circ \star_V^1,$$

and therefore

$$\begin{aligned}
(\operatorname{Cof} S)^T &= (-1)^{d-1} (\star_V^1)^T \circ (\bigwedge^{d-1} S)^T \circ (\star_W^{d-1})^T \\
&= (-1)^{d-1} (\star_V^{d-1}) \circ (\bigwedge^{d-1} S^T) \circ \star_W^1 \\
&= \operatorname{Cof} S^T.
\end{aligned}$$

■

Proposition A.7 The following identities holds:

$$\operatorname{Det} A \ \mathrm{Id}_V = A^T \circ \operatorname{Cof} A = (\operatorname{Cof} A)^T \circ A,$$

and

$$\operatorname{Det} A \ \mathrm{Id}_W = A \circ (\operatorname{Cof} A)^T = \operatorname{Cof} A \circ A^T.$$

Proof: Let $v, u \in V$. Then,

$$
\begin{aligned}
(A^T \circ \operatorname{Cof} A(v), u)_V &= (\operatorname{Cof} A(v), Au)_W \\
&= (-1)^{d-1} (\star_W^{d-1} \bigwedge^{d-1} A \star_V^1 v, Au)_W \\
&= (-1)^{d-1} \star_W^d \left(Au \wedge \star_W^1 \star_W^{d-1} \bigwedge^{d-1} A \star_V^1 v \right) \\
&= \star_W^d (Au \wedge \bigwedge^{d-1} A \star_V^1 v) \\
&= \star_W^d \left(\bigwedge^d A \left(u \wedge \star_V^1 v \right) \right) \\
&= \star_W^d \left(\bigwedge^d A \star_V^0 \star_V^d \left(u \wedge \star_V^1 v \right) \right) \\
&= \left(\star_W^d \bigwedge^d A \star_V^0 \right) \langle u, v \rangle_V \\
&= \operatorname{Det} A \, \langle u, v \rangle_V \,,
\end{aligned}
$$

where the passage to third line follows from the identity

$$\langle v, w \rangle_{\Lambda_p(V)} = \star_V^d (v \wedge \star_V^p w)$$

for $v, w \in \Lambda_p(V)$. Hence, for every $v \in V$,

$$A^T \circ \operatorname{Cof} A(v) = \operatorname{Det} A \ \operatorname{Id}(v).$$

The other equalities follow by transposition, using the fact that $\operatorname{Det} A^T = \operatorname{Det} A$ (Corollary A.3) and $(\operatorname{Cof} A)^T = \operatorname{Cof}(A^T)$ (Corollary A.6). ■

Corollary 2.6 is immediate from Proposition A.7.

The following lemma is useful for proving the weak convergence of $\operatorname{Cof} df_n$ and $\operatorname{Det} df_n$:

Lemma A.8 Let $(V, \mathfrak{g})$ *and* $(W, \mathfrak{h})$ *be d-dimensional oriented inner-product spaces. Let* $b = (b_1, \dots, b_d)$ *and* $c = (c_1, \dots, c_d)$ *be arbitrary bases for V and W. Let* $F \in \operatorname{Hom}(V, W)$*, and let A be its matrix representation in the given bases. Denote by* A^T, $\operatorname{Cof} A$ *and* $\operatorname{Det} A$ *the matrix representations of* F^T, $\operatorname{Cof} F$ *and* $\operatorname{Det} F$ *in the given bases. Denote by* A^t, $\operatorname{cof} A$ *and* $\operatorname{det} A$ *the transpose, cofactor and determinant of the* matrix *A (that is, the "standard" linear-algebraic meaning of these notions). Denote by G and H the matrix representations of* $\mathfrak{g}$ *and h. Then,*

$$A^T = G^{-1} A^t H, \tag{A.3}$$

$$\operatorname{Det} A = \sqrt{\frac{\det H}{\det G}} \det A, \tag{A.4}$$

and

$$\operatorname{Cof} A = \sqrt{\frac{\det H}{\det G}} H^{-1} \operatorname{cof} A\, G. \tag{A.5}$$

Proof: Let $v \in V$ and $w \in W$. By definition $\mathfrak{h}(Fv, w) = \mathfrak{g}(v, F^T w)$. Moving to coordinates and writing this in matrix form, this reads

$$v^t A^t H w = v^t G A^T w,$$

from which (A.3) follows immediately. Equation A.4 follows from Proposition A.2. Using these two identities, (A.5) follows from Proposition A.7 by a direct calculation. ■

Proposition A.9 *Let $\mathcal{M}$ be a compact d-dimensional manifold. Let E an oriented d-dimensional vector bundle endowed with a Riemannian metric. Let $f_n \in W^{1,p}(\mathcal{M}; E)$ with $p > d$. If $f_n \rightharpoonup f$ in $W^{1,p}(\mathcal{M}; E)$, then*

$$\operatorname{Det} df_n \rightharpoonup \operatorname{Det} df \qquad \text{in } L^{p/d}(\mathcal{M}),$$

and

$$\operatorname{Cof} df_n \rightharpoonup \operatorname{Cof} df \qquad \text{in } L^{p/(d-1)}(\mathcal{M}; E).$$

Proof: The case $\mathcal{M} \subset \mathbb{R}^d$, $E = \mathbb{R}^d$ is a classical result in the theory of Sobolev mappings, see e.g. [Eva98, Section 8.2.4] (this reference only considers the determinant, however the same proof applies for the cofactor matrix).

Still in a compact Euclidean setting, if a sequence $g_n \in L^q(\Omega)$ weakly converges to g in $L^q(\Omega)$ for some q, then $\phi g_n \rightharpoonup \phi g$ in $L^q(\Omega)$ for every smooth function ϕ. The proposition follows now from Lemma A.8 by working in local coordinates and using the Euclidean result, since H, G, their inverses and determinants are all smooth functions of the coordinates. ■

We conclude this section by summarizing some of the important formulae mentioned above, written in coordinates. These formulae are important when considering weak harmonicity.

Let (v_i) be a basis of V and let (w_α) be a basis of W, with (v^i) and (w^α) their dual bases. A mapping $A : V \to W$ is represented in coordinates by a matrix A_i^α.

Denote the entries of the metrics on V and W by $\mathfrak{g}_{ij}$ and $\mathfrak{h}_{\alpha\beta}$, and let $\mathfrak{g}^{ij}$ and $\mathfrak{h}^{\alpha\beta}$ denote their inverses (the metrics on the dual spaces). We denote by $|\mathfrak{g}|$ the determinant of $\mathfrak{g}_{ij}$, and similarly for $\mathfrak{h}$.

Invoking Einstein summation convention, and using (A.3), Proposition A.7 reads

$$A_i^\alpha \mathfrak{g}^{ij} (\operatorname{Cof} A)_j^\gamma \mathfrak{h}_{\gamma\beta} = \operatorname{Det} A \, \delta_\beta^\alpha, \tag{A.6}$$

or equivalently

$$A_i^\alpha \mathfrak{g}^{ij} (\operatorname{Cof} A)_j^\beta = \operatorname{Det} A \, \mathfrak{h}^{\alpha\beta}. \tag{A.7}$$

Eq. (A.5) in Proposition A.8 reads

$$(\operatorname{Cof} A)_i^\alpha = \frac{\sqrt{|\mathfrak{h}|}}{\sqrt{|\mathfrak{g}|}} h^{\alpha\beta} \delta_{\beta\gamma} (\operatorname{cof} A)_k^\gamma \delta^{kj} \mathfrak{g}_{ji}. \tag{A.8}$$

or equivalently

$$\mathfrak{h}_{\alpha\beta} (\operatorname{Cof} A)_j^\beta \mathfrak{g}^{ji} = \frac{\sqrt{|\mathfrak{h}|}}{\sqrt{|\mathfrak{g}|}} \delta_{\alpha\beta} (\operatorname{cof} A)_j^\beta \delta^{ji}. \tag{A.9}$$

Finally,

$$\frac{(\operatorname{Cof} A)^T}{\operatorname{Det} A} = \frac{(\operatorname{cof} A)^t}{\det A}.$$

B Proofs for Section 2.4

Proof of Lemma 2.9: For $\alpha \in \Lambda_k(V)$ and $w \in W$,

$$\begin{aligned}
\star_{V,W}^{d-k} \star_{V,W}^{k} (\alpha \otimes w) &= \star_{V,W}^{d-k}((\star_V^k \alpha) \otimes w) \\
&= (\star_V^{d-k} \star_V^k \alpha) \otimes w \\
&= (-1)^{k(d-k)} (\alpha \otimes \omega).
\end{aligned}$$

■

Proof of Lemma 2.10: Since both sides of equation (2.5) are bilinear in α, β, it is enough to prove the identity for simple tensors, i.e., we can assume that $\alpha = v_1 \otimes w_1$ and $\beta = v_2 \otimes w_2$, whewe $v_1, v_2 \in \Lambda_k(V)$ and $w_1, w_2 \in W$.

On the one hand,

$$\begin{aligned}
\operatorname{tr}_W\left(\alpha \wedge \star^k_{V,W}\beta\right) &= \operatorname{tr}_W\left((v_1 \otimes w_1) \wedge \star^k_{V,W}(v_2 \otimes w_2)\right) \\
&= \operatorname{tr}_W\left((v_1 \otimes w_1) \wedge \left(\star^k_V v_2 \otimes w_2\right)\right) \\
&= \operatorname{tr}_W\left(\left(v_1 \wedge \star^k_V v_2\right) \otimes (w_1 \otimes w_2)\right) \\
&= \left(v_1 \wedge \star^k_V v_2\right) \operatorname{tr}_W(w_1 \otimes w_2) \\
&= \left(v_1 \wedge \star^k_V v_2\right) (w_1, w_2)_W.
\end{aligned} \tag{B.1}$$

On the other hand,

$$\begin{aligned}
(\alpha, \beta)_{V,W}\,(\star^0_V 1) &= (v_1 \otimes w_1, v_2 \otimes w_2)_{\Lambda_k(V),W}\,(\star^0_V 1) \\
&= (v_1, v_2)_{\Lambda_k(V)}\,(w_1, w_2)_W\,(\star^0_V 1) \\
&= \left(v_1 \wedge \star^k_V v_2\right) (w_1, w_2)_W,
\end{aligned} \tag{B.2}$$

where the last equality follows from the definition of the Hodge-star operator $\star^k_V$. Equations (B.1),(B.2) imply identity (2.5). ■

As a step towards Lemma 2.11, we first prove a similar commutation relation between covectors and the Hodge-dual operator:

Lemma B.1 (Covectors commute with the Hodge-dual) Let V be a d-dimensional oriented inner-product space. Let $\alpha \in V^* \simeq \Lambda_1(V^*)$. Then,

$$(\star^1_{V^*}\alpha)(v_1, \dots, v_{d-1}) = (-1)^{d-1}\alpha\left(\star^{d-1}_V(v_1 \wedge \dots \wedge v_{d-1})\right). \tag{B.3}$$

Proof: Let $(v_i)_{i=1}^n$ be a positively-oriented orthonormal basis for V and let $(\alpha^i)_{i=1}^n$ denote the dual basis for V^*. By linearity, it suffices to prove (B.3) for $\alpha = \alpha^j$ and $(v_1, \dots, v_{d-1})$ basis elements; since $\star^{d-1}_V(v_1 \wedge \dots \wedge v_{d-1}) = v_d$, it suffices to prove that

$$(\star^1_{V^*}\alpha^j)(v_1, ..., v_{d-1}) = (-1)^{d-1}\alpha^j(v_d).$$

Since α^j is a positively-oriented orthonormal basis for V^*,

$$(\star^1_{V^*}\alpha^j) = (-1)^{j-1}\alpha^1 \wedge \dots \hat{\alpha}^j \dots \wedge \alpha^d,$$

where the hat above α^j indicates that this term is omitted from the product. Hence,

$$(\star^1_{V^*}\alpha^j)(v_1, ..., v_{d-1}) = \begin{cases} 0 & \text{if } j \neq d \\ (-1)^{d-1} & \text{if } j = d \end{cases} = (-1)^{d-1}\alpha_j(v_d),$$

which completes the proof. ∎

Proof of Lemma 2.11: Since both sides of the equality are linear in A, we may assume that $A = \alpha \otimes w$, where $\alpha \in V^* \simeq \Lambda_1(V^*)$ and $w \in W$. Using Definition 2.8,

$$\begin{aligned}(\star^1_{V^*,W} A)(v_1, \ldots, v_{d-1}) &= \left(\left(\star^1_{V^*}\alpha\right) \otimes w\right)(v_1, \ldots, v_{d-1}) \\ &= \left(\star^1_{V^*}\alpha\right)(v_1, \ldots, v_{d-1})\, w.\end{aligned} \tag{B.4}$$

On the other hand,

$$\begin{aligned}A\left(\star^{d-1}_V (v_1 \wedge \cdots \wedge v_{d-1})\right) &= (\alpha \otimes w)\left(\star^{d-1}_V (v_1 \wedge \cdots \wedge v_{d-1})\right) \\ &= \alpha\left(\star^{d-1}_V (v_1 \wedge \cdots \wedge v_{d-1})\right) w.\end{aligned} \tag{B.5}$$

Equations (B.4), (B.5), together with Lemma B.1 imply the desired equality.

∎

Proof of Lemma 2.12: We first show that it suffices to prove this lemma for $k = 0$. That is, assume that for every $\xi \in C^\infty(\mathcal{M}) \simeq \Gamma(\Lambda_0(E))$ and $X \in \Gamma(T\mathcal{M})$,

$$\star^0_E (\nabla^E_X(\xi)) = \nabla^E_X(\star^0_E(\xi)). \tag{B.6}$$

Let $\alpha, \beta \in \Gamma(\Lambda_k(E))$ and let $X \in \Gamma(T\mathcal{M})$. By the Leibniz rule for covariant differentiation and the definition of the Hodge-dual,

$$\begin{aligned}\nabla^E_X(\alpha \wedge \star^k_E\beta) &= \nabla^E_X\alpha \wedge \star^k_E\beta + \alpha \wedge \nabla^E_X(\star^k_E\beta) \\ &= \star^0_E(\nabla^E_X\alpha, \beta)_{\mathfrak{h}} + \alpha \wedge \nabla^E_X(\star^k_E\beta).\end{aligned} \tag{B.7}$$

On the other hand,

$$\begin{aligned}\nabla^E_X(\alpha \wedge \star^k_E\beta) &= \nabla^E_X(\star^0_E(\alpha, \beta)_{\mathfrak{h}}) \\ &= \star^0_E\left(\nabla^E_X(\alpha, \beta)_{\mathfrak{h}}\right) \\ &= \star^0_E\left((\nabla^E_X\alpha, \beta)_{\mathfrak{h}} + (\alpha, \nabla^E_X\beta)_{\mathfrak{h}}\right) \\ &= \star^0_E(\nabla^E_X\alpha, \beta)_{\mathfrak{h}} + \alpha \wedge \star^k_E(\nabla^E_X\beta),\end{aligned} \tag{B.8}$$

where the passage from the first to the second line uses (B.6) for $\xi = (\alpha, \beta)_{\mathfrak{h}}$. Equalities (B.7) and (B.8) imply that

$$\alpha \wedge \nabla^E_X(\star^k_E\beta) = \alpha \wedge \star^k_E(\nabla^E_X\beta).$$

Since this holds for every $\alpha \in \Gamma(\Lambda_k(E))$, we conclude that $\nabla^E_X(\star^k_E\beta) = \star^k_E(\nabla^E_X\beta)$.
Thus, we turn to prove (B.6). Let $\beta \in C^\infty(\mathcal{M}) \simeq \Gamma(\Lambda_0(E))$, and note that

$$\star^0_E\,(\nabla^E_X\beta) = (\nabla^E_X\beta)\ \star^0_E(1), \tag{B.9}$$

where $\star^0_E(1)$ is the positive unit d-dimensional multivector. Likewise,

$$\nabla^E_X(\star^0_E(\beta)) = \nabla^E_X(\beta\ \star^0_E(1)) = (\nabla^E_X\beta)\ \star^0_E(1) + \beta\,\nabla^E_X(\star^0_E(1)). \tag{B.10}$$

Comparing (B.9) and (B.10), we conclude that (B.6) holds for every β if and only if

$$\nabla^E_X(\star^0_E(1)) = 0, \tag{B.11}$$

which is indeed the case, because $\star^0_E(1)$ is the unit d-dimensional multivector and ∇^E is consistent with the metric. ■

In the last part of this section we prove Lemma 2.15. We start by several intermediate steps.

Definition B.2 Let E and F be vector bundles over a smooth manifold $\mathcal{M}$. Suppose that E and F are endowed with connections ∇^E, ∇^F. Let $\Phi : E \to F$ be a vector bundle morphism. We say that Φ respects the connections if

$$\nabla^F_X\Phi(s) = \Phi(\nabla^E_X s)$$

for every $s \in \Gamma(E)$ and for every vector field $X \in \Gamma(T\mathcal{M})$.

Proposition B.3 (Covariant exterior derivative commutes with vector bundle morphisms) Let $(E, \nabla^E), (F, \nabla^F)$ be vector bundles with connections over $\mathcal{M}$. Let $\Phi : E \to F$ be a vector bundle morphism respecting the connections. Φ induces maps

$$\Phi_* : \Omega^k(\mathcal{M}, E) \to \Omega^k(\mathcal{M}, F)$$

by acting on the values of forms. Then, Φ_* is compatible with the covariant exterior derivatives, namely,

$$d_{\nabla^F}(\Phi_*\alpha) = \Phi_*(d_{\nabla^E}\alpha)$$

for every $\alpha \in \Omega^k(\mathcal{M}, E)$.

Proof: We use formula (2.7) for the exterior derivative: Let $X_1, \dots, X_{k+1} \in \Gamma(T\mathcal{M})$ and $\beta \in \Omega^k(\mathcal{M}, F)$. Then,

$$
\begin{aligned}
(d_{\nabla^F}\beta)(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla^F_{X_i} \beta(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) \\
&+ \sum_{i<j} (-1)^{i+j} \beta([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1})
\end{aligned}
$$

Set $\beta = \Phi_* \alpha$. It follows from the compatibility of Φ with the connections and the linearity of Φ_* that

$$
\begin{aligned}
(d_{\nabla^F}(\Phi_*\alpha))(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} (\Phi_*(\nabla^E_{X_i}\alpha))(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) \\
&\quad + \sum_{i<j} (-1)^{i+j} (\Phi_*\alpha)([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}) \\
&= (\Phi_*(d_{\nabla^E}\alpha))(X_1, \dots, X_{k+1}),
\end{aligned}
$$

which concludes the proof. ■

Corollary B.4 (Covariant exterior derivative commutes with contraction) Let $(E, \mathcal{M})$ be defined as above with a metric $\mathfrak{h}$ and a metric connection ∇^E; note that ∇^E induces a connection on $E \otimes E$. Let $\sigma \in \Omega^k(\mathcal{M}, E \otimes E)$ be an $E \otimes E$-valued differential form of degree k. Then,

$$
\mathrm{tr}_{\mathfrak{h}}(d_{\nabla^{E\otimes E}}\sigma) = d(\mathrm{tr}_{\mathfrak{h}}(\sigma)),
$$

where d is the standard exterior derivative.

Proof: We apply Proposition B.3, with $E \mapsto E \otimes E$, $F \mapsto \mathcal{M} \times \mathbb{R}$ with the trivial connection (so the covariant exterior derivative becomes the standard exterior derivative) and $\Phi = tr_{\mathfrak{h}}$.

We need to show that tr_η respects the connections, i.e. to show that

$$
X\left(\mathrm{tr}_{\mathfrak{h}}(\sigma)\right) = \mathrm{tr}_{\mathfrak{h}}(\nabla^{E\otimes E}_X \sigma)
$$

for all $\sigma \in \Gamma(E \otimes E)$ and for any vector field $X \in \Gamma(T\mathcal{M})$. This equality is, in fact, equivalent to the assertion that the connection is metric. Since the assertion is local, we can set $\sigma = \alpha \otimes \beta$, where $\alpha, \beta \in \Gamma(E)$. Then,

$$
\begin{aligned}
\mathrm{tr}_{\mathfrak{h}}(\nabla^{E\otimes E}_X(\alpha\otimes\beta)) &= \mathrm{tr}_{\mathfrak{h}}(\nabla^E_X\alpha\otimes\beta) + \mathrm{tr}_{\mathfrak{h}}(\alpha\otimes\nabla^E_X\beta) \\
&= (\nabla^E_X\alpha,\beta)_{\mathfrak{h}} + (\alpha,\nabla^E_X\beta)_{\mathfrak{h}} \\
&= X(\alpha,\beta)_{\mathfrak{h}} \\
&= X\,\mathrm{tr}_{\mathfrak{h}}(\alpha\otimes\beta).
\end{aligned}
$$

■

Proof of Lemma 2.15: Let $\rho \in \Omega^k(\mathcal{M}; E)$ and let $\sigma \in \Omega^{k-1}(\mathcal{M}; E)$ have compact support. Using Lemma 2.10,

$$
\begin{aligned}
\int_{\mathcal{M}} \langle d_{\nabla^E}\sigma, \rho\rangle_{\mathfrak{g},\mathfrak{h}} \,\mathrm{Vol}_{\mathfrak{g}} &= \int_{\mathcal{M}} \mathrm{tr}_{\mathfrak{h}}(d_{\nabla^E}\sigma \wedge \star^k_{T^*\mathcal{M},E}\rho) \\
&= \int_{\mathcal{M}} \mathrm{tr}_{\mathfrak{h}}\left(d^{\nabla^E}\left(\sigma\wedge\star^k_{T^*\mathcal{M},E}\rho\right)\right) - \int_{\mathcal{M}} (-1)^{k-1}\,\mathrm{tr}_{\mathfrak{h}}(\sigma\wedge d_{\nabla^E}\star^k_{T^*\mathcal{M},E}\rho) \\
&\overset{(1)}{=} \int_{\mathcal{M}} d\,\mathrm{tr}_{\mathfrak{h}}(\sigma\wedge\star^k_{T^*\mathcal{M},E}\rho) + (-1)^k\int_{\mathcal{M}} \mathrm{tr}_{\mathfrak{h}}(\sigma\wedge d_{\nabla^E}\star^k_{T^*\mathcal{M},E}\rho) \\
&= (-1)^k(-1)^{(d-k+1)(k-1)}\int_{\mathcal{M}} \mathrm{tr}_{\mathfrak{h}}(\sigma\wedge\star^{k-1}_{T^*M,E}\star^{d-k+1}_{T^*M,E} d_{\nabla^E}\star^k_{T^*\mathcal{M},E}\rho) \\
&= (-1)^k(-1)^{(d-k+1)(k-1)}\int_{\mathcal{M}} \left\langle\sigma, \star^{d-k+1}_{T^*M,E} d_{\nabla^E}\star^k_{T^*\mathcal{M},E}\rho\right\rangle_{\mathfrak{g},\mathfrak{h}} \mathrm{Vol}_{\mathfrak{g}}.
\end{aligned}
$$

The passage from the second to the third line follows from Corollary B.4; the vanishing of the first integral on the third line follows from Stokes theorem along with the fact σ has compact support. Comparing this equality with Definition 2.14 of the covariant coderivative, we obtain the desired result.

■

C Volume distortion and $\mathrm{dist}(\cdot, \mathrm{SO}_d)$

Let $A \in M_d$ be a linear transformation. A maps the unit cube (which has volume 1 in $\mathbb{R}^d$) into a body whose volume is $\det A$. We may therefore view $|\det A - 1|$ as a measure of **volume distortion** caused by the action of A. Intuitively, when A is close to an (orientation-preserving) isometry, its volume distortion should be small. The following lemma is a quantitative formulation of this claim:

Lemma C.1 *Let $A \in M_d$. Then*

$$|\det A - 1| \leq (\operatorname{dist}(A, \mathrm{SO}_d) + 1)^d - 1$$

Proof: Let $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_d$ be the singular values of A, and define $r_1 = \operatorname{sgn}(\det A)\,\sigma_1$, $r_i = \sigma_i$ for $i = 2, \ldots, d$.
We then have $\det A = \Pi_{i=1}^d r_i$ and for every $1 \leq i \leq d$,

$$\operatorname{dist}(A, \mathrm{SO}_d) = \sqrt{\sum_{j=1}^{d} (r_j - 1)^2} \geq |r_i - 1|.$$

We will show that

$$|\Pi_{i=1}^d r_i - 1| \leq \Pi_{i=1}^d (|r_i - 1| + 1) - 1, \tag{C.1}$$

which will complete the proof since it will follow that

$$|\det A - 1| \leq \Pi_{i=1}^d (|r_i - 1| + 1) - 1 \leq (\operatorname{dist}(A, \mathrm{SO}_d) + 1)^d - 1.$$

We turn to prove (C.1).
Bounding from above is trivial:

$$\Pi_{i=1}^d r_i \leq \Pi_{i=1}^d (|r_i - 1| + 1)$$

The less trivial part is bounding from below. We need to show:

$$\Pi_{i=1}^d r_i - 1 \geq -\left(\Pi_{i=1}^d (|r_i - 1| + 1) - 1\right) = 1 - \Pi_{i=1}^d (|r_i - 1| + 1)$$

which is equivalent to:

$$2 \leq \Pi_{i=1}^d r_i + \Pi_{i=1}^d (|r_i - 1| + 1) \tag{C.2}$$

First, assume $A \in \mathrm{GL}_d^+$. Note that if $r_j \geq 1$ for some j,

$$\Pi_{i=1}^d r_i + \Pi_{i=1}^d (|r_i - 1| + 1) \geq \Pi_{i \neq j} r_i + \Pi_{i \neq j} (|r_i - 1| + 1).$$

Therefore, it is enough to prove (C.2) under the assumption that $r_i \in (0, 1)$ for all i, that is, to prove that

$$f(r_1, \ldots, r_d) = \Pi_{i=1}^d r_i + \Pi_{i=1}^d (2 - r_i) \geq 2.$$

Notice that the inequality holds on the boundary of $[0,1]^d$, and therefore it is enough to prove that f has no local minima at $(0,1)^d$. Indeed, if $r = (r_1, \dots, r_d) \in \partial([0,1]^d)$ then there exists some i such that $r_i = 0$ or $r_i = 1$. If $r_i = 1$ the inequality holds by induction on the dimension. If $r_i = 0$, the inequality reduces to $\Pi_{j\neq i}^d (2 - r_j) \geq 1$ which holds by the assumption $r_i \in (0,1)$.
Differentiating in the interior $(0,1)^d$ we obtain

$$\frac{\partial f}{\partial r_j} = \Pi_{i\neq j} r_i - \Pi_{i\neq j}^d (2 - r_i) < 0,$$

since $r_i \in (0,1)$ for every i. Therefore there are no local minima at $(0,1)^d$, which completes the proof for $A \in \mathrm{GL}_d^+$.
For $A \notin \mathrm{GL}_d^+$, we need to prove (C.2). Note that in this case $r_1 \leq 0$, and therefore $|r_1 - 1| + 1 = 2 - r_1$. We obtain that

$$\begin{aligned}\Pi_{i=1}^d (|r_i - 1| + 1) + \Pi_{i=1}^d r_i &= 2\Pi_{i=2}^d (|r_i - 1| + 1) - r_1 \left(\Pi_{i=2}^d (|r_i - 1| + 1) - \Pi_{i=2}^d r_i \right) \\ &\geq 2 - r_1 \left(\Pi_{i=2}^d (|r_i - 1| + 1) - \Pi_{i=2}^d r_i \right)\end{aligned}$$

Now, the term in the parentheses is non-negative and $-r_1 \geq 0$, and therefore (C.2) holds.

■

Lemma C.2 Let $f : (\mathcal{M}, \mathfrak{g}) \to (\mathcal{N}, \mathfrak{h})$ be an orientation-preserving diffeomorphism between compact manifolds. Then

$$|Vol_{\mathfrak{h}}(\mathcal{N}) - Vol_{\mathfrak{g}}(\mathcal{M})| \leq \int_{\mathcal{M}} \left[(\operatorname{dist}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) + 1)^d - 1 \right] dVol_{\mathfrak{g}}$$

Proof:

$$\mathrm{Vol}_{\mathfrak{h}} \mathcal{N} = \int_{\mathcal{N}} d\mathrm{Vol}_{\mathfrak{h}} = \int_{\mathcal{M}} f^\star (d\mathrm{Vol}_{\mathfrak{h}}) = \int_{\mathcal{M}} (\mathrm{Det}\, df)\, d\mathrm{Vol}_{\mathfrak{g}}.$$

Let $p \in \mathcal{M}$ and let v_i, w_i be positively oriented orthonormal bases for $T_p\mathcal{M}$ and $T_{f(p)}\mathcal{N}$. Let A be the representing matrix of df_p in these bases. Then, (i) $\det A > 0$ since f is orientation-preserving, (ii) $\mathrm{Det}\, df = \det A$ and (iii)

$$\operatorname{dist}_{(\mathfrak{g}, f^*\mathfrak{h})} (df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) = \operatorname{dist}_{\mathfrak{e}} (A, \mathrm{SO}_d),$$

where $\mathfrak{e}$ is the Euclidean metric. Thus

$$\begin{aligned}
|\mathrm{Vol}(\mathcal{N}) - \mathrm{Vol}(\mathcal{M})| &= \left|\int_{\mathcal{M}} (\mathrm{Det}\, df - 1)\, d\mathrm{Vol}_{\mathfrak{g}}\right| \\
&= \left|\int_{\mathcal{M}} (\det A - 1)\, d\mathrm{Vol}_{\mathfrak{g}}\right| \\
&\leq \int_{\mathcal{M}} |\,\mathrm{Det}\, A - 1| d\mathrm{Vol}_{\mathfrak{g}} \\
&\leq \int_{\mathcal{M}} \left[(\mathrm{dist}_{\mathfrak{e}}\,(A, \mathrm{SO}_d) + 1)^d - 1\right] d\mathrm{Vol}_{\mathfrak{g}} \\
&= \int_{\mathcal{M}} \left[(\mathrm{dist}\,(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) + 1)^d - 1\right] d\mathrm{Vol}_{\mathfrak{g}},
\end{aligned}$$

where the the passage to the fourth line follows from Lemma C.1.

■

D Technical lemmas

D.1 Measurability of $\mathrm{dist}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))$

Lemma D.1 Let $\Omega \subset \mathbb{R}^d$ be an open domain and let $f : \Omega \to \mathbb{R}^d$ be a weakly differentiable function. Then the function $\mathrm{dist}(\nabla f, \mathrm{SO}(d)) : \Omega \to \mathbb{R}$ *is measurable.*

Proof: The function f is weakly differentiable, so by definition, its weak derivative ∇f is a measurable function. The function $\mathrm{dist}(A, \mathrm{SO}(d))$ defined on the space of matrices $M_n = \mathbb{R}^{d^2}$ is 1-Lipschitz. Thus, $\mathrm{dist}(\nabla f, \mathrm{SO}(d))$ is measurable as a composition of a continuous function and a measurable function. ■

Proposition D.2 Let $(\mathcal{M}, \mathfrak{g})$ *and* $(\mathcal{N}, \mathfrak{h})$ *be d-dimensional Riemannian manifolds. Let $p \geq 1$ and $f \in W^{1,p}(\mathcal{M}; \mathcal{N})$. Then the function* $\mathrm{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h})) : \mathcal{M} \to \mathbb{R}$ *is measurable.*

Proof: Since measurability of a function $\mathcal{M} \to \mathbb{R}$ is a local property, we can assume without loss of generality that $\mathcal{M} = \mathbb{R}^d$ (with a metric $\mathfrak{g}$). Let $A = \sqrt{\mathfrak{g}}$ be the unique symmetric positive-definite matrix satisfying $A^2 = \mathfrak{g}$. Then $A \in \mathrm{SO}(\mathfrak{g}, \mathfrak{e})$.

Let $(U_k)_k$ be an atlas of $\mathcal{N}$, and identify U_k with a subset of $\mathbb{R}^d$ endowed with a metric $\mathfrak{h}_k$ (for each U_k the metric is different, as it represents in coordinates the metric $\mathfrak{h}$ on a different part of $\mathcal{N}$.). Let $B_k = \sqrt{\mathfrak{h}_k}$, so that $B_k \in \mathrm{SO}(\mathfrak{h}_k, \mathfrak{e})$. Set $V_k = f^{-1}(U_k)$ (this is a measurable subset of $\mathbb{R}^d$).

A straightforward calculation shows that on V_k,

$$\mathrm{dist}_{(\mathfrak{g}, f^*\mathfrak{h})}(df, \mathrm{SO}(\mathfrak{g}, f^*\mathfrak{h}))|_x = \mathrm{dist}_{(\mathfrak{e},\mathfrak{e})}(B_k(f(x)) \circ df(x) \circ A^{-1}(x), \mathrm{SO}(d)).$$

Thus, we need to prove that the mapping $V_k \to \mathbb{R}^{d^2}$ defined by $x \mapsto B_k(f(x)) \circ df(x) \circ A^{-1}(x)$ is measurable. This composition is a multiplication of three matrices, so it suffices to show that each component is measurable. Indeed, the mapping $x \mapsto A^{-1}(x)$ is smooth, $x \mapsto B_k(f(x))$ is measurable since B_k is smooth and $f|_{V_k}$ is measurable. As for df, it is measurable, by definition, as a mapping $(x, v) \mapsto df(x)v$ (see [CS16]). Since it is obviously a normal integrand (even a Carathéodory function), this implies immediately that $x \mapsto df(x)$ is measurable (see [KM14, Appendix A] for information on normal integrands in such settings). ■

D.2 Equivalence of metrics

Lemma D.3 *Let $A \subseteq \mathbb{R}^d$ be a compact set, and let $\mathfrak{g} : A \to \mathrm{Psym}_d$ be a continuous metric on A. Then $\mathfrak{g}$ is strongly equivalent to the Euclidean metric $\mathfrak{e}$. Specifically,*

$$\|\mathfrak{g}^{-1}\|_\infty^{-1} \cdot \mathfrak{e}(v, v) \leq \mathfrak{g}(v, v) \leq \|\mathfrak{g}\|_\infty \cdot \mathfrak{e}(v, v),$$

where $\|\mathfrak{g}\|_\infty = \sup_{p\in A} |\mathfrak{g}_p|_{op}$ and $|\cdot|_{op}$ is the standard operator norm of matrices.

Proof: By definition of the operator norm, for $p \in A$

$$\mathfrak{g}_p(v, v) = v^T \mathfrak{g}_p v = \mathfrak{e}(v, \mathfrak{g}_p v) \leq |\mathfrak{g}_p|_{op} \mathfrak{e}(v, v) \leq \|\mathfrak{g}\|_\infty \, \mathfrak{e}(v, v). \tag{D.1}$$

Set $w = \sqrt{\mathfrak{g}_p} v$, where $\sqrt{\mathfrak{g}_p}$ is the unique symmetric positive-definite square root of $\mathfrak{g}$. Substituting $v \mapsto w$ and $\mathfrak{g}_p \to \mathfrak{g}_p^{-1}$ in (D.1) we obtain

$$\mathfrak{e}(v, v) = w^T \mathfrak{g}_p^{-1} w \leq \mathfrak{e}(w, w) \, |\mathfrak{g}_p^{-1}|_{op} = w^T w \, |g_p^{-1}|_{op} = v^T \mathfrak{g}_p v \, |\mathfrak{g}_p^{-1}|_{op},$$

which implies

$$\mathfrak{g}_p\,(v, v) \geq \left(|\mathfrak{g}_p^{-1}|_{op}\right)^{-1} \mathfrak{e}(v, v) \geq \|\mathfrak{g}^{-1}\|_\infty^{-1} \, \mathfrak{e}(v, v). \tag{D.2}$$

Since D.1 and D.2 hold for every $p \in A$, we obtain the desired result.

■

References

[AF03] R. A. Adams and J.J.F. Fournier, *Sobolev spaces*, 2nd ed., Academic Press, 2003.

[Bal89] J. M. Ball, *A version of the fundamental theorem of Young measures*, Proceedings of conference on 'Partial differential equations and continuum models of phase transitions' (M. Rascle, D. Serre, and M. Slemrod, eds.), Springer Lecture Notes in Physics, 1989, pp. 3–16.

[BBI01] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, American Mathematical Society, 2001.

[CM16] P. G. Ciarlet and S. Mardare, *Nonlinear Korn inequalities in $\mathbb{R}^n$ and immersions in $W^{2,p}, p > n$, considered as functions of their metric tensors in $W^{1,p}$*, J. Math. Pures Appl. **105** (2016), 873–906.

[Coh] M. Cohen, *Equivalence of intrinsic and extrinsic metrics of embedded manifolds*, `http://math.stackexchange.com/a/566430/104576`.

[CS16] A. Convent and J. Van Schaftingen, *Intrinsic colocal weak derivatives and Sobolev spaces between manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **16** (2016), no. 1, 97–128.

[EG15] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, revised ed., CRC Press, 2015.

[ESK09] E. Efrati, E. Sharon, and R. Kupferman, *Buckling transition and boundary layer in non-Euclidean plates*, PRE **80** (2009), 016602.

[ESK13] ______, *The metric description of elasticity in residually stressed soft materials*, Soft Matter **8** (2013), 8187.

[Eva98] L. C. Evans, *Partial differential equations*, American Mathematical Society, 1998.

[FJM02] G. Friesecke, R. D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity*, Comm. Pure Appl. Math. **55** (2002), 1461–1506.

[Gro86] M. Gromov, *Partial differential relations*, Springer-Verlag, Berin-Heidelberg, 1986.

[Haj09] P. Hajłasz, *Sobolev mappings between manifolds and metric spaces*, Sobolev Spaces In Mathematics I, Springer New York, 2009, pp. 185–222.

[Hei05] J. Heinonen, *Lectures on Lipschitz analysis*, Jyväskylän Yliopistopaino, 2005.

[Hél02] F. Hélein, *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge University Press, 2002.

[HW08] F. Hélein and J. C. Wood, *Harmonic maps*, Handbook of global analysis (Demeter Krupka and David Saunders, eds.), Elsevier, 2008, pp. 417–492.

[JK90] R. D. James and D. Kinderlehrer, *Theory of diffusionless phase transformations*, Lecture Notes in Physics **344** (1990), 51–84.

[KM14] R. Kupferman and C. Maor, *A Riemannian approach to the membrane limit of non-Euclidean elasticity*, Comm. Contemp. Math. **16** (2014), no. 5, 1350052.

[KM15] ———, *The emergence of torsion in the continuum limit of distributed dislocations*, Journal of Geometric Mechanics **7** (2015), no. 3, 361–387.

[KM16a] ———, *Limits of elastic models of converging Riemannian manifolds*, Calc. Variations and PDEs **55** (2016), 1–22, http://link.springer.com/article/10.1007/s00526-016-0979-6.

[KM16b] ———, *Riemannian surfaces with torsion as homogenization limits of locally-Euclidean surfaces with dislocation-type singularities*, Proc. Roy. Soc. Edin. A **146** (2016), no. 04, 741–768.

[KOS] R. Kupferman, E. Olami, and R. Segev, *Continuum dynamics on manifolds: Application to non-Euclidean elasticity*, to appear in Journal of Elasticity.

[KS12] R. Kupferman and Y. Shamai, *Incompatible elasticity and the immersion of non-flat Riemannian manifolds in Euclidean space*, Israel Journal of Mathematics **190** (2012), no. 1, 135–156.

[Lio50] J. Liouville, *Théoréme sur l'équation* $dx^2 + dy^2 + dz^2 = \lambda(d\alpha^2 + d\beta^2 + d\gamma^2)$, J. Math. Pures Appl. (1850).

[LL01] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., American Mathematical Soc., 2001.

[Lor13] A. Lorent, *On functions whose symmetric part of gradient agree and a generalization of Reshetnyak's compactness theorem*, Calc. Variations and PDEs **48** (2013), no. 3, 625–665.

[Lor16] ———, *Rigidity of pairs of quasiregular mappings whose symmetric part of gradient are close*, Ann. Inst. H. Poincaré Anal. Nonlin. **33** (2016), no. 1, 23–65.

[LP11] M. Lewicka and M. R. Pakzad, *Scaling laws for non-Euclidean plates and the $W^{2,2}$ isometric immersions of Riemannian metrics*, ESAIM: Control, Optimisation and Calculus of Variations **17** (2011), 1158–1173.

[Res67a] Yu. G. Reshetnyak, *Liouville's theorem on conformal mappings for minimal regularity assumptions*, Sibirskii Matematicheskii Zhurnal **8** (1967), no. 4, 835–840.

[Res67b] ———, *On the stability of conformal mappings in multidimensional spaces*, Sibirskii Matematicheskii Zhurnal **8** (1967), no. 1, 91–114.

[War71] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, Scott, Foresman, 1971.

[Weh04] K. Wehrheim, *Uhlenbeck compactness*, European Mathematical Society, 2004.