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Riemannian surfaces with torsion as homogenization limits of locally Euclidean surfaces with dislocation-type singularities

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We reconcile two classical models of edge dislocations in solids. The first, from the early 1900s, models isolated edge dislocations as line singularities in locally Euclidean manifolds. The second, from the 1950s, models continuously distributed edge dislocations as smooth manifolds endowed with non-symmetric affine connections (equivalently, endowed with torsion fields). In both models, the solid is modelled as a Weitzenböck manifold. We prove, using a weak notion of convergence, that the second model can be obtained rigorously as a homogenization limit of the first model as the density of singular edge dislocation tends to infinity.

Keywords: dislocations; homogenization; Weitzenböck manifolds; torsion; Gromov–Hausdorff convergence

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1. Introduction and main results

Weitzenböck manifolds are Riemannian manifolds endowed with a flat (not necessarily symmetric) metric connection. They have been used in general relativity theory in the context of teleparallelism, and in material science in the context of continuous distributions of dislocations (see, for example, [3, 6]).

In the theory of dislocations, the material body is modelled as a smooth manifold endowed with a Riemannian metric that represents the intrinsic distance between neighbouring material elements. Material defects (e.g. dislocations, disclinations and point defects) are viewed as singularities in the manifold. Edge dislocations can be modelled as curvature dipoles: a pair of cone singularities of equal magnitudes and opposite signs. Even if a neighbourhood of the line connecting the pair of singular points (the dislocation line) is removed, thus leaving a smooth locally flat manifold, the resulting manifold retains the same defect and cannot be isometrically embedded in the Euclidean plane (more precisely, the defect remains in the sense that the non-trivial monodromy of the manifold does not change; see [8] for details). Therefore, mathematically speaking, we can always remove a neighbourhood of the dislocation line, resulting in a smooth manifold with boundary and non-trivial topology.

Real materials often contain a large number of distributed defects. In such cases, one would like to smear out the singularities (or the holes) and represent the dislocations by a smooth field. Such a representation has been in use since the 1950s, with the density of dislocations represented by the torsion field of a Weitzenböck manifold.

The smearing out of discrete entities (known as homogenization) is a central theme in the mathematics of material science. In the context of distributed dislocations, an interesting question is how a smooth Riemannian manifold endowed with a non-symmetric connection emerges as a limit of manifolds with singularities or holes, whose only intrinsic connection is the (symmetric) Riemannian connection. On the one hand, we seek a notion of limit that involves a smooth structure, and therefore is beyond the scope of convergence of metric spaces (e.g. Gromov–Hausdorff convergence; see § 5 for details). On the other hand, standard notions of convergence for Riemannian manifolds, such as Hölder convergence, require the sequence of converging manifolds to be of the same diffeomorphism class as the limit. This is clearly not the case for manifolds with singularities or holes converging to a smooth, simply connected manifold.

In [7] we defined a new weak notion of convergence for Weitzenböck manifolds that encompasses the presence of edge dislocations.

Definition 1.1. Let $(\mathcal{M}_n, \mathfrak{g}_n, \nabla_n)$, $(\mathcal{M}, \mathfrak{g}, \nabla)$ be compact d-dimensional Weitzenböck manifolds with corners. We say that the sequence $(\mathcal{M}_n, \mathfrak{g}_n, \nabla_n)$ converges to $(\mathcal{M}, \mathfrak{g}, \nabla)$ with $p \in [d, \infty)$, if there exists a sequence of embeddings $F_n\colon \mathcal{M}_n \to \mathcal{M}$ such that the following hold.

(1) F_n is asymptotically surjective:
$$\lim_{n\to\infty} \mathrm{Vol}_{\mathfrak{g}}(\mathcal{M} \setminus F_n(\mathcal{M}_n)) = 0.$$

(2) F_n are approximate isometries: the distortion vanishes asymptotically, i.e.
$$\lim_{n\to\infty} \operatorname{dis} F_n = 0.$$

(3) F_n are asymptotically rigid in the mean:
$$\lim_{n\to\infty} \int_{F_n(\mathcal{M}_n)} \mathrm{dist}^p(\mathrm{d}F_n^{-1}, \mathrm{SO}(\mathfrak{g}, \mathfrak{g}_n))\, \mathrm{dVol}_{\mathfrak{g}} = 0,$$
where $\mathrm{SO}(\mathfrak{g}, \mathfrak{g}_n)$ denotes the set of metric- and orientation-preserving linear maps $T\mathcal{M}|_{F_n(\mathcal{M}_n)} \to (F_n^{-1})^* T\mathcal{M}_n$, and $\mathrm{dVol}_{\mathfrak{g}}$ denotes the volume form induced by the metric $\mathfrak{g}$.

(4) The parallel transport converges in the mean in the following sense: every point in $\mathcal{M}$ has a neighbourhood $U \subset \mathcal{M}$, with

(a) a ∇-parallel frame field E on U, and

(b) a sequence of ∇_n-parallel frame fields E_n on $F_n^{-1}(U)$ such that
$$\lim_{n\to\infty} \int_{U\cap F_n(\mathcal{M}_n)} |(F_n)_\star E_n - E|_{\mathfrak{g}}^p \, \mathrm{dVol}_{\mathfrak{g}} = 0,$$
where $(F_n)_\star E_n$ is the push forward of E_n to $T\mathcal{M}$.

In this definition, the fact that the $\mathcal{M}_n$ are not diffeomorphic to the limit $\mathcal{M}$ (the mappings F_n are only asymptotically surjective) allows for the presence of holes. Statements 2 and 3 define a weak notion of convergence of Riemannian manifolds, slightly stronger than Gromov–Hausdorff (GH) convergence. Statement 4 defines the convergence of the connection. This convergence is weak in the sense that it applies to the parallel transport but not to the connection as a derivation (i.e. the Christoffel symbols may not converge).

In [7] we showed that this sense of convergence may indeed give rise to torsion as a limit of defects. We constructed a particular sequence of manifolds with edge dislocations (and no torsion) that converges to a smooth Weitzenböck manifold with non-zero torsion.

A natural question is whether Weitzenböck manifolds can be constructed generically as limits of smooth Riemannian manifolds with torsion-free connections. In the material science context, this question amounts to whether any body that falls within the 1950s model of continuously distributed dislocation is a limit of bodies with finitely many dislocations. In this paper we show that this is the case for essentially any compact, oriented two-dimensional Weitzenböck manifold. Our main theorem is the following.

THEOREM 1.2. *Let $(\mathcal{N}, \mathfrak{g}, \nabla)$ be a compact, oriented two-dimensional Weitzenböck manifold with corners and with a Lipschitz-continuous boundary. The connection ∇ is, by definition, flat and metrically consistent with the metric $\mathfrak{g}$. Then, there exists a sequence of compact locally Euclidean Riemannian manifolds $(\mathcal{M}_n, \mathfrak{g}_n)$ with a trivial holonomy, such that $(\mathcal{M}_n, \mathfrak{g}_n, \nabla_n)$ converges to $(\mathcal{N}, \mathfrak{g}, \nabla)$ in the sense of definition 1.1 for every $p \in [2, \infty)$, where ∇_n is the Levi-Cività connection of $(\mathcal{M}_n, \mathfrak{g}_n)$.*

The fact that the manifolds $(\mathcal{M}_n, \mathfrak{g}_n)$ have trivial holonomy implies that the parallel transport of ∇_n is path independent. Less formally, it implies that there is no curvature hidden in the 'holes' of the manifold (such as the curvature 'charge' on the tip of a cone). The sequence itself will be constructed using edge dislocations, i.e. pairs of cone singularities of equal magnitudes and opposite signs, so the total curvature in every defect is indeed zero.

EXAMPLE 1.3. Weitzenböck manifolds that can be obtained as a limit as described in theorem 1.2 include the following cases.

- Let $D \subset \mathbb{R}^2$ be an open subset of the Euclidean plane, with a smooth boundary, and let (e_1, e_2) be an orthonormal frame field on $\bar{D}$. By declaring this frame field to be parallel we obtain a flat connection ∇ that is metrically consistent with the Euclidean metric $\mathfrak{e}$ on $\bar{D}$, and the triplet $(\bar{D}, \mathfrak{e}, \nabla)$ is a Weitzenböck manifold satisfying the conditions of theorem 1.2. The example in [7] is of this type, with D a sector of an annulus endowed with the connection obtained by declaring the orthonormal frame field $(\partial r, r^{-1}\partial\theta)$ parallel, where r and θ are polar coordinates (figure 1(a)).

- Let $\mathbb{T}^2 \subset \mathbb{R}^3$ be the two-dimensional torus embedded in $\mathbb{R}^3$, with the metric $\mathfrak{g}$ induced by this embedding. From the standard representation of $\mathbb{T}$ as $\mathbb{R}^2/\mathbb{Z}^2$ we obtain an orthogonal frame field $(\partial x, \partial y)$. Normalizing it and declaring it

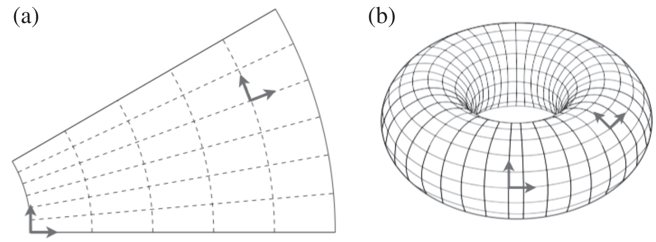

Figure 1. Examples of Weitzenböck manifolds for which theorem 1.2 applies. The arrows indicate a frame field that is parallel with respect to the metrically consistent, non-symmetric connection on each manifold.

to be parallel, we obtain a non-symmetric metrically consistent connection ∇. This is equivalent to declaring the directions of the meridians and parallels on the torus to be parallel (figure 1(b)). By theorem 1.2, the Weitzenböck manifold $(\mathbb{T}, \mathfrak{g}, \nabla)$ can be obtained as a limit of locally Euclidean manifolds with no curvature 'charges'. This is somewhat surprising, as the non-zero Gaussian curvature of $(\mathbb{T}^2, \mathfrak{g})$ seems to come out of nowhere. However, we shall see that the fact that the total Gaussian curvature of $(\mathbb{T}^2, \mathfrak{g})$ is zero plays a crucial role here.

The structure of this paper is as follows: in § 2 we present properties of geodesic curves in Weitzenböck manifolds. These geodesics play an important part in our construction. Since geodesics (of a general affine connection) are generally not locally length minimizing, the classical treatment of geodesics in the Riemannian setting needs to be extended. In § 3 we construct the sequence $\mathcal{M}_n$ of manifolds with edge dislocations, and in § 4 we construct the embeddings $F_n \colon \mathcal{M}_n \to \mathcal{N}$ and establish some of their properties. In § 5 we show that the distortion of F_n vanishes asymptotically, which, together with asymptotic surjectivity, implies in particular that $\mathcal{M}_n$ GH-converges to $\mathcal{N}$. Finally, in § 6 we complete the proof of theorem 1.2 by showing the convergence of the connections.

2. Geodesics of flat metric connections on the Euclidean plane

In this section we describe properties of geodesics of flat connections. The main results concern the existence of geodesic triangles and their properties (corollary 2.7).

Given a general Riemannian manifold $(\mathcal{N}, \mathfrak{g})$, every metrically consistent connection is defined by the torsion tensor T (via a generalization of the Koszul formula [10, p. 26]). For two-dimensional manifolds, the connection can be defined equivalently by a vector field V on $\mathcal{N}$ via the formula

$$\nabla_X Y = \nabla^{\mathfrak{g}}_X Y + \mathfrak{g}(X, Y)V - \mathfrak{g}(V, Y)X, \tag{2.1}$$

where $\nabla^{\mathfrak{g}}$ is the Riemannian Levi-Cività connection (see, for example, [1]). The torsion is then related to V by

$$T(X, Y) = \mathfrak{g}(V, X)Y - \mathfrak{g}(V, Y)X.$$

The fact that a flat connection is metrically consistent with a metric does not imply that the metric is flat (a metric is flat if the curvature tensor of the Levi-Cività connection vanishes). The following proposition relates the Gaussian curvature K of ∇ to the Gaussian curvature $K^{\mathfrak{g}}$ of $\nabla^{\mathfrak{g}}$.

PROPOSITION 2.1. *Let $(\mathcal{N}, \mathfrak{g})$ be a two-dimensional Riemannian manifold. Let ∇ be a metrically consistent connection defined as in (2.1) by a vector field V. Let K and $K^{\mathfrak{g}}$ denote the connection and the Riemannian Gaussian curvatures. Then,*

$$K \,\mathrm{dVol}_{\mathfrak{g}} = K^{\mathfrak{g}} \,\mathrm{dVol}_{\mathfrak{g}} - d \star V^{\flat},$$

where $\star$ is the Hodge dual and $\flat$ denotes the musical isomorphism operator. In particular, a Riemannian manifold endowed with a metrically consistent connection $(\mathcal{N}, \mathfrak{g}, \nabla)$ is a Weitzenböck manifold, if and only if the curl of V (viewed as a scalar) is equal to the Gaussian curvature of $(\mathcal{N}, \mathfrak{g})$.

The proof is given in appendix A. An immediate corollary of proposition 2.1 and the Gauss–Bonnet theorem is that a closed, oriented 2-manifold can be endowed with a metrically consistent flat connection (i.e. a structure of a Weitzenböck manifold) only if its genus is 1, that is, it is diffeomorphic to the torus. This is because the equation $d \star V^{\flat} = K^{\mathfrak{g}} \,\mathrm{dVol}_{\mathfrak{g}}$ cannot be solved for V unless $\int_{\mathcal{N}} K^{\mathfrak{g}} \,\mathrm{dVol}_{\mathfrak{g}} = 0$. As mentioned in § 1, the torus can indeed be endowed with a structure of a Weitzenböck manifold. Of course, there exist other compact manifolds with boundary or corners that can be endowed with such a structure.

Let $(\mathcal{N}, \mathfrak{g})$ be a Riemannian manifold, let ∇ be a metrically consistent connection and let γ be a curve in $\mathcal{N}$. γ may be a geodesic with respect to ∇ or $\nabla^{\mathfrak{g}}$ (i.e. $\nabla_{\dot\gamma}\dot\gamma = 0$ or $\nabla^{\mathfrak{g}}_{\dot\gamma}\dot\gamma = 0$, respectively). We call γ a *geodesic* in the former case, and a *segment* in the latter case (with a slight abuse of terminology, we also call any length-minimizing curve a segment, even if it hits the boundary). Note that, while a segment is locally a length minimizer, a geodesic need not be. For an arbitrary curve γ we call $\kappa = |\nabla^{\mathfrak{g}}_{\dot\gamma}\dot\gamma|/|\dot\gamma|^2$ the *curvature* of γ (it is the geodesic curvature with respect to the Levi-Cività connection, but in order to avoid confusion we use the term 'geodesic' only with respect to the connection ∇). Note that γ has zero curvature if and only if it is a segment, and, in particular, a geodesic may have a non-zero curvature.

In [1] it is shown that if ∇ is defined by a vector field V, the curvature of a geodesic γ is given by

$$\kappa^2 = |V|^2 - \frac{\mathfrak{g}(V, \dot\gamma)^2}{|\dot\gamma|^2}. \tag{2.2}$$

In particular, $\kappa \leqslant |V|$.

Henceforth, the Riemannian manifold $(\mathcal{N}, \mathfrak{g})$ is compact, oriented, with a Lipschitz-continuous boundary, endowed with a flat metrically consistent connection ∇ defined by a smooth vector field V. We define $\Lambda = \max_{\mathcal{N}} |V|$.

The following proposition provides a quantitative version of the fact that short curves with curvature below a given bound are almost length minimizers.

Proposition 2.2. *Let $(\mathcal{N},\mathfrak{g})$ be a two-dimensional Riemannian manifold whose Gaussian curvature satisfies $K \leqslant \bar{\mathcal{K}}$. Let $\gamma\colon [0,\ell] \to \mathcal{N}$ be a curve in arc-length parametrization with bounded curvature, $\kappa \leqslant \Lambda$. Set $d = d(\gamma(0),\gamma(\ell))$. Then, there exists an $L(\bar{\mathcal{K}},\Lambda) > 0$, such that if $\ell < L(\bar{\mathcal{K}},\Lambda)$, the following hold.*

(1) *$d(\gamma(0),\gamma(t))$ is an increasing function of t.*

(2)
$$d \leqslant \ell \leqslant \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d),$$
where
$$\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(x) = \begin{cases} \dfrac{2}{\sqrt{\Lambda^2+\bar{\mathcal{K}}}} \sin^{-1}\left(\sqrt{1+\dfrac{\Lambda^2}{\bar{\mathcal{K}}}} \sin \dfrac{\sqrt{\bar{\mathcal{K}}}x}{2}\right), & \bar{\mathcal{K}} > 0, \\ \dfrac{2}{\Lambda} \sin^{-1}\left(\dfrac{\Lambda x}{2}\right), & \bar{\mathcal{K}} = 0, \\ \dfrac{2}{\sqrt{\Lambda^2-\bar{\mathcal{K}}}} \sin^{-1}\left(\sqrt{1-\dfrac{\Lambda^2}{\bar{\mathcal{K}}}} \sinh \dfrac{\sqrt{-\bar{\mathcal{K}}}x}{2}\right), & \bar{\mathcal{K}} < 0. \end{cases}$$
In particular, for small x,
$$\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(x) = x + O(x^3).$$

Proof. Let
$$L_1(\bar{\mathcal{K}},\Lambda) = \begin{cases} \dfrac{2}{k}\tan^{-1}\dfrac{k}{\Lambda}, & \bar{\mathcal{K}} > 0, \\ \dfrac{2}{\Lambda}, & \bar{\mathcal{K}} = 0, \\ \dfrac{2}{k}\tanh^{-1}\dfrac{k}{\Lambda}, & \bar{\mathcal{K}} < 0, \end{cases}$$
where $k = \sqrt{|\bar{\mathcal{K}}|}$, and
$$L(\bar{\mathcal{K}},\Lambda) = \min(\tfrac{1}{2}\operatorname{inj}(\mathcal{N},\mathfrak{g}), L_1(\bar{\mathcal{K}},\Lambda)),$$
where inj denotes the injectivity radius; for $\bar{\mathcal{K}} > 0$, $\operatorname{inj}(\mathcal{N},\mathfrak{g}) \leqslant \pi/\sqrt{\bar{\mathcal{K}}}$. Consider a semi-geodesic (polar) parametrization (r,θ) around $\gamma(0)$, in which the metric has the following form:
$$\mathfrak{g}(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi^2(r,\theta) \end{pmatrix},$$
where r is the distance from the origin $\gamma(0)$, $\theta \in [0,2\pi)$ and φ is monotonically increasing in r, with initial conditions $\varphi(0) = 0$ and $\varphi_r(0) = 1$. Since, for $\bar{\mathcal{K}} > 0$,
$$d(\gamma(t),\gamma(0)) \leqslant L(\bar{\mathcal{K}},\Lambda) \leqslant \tfrac{1}{2}\operatorname{inj}(\mathcal{N},\mathfrak{g}), \tag{2.3}$$
it follows that γ lies within the domain of this parametrization.

The Gaussian curvature $K(r,\theta)$ is related to the function $\varphi(r,\theta)$ by the well-known formula
$$K = -\frac{\varphi_{rr}}{\varphi}.$$

Define

$$\psi(r) = \begin{cases} \dfrac{1}{k}\sin(kr), & \bar{\mathcal{K}} > 0, \\ r, & \bar{\mathcal{K}} = 0, \\ \dfrac{1}{k}\sinh(kr), & \bar{\mathcal{K}} < 0. \end{cases}$$

The case $\varphi = \psi$ corresponds to a surface of constant Gaussian curvature $\bar{\mathcal{K}}$. It is easy to see that, for all (r, θ), in the domain of parametrization,

$$\frac{\varphi_r}{\varphi} \geqslant \frac{\psi_r}{\psi}. \tag{2.4}$$

The equations of a curve $\gamma(t) = (r(t), \theta(t))$ whose (signed) curvature is $\kappa(t)$ are given by

$$\ddot{r} - \frac{\varphi_r}{\varphi}(\varphi\dot{\theta})^2 = -\kappa\varphi\dot{\theta},$$
$$\varphi\ddot{\theta} + \frac{\varphi_\theta}{\varphi^2}(\varphi\dot{\theta})^2 + 2\frac{\varphi_r}{\varphi}\dot{r}\varphi\dot{\theta} = \kappa\dot{r}.$$

For a curve in arc-length parametrization,

$$\dot{r}^2 + \varphi^2\dot{\theta}^2 = 1. \tag{2.5}$$

Taking the equation for r, using the bound (2.4), the bound $|\kappa| \leqslant \Lambda$ and (2.5), we obtain the inequality

$$\ddot{r} - \frac{\psi_r}{\psi}(1 - \dot{r}^2) \geqslant -\Lambda\sqrt{1 - \dot{r}^2}.$$

Introducing $G = \psi\sqrt{1 - \dot{r}^2}$, it follows that

$$\dot{G} \leqslant \Lambda\psi\dot{r}.$$

Setting

$$\Psi(r) = \int_0^r \psi(s)\,\mathrm{d}s = \begin{cases} \dfrac{1}{k^2}(1 - \cos(kr)), & \bar{\mathcal{K}} > 0, \\ \dfrac{r^2}{2}, & \bar{\mathcal{K}} = 0, \\ \dfrac{1}{k^2}(\cosh(kr) - 1), & \bar{\mathcal{K}} < 0, \end{cases}$$

we get, upon a first integration,

$$\psi\sqrt{1 - \dot{r}^2} \leqslant \Lambda\Psi.$$

Thus,

$$\dot{r}^2 \geqslant 1 - \Lambda^2\frac{\Psi^2}{\psi^2}.$$

In particular, $\dot{r}$ does not change sign as long as $\varLambda\varPsi < \psi$, i.e. as long as

$$r < L_1(\bar{\mathcal{K}}, \varLambda),$$

which holds since $r < L(\bar{\mathcal{K}}, \varLambda) \leqslant L_1(\bar{\mathcal{K}}, \varLambda)$. This proves statement 1. By isolating $\dot{r}$ and integrating the resulting inequality once more, we obtain statement 2. □

Remark 2.3.

(1) The bound $L(\bar{\mathcal{K}}, \varLambda)$ is not optimal, but it is sufficient for our construction.

(2) In the above proof we did not consider the case in which either γ or the segment between its endpoints intersects the boundary. While this is immaterial for the rest of the construction, the proof can be slightly modified to include these cases too.

(3) The bound in statement 2 can also be obtained by proving that if $\ell < L(\bar{\mathcal{K}}, \varLambda)$, then γ is contained in a ball of radius $\frac{1}{2}d$ around the midpoint of the segment connecting $\gamma(0)$ and $\gamma(\ell)$, and then using the main result in [4].

Corollary 2.4. *Under the assumptions of proposition 2.2 and a lower bound $K \geqslant \underline{\mathcal{K}}$ on the Gaussian curvature, the Hausdorff distance between γ and the segment σ that connects $\gamma(0)$ with $\gamma(\ell)$ is $O(d^2) = O(\ell^2)$.*

Proof. Reparametrize γ and σ such that they are defined on the interval $[0, 1]$ with constant speed, i.e. $|\dot{\gamma}| = \ell$ and $|\dot{\sigma}| = d$. By proposition 2.2, $\ell = d + O(d^3)$, and, for every $t \in [0, 1]$,

$$d(\gamma(0), \gamma(t)) = td + O(d^3), \qquad d(\gamma(t), \gamma(1)) = (1 - t)d + O(d^3).$$

Consider the segment triangle with vertices $\gamma(0)$, $\gamma(t)$ and $\gamma(1)$. By Rauch's comparison theorem [5, p. 215], the angles $\alpha(0)$, $\alpha(t)$, $\alpha(1)$ of this triangle are smaller than the angles $\alpha_{\bar{\mathcal{K}}}(0)$, $\alpha_{\bar{\mathcal{K}}}(t)$, $\alpha_{\bar{\mathcal{K}}}(1)$ of a segment triangle with same edge lengths in a space of constant curvature $\bar{\mathcal{K}}$. Assume $\bar{\mathcal{K}} > 0$ (the other cases are analogous), and set $\bar{k} = \sqrt{\bar{\mathcal{K}}}$. By the law of cosines for a space of constant positive curvature [10, p. 340], we obtain

$$\begin{aligned}\cos(\alpha_{\bar{\mathcal{K}}}(0)) &= \frac{\cos(\bar{k}d(\gamma(t), \gamma(1))) - \cos(\bar{k}d(\gamma(0), \gamma(1)))\cos(\bar{k}\,\mathrm{d}(\gamma(0), \gamma(t)))}{\sin(\bar{k}d(\gamma(0), \gamma(1)))\sin(\bar{k}d(\gamma(0), \gamma(t)))} \\ &= \frac{t + O(d^2)}{t + O(d^2)}.\end{aligned}$$

And, similarly,

$$\cos(\alpha_{\bar{\mathcal{K}}}(1)) = \frac{1 - t + O(d^2)}{1 - t + O(d^2)}.$$

Either t or $1-t$ is of order 1. Assume that t is of order 1 (the other case is analogous). Then, $\alpha(0) \leqslant \alpha_{\bar{\mathcal{K}}}(0) = O(d)$. Consider next the segment triangle whose vertices are $\gamma(0)$, $\gamma(t)$ and $\sigma(t)$. By Toponogov's comparison theorem for hinges [5, p. 215], it follows that $d(\gamma(t), \sigma(t))$ is bounded from above by the length of an edge in a triangle whose two other edges are of length $d(\gamma(0), \gamma(t)) = td + O(d^3)$ and $d(\gamma(0), \sigma(t)) = td$

and enclose an angle $\alpha(0) = O(d)$, in a space of constant Gaussian curvature $\underline{\mathcal{K}}$. It follows from the law of cosines that this distance is $O(d^2)$, with a bound independent of t; hence,

$$d_H(\sigma, \gamma) \leqslant \sup_{t\in[0,1]} d(\sigma(t), \gamma(t)) = O(d^2).$$

$\square$

Proposition 2.5. *Let $p, q \in \mathcal{N}$. Assume that there exists a geodesic γ of length less than $L(\bar{\mathcal{K}}, \Lambda)$ connecting p and q. Then the angle θ at p between γ and the segment σ connecting p and q satisfies the bound*

$$\theta \leqslant \Lambda \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d(p, q)) + O(d(p, q)^3) = \Lambda d(p, q) + O(d(p, q)^3).$$

Proof. It suffices to prove the proposition for the case where γ does not intersect σ. Indeed, assuming that we prove the proposition for this case, if γ intersects σ, let q' be the first point of intersection. Then,

$$\theta \leqslant \Lambda \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d(p, q')) + O(d(p, q')^3) \leqslant \Lambda \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d(p, q)) + O(d(p, q)^3),$$

where we used the fact that $\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}$ is monotonic.

Let θ_p and θ_q denote the signed angles between γ and σ at the points p and q. Then, the Gauss–Bonnet theorem implies that

$$\theta_q - \theta_p = \int_{\gamma\cup\sigma} k(t)\,\mathrm{d}t + \int_A K\,\mathrm{dVol}_{\mathfrak{g}},$$

where A is the area enclosed by γ and σ. Therefore,

$$|\theta_q - \theta_p| \leqslant \Lambda L(\gamma) + \sup|K|\,\mathrm{Vol}(A) \leqslant \Lambda \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d(p, q)) + O(d(p, q)^3),$$

where the last inequality follows from proposition 2.2 and corollary 2.4. Since γ does not intersect σ, θ_p and θ_q have opposite signs. Therefore,

$$\theta = |\theta_p| \leqslant |\theta_p| + |\theta_q| = |\theta_q - \theta_p| \leqslant \Lambda \mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(d(p, q)) + O(d(p, q)^3).$$

$\square$

Proposition 2.6. *Let $p, q \in \mathcal{N}$ satisfy*

$$d(p, q) \leqslant \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(L(\bar{\mathcal{K}}, \Lambda)).$$

Then there exists a unique geodesic γ connecting p and q whose length is less than $L(\bar{\mathcal{K}}, \Lambda)$.

Proof. Denote by $\exp_p^\nabla : \mathcal{B}(0, L(\bar{\mathcal{K}}, \Lambda)) \subset T_p\mathcal{N} \to \mathcal{N}$ the exponential map with respect to ∇. That is, $\exp_p^\nabla(v) = \sigma(1)$, where $\sigma : I \to \mathcal{N}$ is a geodesic with $\sigma(0) = p$ and $\dot{\sigma}(0) = v$. Proposition 2.2 implies that, for all $t \leqslant L(\bar{\mathcal{K}}, \Lambda)$ and unit vectors $\xi \in T_p\mathcal{N}$, $d(p, \exp_p^\nabla(t\xi))$ is monotonically increasing, and

$$\partial \exp_p^\nabla(\mathcal{B}(0, t)) \subset \mathcal{B}(p, t) \setminus \mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(t)), \tag{2.6}$$

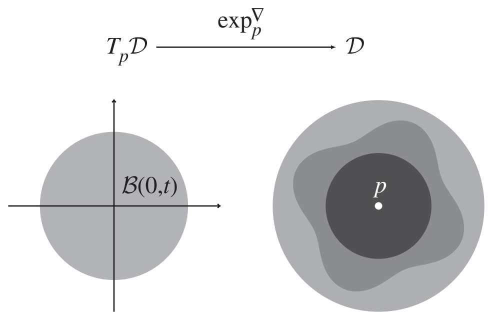

Figure 2. Properties of the image of the exponential map as established in proposition 2.6. Black shading shows $\mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(t))$. Dark grey shading shows $\exp_p^\nabla(\mathcal{B}(0,t))$. Pale grey shading shows $\mathcal{B}(p,t)$.

where $\mathcal{B}$ denotes a ball in $\mathcal{N}$ or $T_p\mathcal{N}$, according to the context. A classical argument about Riemannian geodesics implies that $d\exp_p^\nabla|_0 = \mathrm{id}$; hence, $\exp_p^\nabla$ is a local diffeomorphism at the origin. It follows that there exists $\epsilon > 0$, such that the image under $\exp_p^\nabla$ of $\mathcal{B}(0,\epsilon)$ is a simply connected domain that contains p. Since

$$\partial \exp_p^\nabla(\mathcal{B}(0,\epsilon)) \subset \mathcal{B}(p,\epsilon) \setminus \mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(\epsilon)),$$

it follows that

$$\exp_p^\nabla(\mathcal{B}(0,\epsilon)) \supset \mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(\epsilon)). \tag{2.7}$$

Equations (2.6) and (2.7) imply that, for all $t \leqslant L(\bar{\mathcal{K}},\Lambda)$,

$$\exp_p^\nabla(\mathcal{B}(0,t)) \supset \mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(t))$$

(see figure 2). In particular, if $q \in \mathcal{B}(p, \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(L(\bar{\mathcal{K}},\Lambda)))$, then

$$q \in \exp_p^\nabla(\mathcal{B}(0, L(\bar{\mathcal{K}},\Lambda))),$$

and there exists a geodesic of length up to $L(\bar{\mathcal{K}},\Lambda)$ from p to q.

As for uniqueness, two geodesics that emanate from a point p cannot intersect as long as they bound a simply connected subset of $\mathcal{N}$. Indeed, assume that q is their first point of intersection. By the Gauss–Bonnet theorem for Weitzenböck manifolds (see appendix B), the interior angles of the geodesic '2-gon' sum to zero, which implies that the two geodesics coincide. □

The following corollary applies the results of this section to geodesic triangles.

Corollary 2.7. *Let $A, B, C \in \mathcal{N}$ be the vertices of a triangle whose edges are segments of lengths a, b, c satisfying*

$$a, b, c < \mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(L(\bar{\mathcal{K}},\Lambda)).$$

Then, the following claims are satisfied (see figure 3).

(1) *Every pair of vertices is connected by a unique geodesic.*

(2) *These geodesics do not intersect.*

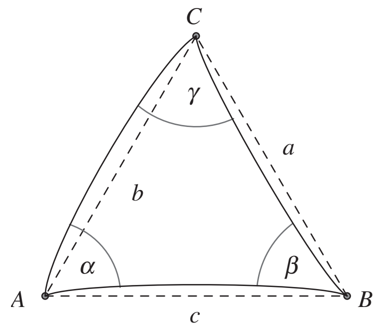

Figure 3. A geodesic triangle: the vertices are connected by segments (dashed lines) whose lengths are a, b, c. The edges are ∇-geodesics and are represented by solid lines. The angles α, β, γ between the edges sum to π. For $a, b, c = O(l)$ the lengths of the edges differ from the distances between the vertices by $O(l^3)$ and the angles between the edges deviate from the angles between the corresponding segments by $O(l)$.

(3) *The lengths of the geodesics are bounded by* $\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(a)$, $\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(b)$ *and* $\mathfrak{L}_{\bar{\mathcal{K}},\Lambda}(c)$, *respectively.*

(4) *The angle* α *between the pair of geodesics emanating from* A *and the angle* α_0 *between the pair of segments emanating from* A *satisfy*

$$|\alpha - \alpha_0| \leqslant 2\Lambda l + O(l^3),$$

where $l = \max(a, b, c)$. *A similar relation holds for the pairs of angles* β, β_0 *and* γ, γ_0.

(5) $\alpha + \beta + \gamma = \pi$.

Proof. Since $l < L(\bar{\mathcal{K}}, \Lambda)$, claims 1–3 are direct consequences of proposition 2.6. Claim 4 is a direct consequence of proposition 2.5. Finally, claim 5 also follows from the Gauss–Bonnet theorem (see appendix B). □

3. Construction of locally Euclidean manifolds, $\mathcal{M}_n$

In this section we construct the sequence $\mathcal{M}_n$ approximating $\mathcal{N}$ by triangulating $\mathcal{N}$ with geodesic triangles and replacing each triangle with a locally Euclidean triangle that encloses an edge dislocation. The following propositions assert the existence of a regular geodesic triangulation (proposition 3.1) and the existence of a locally Euclidean triangle enclosing an edge dislocation with a given boundary (proposition 3.3). We then use these results to define $\mathcal{M}_n$.

From now on we shall assume that the manifold $\mathcal{N}$ is simply connected. We shall remove this restriction in § 6.

PROPOSITION 3.1 (existence of a regular geodesic triangulation). *For every* $n \in \mathbb{N}$ *large enough, there exists a subdomain* $\mathcal{N}_n \subset \mathcal{N}$ *such that*

$$\mathcal{N} \setminus \mathcal{N}_n \subset \mathcal{B}(\partial\mathcal{N}, 3/n), \tag{3.1}$$

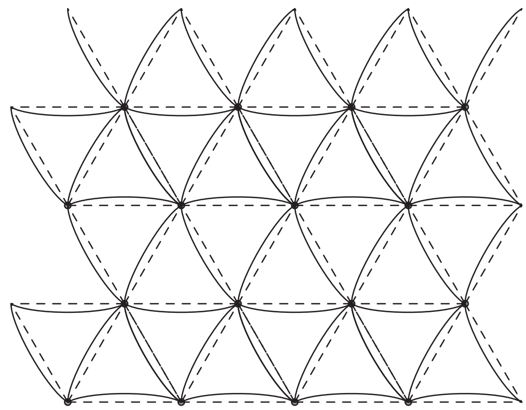

Figure 4. Triangulation of the domain $\mathcal{N}_n$ by geodesic triangles. The dashed lines are segments of length $O(1/n)$ triangulating $\mathcal{N}'_n$. The solid lines are the geodesic curves connecting those vertices.

and $\mathcal{N}_n$ can be triangulated by geodesic triangles whose edge lengths are bounded by

$$\frac{\underline{L}}{n} \leqslant \textit{edge length} \leqslant \frac{\bar{L}}{n}$$

for some constants $c, C > 0$ independent of n. The angles between intersecting edges are bounded uniformly away from 0 and π, i.e.

$$\delta \leqslant \textit{angle size} \leqslant \pi - \delta$$

for some constant $\delta \in (0, \pi)$ independent of n. Moreover, the angles in each triangle sum to π.

Proof. Let $n > 1/\mathfrak{L}^{-1}_{\bar{\mathcal{K}},\Lambda}(L(\bar{\mathcal{K}}, \Lambda))$. First, triangulate a subdomain $\mathcal{N}'_n \subset \mathcal{N}$ by segments of edge length

$$\frac{\underline{L}}{n} \leqslant \text{edge length} \leqslant \frac{\bar{L}}{n}$$

and angles

$$\delta \leqslant \text{angle size} \leqslant \pi - \delta,$$

such that

$$\partial\mathcal{N}'_n \subset \mathcal{B}(\partial\mathcal{N}, 2/n) \setminus \mathcal{B}(\partial\mathcal{N}, 1/n). \tag{3.2}$$

Such a construction is always possible for large enough n (see, for example, [2, p. 157]). Then take $\mathcal{N}_n$ to be the geodesic triangulation based on the graph structure of $\mathcal{N}'_n$ (see figure 4). For large enough n, the geodesics never hit the boundary, as a result of (3.2) and corollary 2.4, which also imply (3.1). The other properties of $\mathcal{N}_n$ for n large enough are direct consequences of corollary 2.7, up (possibly) to a slight adjustment of the constants $\underline{L}$, $\bar{L}$ and δ. □

Corollary 3.2. *In the triangulation described in proposition 3.1, the distance between a vertex and the opposite edge in a triangle is larger than r/n for some $r > 0$ independent of n.*

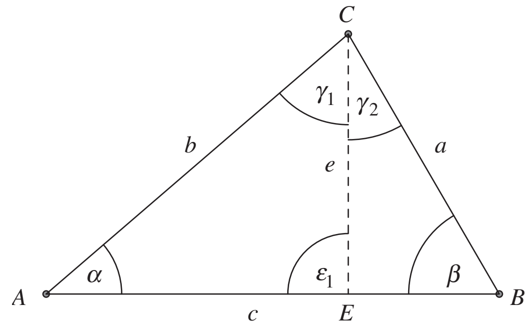

Figure 5. A segment triangle ABC. The edge lengths satisfy $a, b, c \in (\underline{L}/n, \bar{L}/n)$, and the angles satisfy $\alpha, \beta, \gamma_1 + \gamma_2 \in (\delta, \pi - \delta)$. e is a segment that minimizes the distance between the vertex C and the edge c.

Proof. By corollary 2.4, it is sufficient to consider the segment triangulation used in the proof of proposition 3.1 instead of the geodesic triangulation, since the distance between the opposite segment and the opposite edge is $O(n^{-2})$.

Assume by contradiction that there exists a sequence of triangles, with the nth triangle belonging to the nth triangulation, such that the distance between one of the vertices and its opposite edge is $o(1/n)$. Consider the segment triangle ACE in figure 5. By Rauch's comparison theorem for triangles (see [5, p. 215]), the angles $\alpha, \gamma_1, \varepsilon_1$ are smaller than the angles $\alpha', \gamma_1', \varepsilon_1'$ of the segment triangle with same side lengths in a space of constant curvature $\bar{\mathcal{K}}$ (the upper bound on the Gaussian curvature). Assume $\bar{\mathcal{K}} > 0$ (the other cases are analogous). In a space of constant curvature the law of sines reads

$$\frac{\sin(\sqrt{\bar{\mathcal{K}}}d)}{\sin(\sqrt{\bar{\mathcal{K}}}b)} = \frac{\sin(\alpha')}{\sin(\varepsilon_1')} \geqslant \sin(\alpha')$$

Since $b \in (\underline{L}/n, \bar{L}/n)$, it follows that if $d = o(1/n)$, then $\sin(\alpha') = o(1)$. Since $\alpha' \geqslant \alpha > \delta$, it follows that $\pi - \alpha' = o(1)$; hence, $\gamma_1' + \varepsilon_1' = o(1)$ ($\alpha' + \gamma_1' + \varepsilon_1' = \pi + O(1/n^2)$ by the Gauss–Bonnet theorem). This is a contradiction, since

$$\gamma_1' + \varepsilon_1' \geqslant \gamma_1 + \varepsilon_1 = \pi - \alpha + O\left(\frac{1}{n^2}\right) > \delta + O\left(\frac{1}{n^2}\right).$$

□

The next proposition shows that we can associate with any geodesic triangle a locally flat triangle with an edge dislocation (a pair of cone singularities of equal magnitudes and opposite signs) that has the same edge lengths and angles. These triangles will be the building blocks of the approximating sequence of manifolds $\mathcal{M}_n$.

Proposition 3.3. *Let $\theta \in (0, \pi/2)$. Let a, b, c be numbers satisfying*

$$\frac{\underline{L}}{n} \leqslant a, b, c \leqslant \frac{\bar{L}}{n}, \tag{3.3}$$

such that the sum of any two of them is larger than the third. Let $\alpha_0, \beta_0, \gamma_0$ be the angles of a (Euclidean) triangle whose side lengths are a, b, c. Suppose that α, β, γ

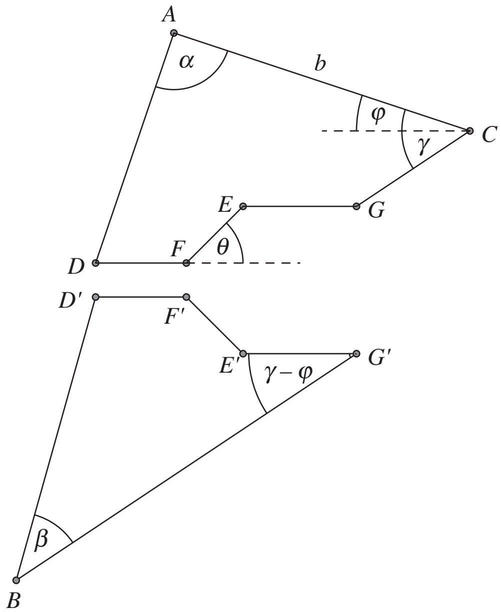

Figure 6. Construction of a Euclidean triangle with an edge dislocation used in the proof of proposition 3.3.

satisfy

$$\alpha + \beta + \gamma = \pi$$

and

$$|\alpha - \alpha_0|, |\beta - \beta_0|, |\gamma - \gamma_0| \leqslant \frac{K}{n}. \tag{3.4}$$

Then, for n large enough, there exists a locally Euclidean triangle of edge lengths a, b, c and respective angles α, β, γ, enclosing an edge dislocation of magnitude $\epsilon = O(1/n^2)$ and a disclination angle θ.

Proof. If a, b, c and α, β, γ are compatible with Euclidean geometry, then the claim is trivial with $\epsilon = 0$ (no dipole). Otherwise, the law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

does not hold.

Figure 6 shows two (Euclidean) polygons, $ADFEGC$ and $BD'F'E'G'$. The dipole is constructed by identifying the edge DF with $D'F'$, edge EF with $E'F'$ and edge EG with $E'G'$; each pair is assumed to be of the same length. Also,

$$\sphericalangle EFD = \sphericalangle E'F'D' = \sphericalangle FEG = \sphericalangle F'E'G' = \pi - \theta.$$

The dipole magnitude is

$$|\epsilon| = 2|EF| \sin\theta.$$

If $\epsilon > 0$ we elongate the edge opposite to γ and shorten the edge opposite to α, and vice versa if $\epsilon < 0$. We need to prove that such a construction is possible under the constraints

$$AD + D'B = c \quad \text{and} \quad CG + G'B = a,$$

$\varphi \in (0,\gamma)$ and $\epsilon = O(1/n^2)$.

By means of figure 6 and straightforward trigonometry we obtain that φ and ϵ are given by

$$\tan\varphi = \frac{b - a\cos\gamma - c\cos\alpha}{a\sin\gamma - c\sin\alpha} \tag{3.5}$$

and

$$\epsilon = \frac{c\sin\alpha - a\sin\gamma}{\cos\varphi}.$$

Consider (3.5). In a Euclidean triangle both the numerator and the denominator vanish. Order the vertices of the triangle such that $a/\sin\alpha$, $b/\sin\beta$ and $c/\sin\gamma$ form a monotone sequence. The sign of the numerator of (3.5) does not change regardless of whether we take an increasing or decreasing sequence. If it is positive, we let $a/\sin\alpha > c/\sin\gamma$; otherwise, we let $a/\sin\alpha < c/\sin\gamma$. Thus, we ensure that $\tan\varphi > 0$.

We now prove that $\varphi < \gamma$. Take, for example, the case where

$$\frac{a}{\sin\alpha} \geqslant \frac{b}{\sin\beta} \geqslant \frac{c}{\sin\gamma},$$

where at least one of the inequalities is strong (the other case is similar). Then,

$$\begin{aligned} a - b\cos\gamma - c\cos\beta &> a - \frac{a\sin\beta}{\sin\alpha}\cos\gamma - \frac{a\sin\gamma}{\sin\alpha}\cos\beta \\ &= \frac{a}{\sin\alpha}(\sin\alpha - \sin(\beta+\gamma)) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\tan\varphi}{\tan\gamma} &= \frac{b\cos\gamma - a\cos^2\gamma - c\cos\alpha\cos\gamma}{a\sin^2\gamma - c\sin\alpha\sin\gamma} \\ &= \frac{b\cos\gamma - a + a\sin^2\gamma + c\cos\beta - c\sin\alpha\sin\gamma}{a\sin^2\gamma - c\sin\alpha\sin\gamma} \\ &= 1 - \frac{1}{\sin\gamma}\frac{a - b\cos\gamma - c\cos\beta}{a\sin\gamma - c\sin\alpha} < 1, \end{aligned}$$

which proves that indeed $\varphi \in (0,\gamma)$.

It remains to show that $\epsilon = O(1/n^2)$. From our bound on φ it follows that $\cos\varphi$ is bounded away from zero for large n. From assumptions (3.3) and (3.4) it follows that $c\sin\alpha - a\sin\gamma = O(1/n^2)$, which implies $\epsilon = O(1/n^2)$. □

REMARK 3.4. The line $EF = E'F'$, whose length is $O(1/n^2)$, is called the *dislocation line*. By shifting it, either horizontally or vertically, we can always modify the construction so that the singular points E and F are located at the centre of the defective triangle, in the sense that their distance from the boundary is of order $1/n$. In particular, this guarantees that when $\epsilon < 0$ the points E and F remain in the interior of the triangle.

DEFINITION 3.5. Let a, b, c, α, β, γ and θ satisfy the conditions of proposition 3.3. We denote by

$$\Delta(a, b, c; \alpha, \beta, \gamma; \theta)$$

the locally Euclidean triangle enclosing an edge dislocation constructed in proposition 3.3. We assume that the dislocation line is at the centre of the triangle as explained in the above remark.

We now turn to the construction of a sequence of locally flat manifolds with path-independent Levi-Cività connections, $(\mathcal{M}_n, \mathfrak{g}_n, \nabla_n)$, that approximate the Weitzenböck manifold $(\mathcal{N}, \mathfrak{g}, \nabla)$. By proposition 3.1, there exists a subdomain $\mathcal{N}_n \subset \mathcal{N}$ that can be triangulated by geodesic triangles $\mathcal{N}_{n,i}$ for $i \in I_n$,

$$\mathcal{N}_n = \bigcup_{i \in I_n} \mathcal{N}_{n,i},$$

whose edges $a_{n,i}, b_{n,i}, c_{n,i}$ and angles $\alpha_{n,i}, \beta_{i,n}, \gamma_{n,i}$ satisfy conditions (3.3) and (3.4). Note that $|I_n| = O(n^2)$.

For every $i \in I_n$ we construct a locally Euclidean triangle with edge dislocation

$$\tilde{\mathcal{M}}_{n,i} = \Delta(a_{n,i}, b_{n,i}, c_{n,i}; \alpha_{n,i}, \beta_{i,n}, \gamma_{n,i}; \theta).$$

We denote by $\tilde{\mathcal{M}}_n$ the amalgam of the $\tilde{\mathcal{M}}_{n,i}$ obtained by replacing each $\mathcal{N}_{n,i}$ by the corresponding $\tilde{\mathcal{M}}_{n,i}$.

$\tilde{\mathcal{M}}_n$ is a compact, simply connected topological manifold with corners. It is smooth and locally Euclidean everywhere except at two singular points within each triangle $\tilde{\mathcal{M}}_{n,i}$. Removing the dislocation line that connects each pair of singular points, the Levi-Cività parallel transport ∇_n is path independent (see [7] for more details about a similar construction). We denote by $\mathfrak{g}_n$ the Riemannian metric on the smooth part of $\tilde{\mathcal{M}}_n$ and by $\tilde{d}_n$ the induced distance function on $\tilde{\mathcal{M}}_n$. The Riemannian metric $\mathfrak{g}_n$ also induces a volume measure on $\tilde{\mathcal{M}}_n$ (including the singular points).

Let $L_{n,i} \subset \tilde{\mathcal{M}}_{n,i}$ be open neighbourhoods of radius $1/n^2$ around the dislocation line, and let

$$L_n = \bigcup_{i \in I_n} L_{n,i}.$$

Set $\mathcal{M}_{n,i} = \tilde{\mathcal{M}}_{n,i} \setminus L_{n,i}$ and define

$$\mathcal{M}_n = \bigcup_{i \in I_n} \mathcal{M}_{n,i} = \tilde{\mathcal{M}}_n \setminus L_n.$$

$(\mathcal{M}_n, \mathfrak{g}_n)$ is a smooth, compact, multiply connected, locally Euclidean manifold with corners. We denote by d_n the distance function induced by $\mathfrak{g}_n$. Since the diameter of $L_{n,i}$ is $O(1/n^2)$, it follows that the Gromov–Hausdorff distance between $(\tilde{\mathcal{M}}_{n,i}, \tilde{d}_n)$ and $(\mathcal{M}_{n,i}, d_n)$ is $O(1/n^2)$, namely,

$$\sup_{p,q \in \mathcal{M}_{n,i}} |\tilde{d}_n(p, q) - d_n(p, q)| = O\left(\frac{1}{n^2}\right).$$

REMARK 3.6.

(1) Note that the distances inside each triangle $\mathcal{M}_{n,i}$ differ only by $O(1/n^2)$ from the distances of a Euclidean triangle with the same edges. For n large enough, $\mathcal{M}_{n,i}$ is convex in $\mathcal{M}_n$, in the sense that segments between points in $\mathcal{M}_{n,i}$ stay inside it (that is, the induced metric on $\mathcal{M}_{n,i}$ from d_n and its intrinsic distance are the same). This is because the only way that a curve between points on an edge of $\mathcal{M}_{n,i}$ could have been shorter than the length of the edge is by moving around a dislocation. However, since the dislocations are located at a distance of order $1/n$ from the edge and the length gain is only order $1/n^2$, such a curve would in fact be longer.

(2) Since the Levi-Cività parallel transport in each $\mathcal{M}_{n,i}$ is path independent (this is one of the main features of the dislocation; see [7] for details) and since the angles around each vertex of each of the triangles $\mathcal{M}_{n,i}$ sum to exactly 2π, the Levi-Cività parallel transport ∇_n in the whole manifold $\mathcal{M}_n$ is path independent.

4. Embeddings of $\mathcal{M}_n$ into the Weitzenböck manifold $\mathcal{N}$

In this section we construct embeddings $F_n \colon \mathcal{M}_n \to \mathcal{N}$ satisfying the conditions of definition 1.1. We denote by X_n the skeleton formed by the union of the boundaries of the triangles with defects $\tilde{\mathcal{M}}_{n,i}$. Likewise, we denote by Y_n the skeleton formed by the union of the boundaries of the geodesic triangles $\mathcal{N}_{n,i}$.

These skeletons have the following properties.

(1) The vertices of X_n form a finite $O(n^{-1})$-net of $\mathcal{M}_n$, and the vertices of Y_n form a finite $O(n^{-1})$-net of $\mathcal{N}$ of the same cardinality. This follows from the construction and, in particular, from the fact that $\mathrm{Vol}(\mathcal{N} \setminus \mathcal{N}_n) \to 0$.

(2) The edges in X_n are curves that are of the same length as the corresponding edges in Y_n.

(3) Y_n consists of ∇-geodesics and X_n consists of ∇_n-geodesics.

It follows that there exists a natural mapping $T_n \colon X_n \to Y_n$ that preserves the intrinsic distances of X_n and Y_n (the intrinsic distances on path-connected subsets differ from the induced distances d and d_n).

Before extending T_n to smooth embeddings $\mathcal{M}_n \to \mathcal{N}$, we need the following technical lemma.

LEMMA 4.1. *Consider the two geometric figures displayed in figure 7. Both figures exhibit a geodesic curve of length a, with at one end a geodesic curve of length b emanating at an angle γ and at the other end a geodesic curve of length c emanating at an angle β. The figure in part (a) is in the Euclidean plane (i.e. the segments pq, pp_1 and qq_1 are Euclidean segments), whereas the figure in part (b) is in $\mathcal{N}$ (i.e. the curves PQ, PP_1 and QQ_1 are ∇-geodesics). It is given that*

$$a, b, c \leqslant \frac{\bar{L}}{n}.$$

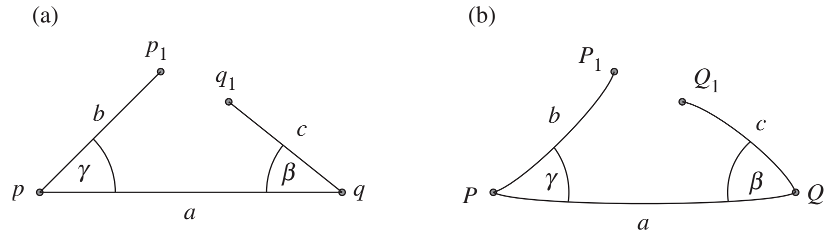

Figure 7. Geometric figures consisting of geodesic curves. In both figures the respective lengths of the geodesics, as well as the respective angles between geodesics, are equal. The figure in part (a) is in the Euclidean plane $\mathbb{R}^2$, whereas the figure in part (b) is in $\mathcal{N}$.

Then, there exists a constant Δ *that depends only on* $\underline{\mathcal{K}}$, $\bar{\mathcal{K}}$, Λ *and* $\bar{L}$ *such that, for large enough* n,

$$|d(P_1, Q_1) - |p_1 - q_1|| \leqslant \frac{\Delta}{n^2}.$$

This lemma can be proved using geometric comparison theorems, but an analytical approach is shorter. In fact, we shall prove the following stronger result.

Proposition 4.2. *Let* $\gamma\colon [0,\ell] \to \mathcal{N}$, $\ell \leqslant 3\bar{L}/n$, *be a curve in arc-length parametrization with geodesic curvature* $k(t)$ *(the geodesic curvature is with respect to the connection* ∇*). Let* $\sigma\colon [0,\ell] \to \mathbb{R}^2$ *be a curve in arc-length parametrization with geodesic curvature* $k(t)$ *(with respect to the Euclidean connection). Then, there exists a constant* $\Delta > 0$*, which depends only on* $\bar{L}$, $\bar{\mathcal{K}}$, $\underline{\mathcal{K}}$ *and* Λ *such that, for every* $t \in [0,\ell]$,

$$|d(\gamma(t), \gamma(0)) - |\sigma(t) - \sigma(0)|| \leqslant \frac{\Delta}{n^2}.$$

The proof is given in appendix C.

Proposition 4.3. *There exist smooth embeddings* $F_n\colon \mathcal{M}_n \to \mathcal{N}$ *satisfying the following properties:*

(1) F_n *extends* $T_n\colon X_n \to Y_n$;

(2) *the image of* $\mathcal{M}_{n,i}$ *is a subset of* $\mathcal{N}_{n,i}$ *for all* $i \in I_n$;

(3) $\mathrm{Vol}(\mathcal{N} \setminus F_n(\mathcal{M}_n)) \to 0$;

(4) $\mathrm{d}F_n$ *and* $\mathrm{d}F_n^{-1}$ *are uniformly bounded in* n *over their domains of definition;*

(5) *we have*

$$\mathrm{dist}(\mathrm{d}F_n, \mathrm{SO}(\mathfrak{g}_n, \mathfrak{g})) = O(1/n)$$

everywhere, except for a set of volume $O(1/n^{1-\varepsilon})$*, where* $0 < \varepsilon < 1$ *can be chosen arbitrarily.*

The smaller the constant ε, the faster the convergence of $\mathcal{M}_n$ to $\mathcal{N}$. With a more elaborate construction we can obtain the same result with $\varepsilon = 0$. Since the convergence rate is immaterial for our main result, we shall make do with $\varepsilon > 0$.

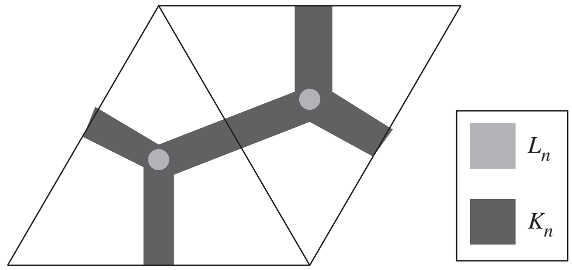

Figure 8. Partition of $\tilde{\mathcal{M}}_n$ used in the construction of F_n. The dislocation lines are surrounded by open neighbourhoods $L_{n,i}$ of radius $O(1/n^2)$ whose union is denoted by L_n; we set $\mathcal{M}_{n,i} = \tilde{\mathcal{M}}_{n,i} \setminus L_{n,i}$. Each $\mathcal{M}_{n,i}$ is partitioned into three connected components by constructing the set K_n.

Proof. For every $n \in \mathbb{N}$ large enough and for every $i \in I_n$ we construct a compact set $K_{n,i} \subset \tilde{\mathcal{M}}_{n,i}$ as shown in figure 8, such that

(1) $\mathrm{Vol}(K_{n,i}) = O(1/n^{3-\varepsilon})$,

(2) the intersection of $K_{n,i}$ with X_n consists of three segments,

(3) the distances between different connected components of $\partial K_{n,i} \setminus \partial\mathcal{M}_{n,i}$ are between $1/n^{2-\varepsilon}$ and $2/n^{2-\varepsilon}$,

(4) if $\mathcal{M}_{n,i}$ and $\mathcal{M}_{n,j}$ share an edge, then the intersections of $K_{n,i}$ and $K_{n,j}$ with this edge coincide, and the boundary of $K_{n,i} \cup K_{n,j}$ is smooth.

Set $K_n = \bigcup_{i \in I_n} K_{n,i}$. We define F_n on $\mathcal{M}_n \backslash K_n$ as follows: let $p \in \mathcal{M}_n \backslash K_n$. Then, by construction, the connected component of $\mathcal{M}_n \setminus K_n$ that contains p contains a single vertex q_p of the grid X_n. Define

$$F_n(p) = \exp^{\nabla}_{T_n(q_p)}(v(p)),$$

where $v(p) \in T_{T_n(q_p)}\mathcal{N}$ is the vector of length $|p - q_p|$ that forms with Y_n the same angle at $T_n(q_p)$ as $p - q_p$ forms with X_n at q_p.

Since the injectivity radius of $\exp^{\nabla}$ is at least $\mathfrak{L}_{\Lambda}^{-1}(L(\bar{\mathcal{K}}, \Lambda))$ (proposition 2.6), it follows that F_n is injective on each connected component of $\mathcal{M}_n \setminus K_n$. We have to show that F_n is globally injective: it maps disjoint connected components of $\mathcal{M}_n \backslash K_n$ into disjoint sets in $\mathcal{N}$, and, moreover, the separation between the images remains of order $1/n^{2-\varepsilon}$. This is an immediate consequence of lemma 4.1, which states that F_n distorts paths in $\mathcal{M}_{n,i}$ by distances of order $O(1/n^2)$; in the application of lemma 4.1 the segment pq corresponds to an edge of $\mathcal{M}_{n,i}$, whereas the geodesic between P and Q corresponds to the matching edge in $\mathcal{N}_{n,i}$. Note that the fact that the domain in lemma 4.1 is $\mathbb{R}^2$ and not $\mathcal{M}_{n,i}$ does not matter, since the difference in distances is only $O(n^{-2})$ by construction. Thus, F_n is injective and the distance between the images of the connected components of $\mathcal{M}_{n,i} \setminus K_{n,i}$ is of order $1/n^{2-\varepsilon}$.

We now consider the derivative of F_n. Since $q = q_p$ is independent of p in any connected component,

$$\mathrm{d}F_n = \mathrm{d}(\exp^{\nabla}_{T_n(q)}) \circ \mathrm{d}v.$$

Note that, by definition, $\mathrm{d}v_p \in \mathrm{SO}(\mathfrak{g}_n, \mathfrak{g})$ for every p. Also,

$$\mathrm{d}(\exp^{\nabla}_{T_n(q)})_0 = \mathrm{Id}.$$

Since $|v| = O(1/n)$ and $\exp^{\nabla}$ and its derivatives only depend on metric $\mathfrak{g}$ and the vector field V defining ∇, we conclude that

$$\mathrm{dist}(\mathrm{d}(\exp^{\nabla}_{T_n(q)})_v, \mathrm{Id}) = O\left(\frac{1}{n}\right),$$

and hence

$$\mathrm{dist}(\mathrm{d}F_n, \mathrm{SO}(\mathfrak{g}_n, \mathfrak{g})) = O\left(\frac{1}{n}\right),$$

and the same estimate holds for $\mathrm{d}F_n^{-1}$. An immediate consequence is

$$|\mathrm{d}F_n|, |\mathrm{d}F_n^{-1}| = 1 + O\left(\frac{1}{n}\right);$$

in particular, F_n are uniformly bi-Lipschitz.

We next extend F_n to $\mathcal{M}_n$, that is, we define F_n on $K_{n,i}$. F_n maps $\partial K_{n,i} \setminus \partial\mathcal{M}_{n,i}$ with an $O(1/n^2)$ distortion and a uniformly bounded derivative. Since we have shown that the dimensions of the domains and images are of the same order, it follows that F_n can be extended smoothly to an embedding $\mathcal{M}_n$ with a uniformly bounded derivative, and with the constraint that $F_n|_{X_n} = T_n$.

It remains to prove that $\mathrm{Vol}(\mathcal{N} \setminus F_n(\mathcal{M}_n)) \to 0$. Note that

$$\mathrm{Vol}(\mathcal{N} \setminus F_n(\mathcal{M}_n)) = \mathrm{Vol}(\mathcal{N} \setminus \mathcal{N}_n) + \mathrm{Vol}(\mathcal{N}_n \setminus F_n(\mathcal{M}_n)).$$

The first term tends to zero by proposition 3.1. For the second term, we note that the length of $\partial L_{n,i}$ is $O(1/n^2)$. Since $\mathrm{d}F_n$ is uniformly bounded, $F_n(\partial L_{n,i})$ is also a curve whose length is $O(1/n^2)$ and thus, by the isoperimetric inequality for manifolds with lower-bounded Gaussian curvature, its volume is $O(1/n^4)$. Hence,

$$\mathrm{Vol}(\mathcal{N}_n \setminus F_n(\mathcal{M}_n)) = O(1/n^2).$$

□

COROLLARY 4.4 (mean asymptotic rigidity). *The embeddings $F_n \colon \mathcal{M}_n \to \mathcal{N}$ satisfy, for every $p \in (1, \infty)$,*

$$\int_{F_n(\mathcal{M}_n)} \mathrm{dist}^p(\mathrm{d}F_n^{-1}, \mathrm{SO}(\mathfrak{g}, \mathfrak{g}_n))\, \mathrm{dVol}_{\mathfrak{g}} \to 0.$$

Proof. This is an immediate consequence of properties 4 and 5 in proposition 4.3. □

5. Gromov–Hausdorff convergence

In this section we prove that F_n satisfies the vanishing distortion property in definition 1.1, i.e. that $\operatorname{dis} F_n \to 0$. We then deduce that $(\mathcal{M}_n, d_n)$ GH converges to $(\mathcal{N}, d)$ as a sequence of compact metric spaces. Recall that d is the distance function on $\mathcal{N}$ induced by the metric $\mathfrak{g}$.

PROPOSITION 5.1.

$$\operatorname{dis} F_n = \max_{p,q\in\mathcal{M}_n} |d_n(p,q) - d(F_n(p), F_n(q))| = O\left(\frac{1}{n^{1-\varepsilon}}\right).$$

Proof. The proof relies on two lemmas. The first shows that the restriction of F_n to a single triangle $\mathcal{M}_{n,i}$ has a distortion of order $O(n^{-2+\varepsilon})$.

LEMMA 5.2. *Let $F_{n,i}$ be the restriction of F_n to $\mathcal{M}_{n,i}$. Then there exists a constant $c > 0$, independent of n and i, such that*

$$\max_{p,q\in\mathcal{M}_{n,i}} |d_n(p,q) - d(F_{n,i}(p), F_{n,i}(q))| < \frac{c}{n^{2-\varepsilon}}.$$

Proof. Let $p, q \in \mathcal{M}_{n,i}$ and let γ be a 'short' path in $\mathcal{M}_{n,i}$ between p and q in the sense that

$$\ell(\gamma) \leqslant d_n(p,q) + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

The extra $O(n^{-2+\varepsilon})$ term in the length of γ enables us to choose γ such that its intersection with $K_{n,i}$ is of length $O(n^{-2+\varepsilon})$.

Let $\sigma = F_n(\gamma)$ be the image of γ in $\mathcal{N}_{n,i}$, which connects $F_n(p)$ and $F_n(q)$. Then,

$$d(F_n(p), F_n(q)) \leqslant \ell(\sigma) \leqslant \ell(\gamma) + O\left(\frac{1}{n^{2-\varepsilon}}\right) \leqslant d_n(p,q) + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

The middle inequality follows from the bounds $\ell(\gamma) = O(1/n)$ and $\ell(\gamma \cap K_{n,i}) = O(n^{-2+\varepsilon})$, and from the fact that $\mathrm{d}F_n$ increases length by not more than a factor of $O(1/n)$ in $\mathcal{M}_n \setminus K_{n,i}$ and at most by a factor of $O(1)$ in $K_{n,i}$.

A similar argument holds in the other direction, by choosing σ to be a 'short' path in $\mathcal{N}_{n,i}$ between $F_n(p)$ and $F_n(q)$ in the sense that

$$\ell(\sigma) \leqslant d(F_n(p), F_n(q)) + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

This time we use the extra $O(n^{-2+\varepsilon})$ term to consider curves inside of $\mathcal{N}_{n,i}$ (unlike $\mathcal{M}_{n,i}$, $\mathcal{N}_{n,i}$ is not convex: a segment between p and q may leave $\mathcal{N}_{n,i}$, but by considering only curves inside $\mathcal{N}_{n,i}$ the length gain is only $O(1/n^2)$), to avoid $\mathcal{N}_{n,i} \setminus F_n(\mathcal{M}_{n,i})$ (where F_n^{-1} is not defined) and guarantee that the intersection of σ with $F_n(K_{n,i})$ is of length $O(n^{-2+\varepsilon})$. □

The second lemma bounds the number of triangles intersected by a length-minimizing curve, thus allowing us to estimate the accumulated distortion along such a curve.

LEMMA 5.3. *For every $n \in \mathbb{N}$ and $p, q \in \mathcal{N}_n$, the shortest path in $\mathcal{N}$ connecting p and q intersects $O(n)$ geodesic triangles $\mathcal{N}_{n,i}$. Likewise, for every (large enough) $n \in \mathbb{N}$ and $p, q \in \mathcal{M}_n$, the shortest path in $\mathcal{M}_n$ connecting p and q intersects $O(n)$ of the $\mathcal{M}_{n,i}$s.*

Proof. Let $p, q \in \mathcal{N}_n$, and let σ be a segment connecting them. Let r be defined as in corollary 3.2, that is, each edge in the triangulation is of distance greater than r/n from an opposite edge. The lower bound on the angles in proposition 3.1 implies that each vertex in Y_n is surrounded by at most $2\pi/\delta$ triangles. Therefore, a ball of radius $r/2n$ intersects at most $2\pi/\delta$ triangles. Therefore, a ball of radius $r/2n$ around a point $p' \in \mathcal{N}_n$ does not intersect more triangles than the total number of triangles that balls of radius $r/2n$ around the four vertices of its (two) adjacent triangles intersect, that is $8\pi/\delta$. Since the length of σ is at most the diameter of $\mathcal{N}$, it follows that σ intersects at most $16\pi n \operatorname{Diam}(\mathcal{N})/r\delta$ triangles.

Now let $p, q \in \mathcal{M}_n$. Let σ be the shortest path in $\mathcal{N}$ connecting $F_n(p)$ and $F_n(q)$. It intersects $O(n)$ geodesic triangles. Denote by $F_n(p) = t_0, t_1, \ldots, t_k = F_n(q)$ the points in $\sigma \cap Y_n$. By lemma 5.2,

$$\begin{aligned} d_n(p, q) &\leqslant \sum_{j=1}^{k} d_n(F_n^{-1}(t_{j-1}), F_n^{-1}(t_j)) \leqslant \sum_{j=1}^{k} \left(d(t_{j-1}, t_j) + \frac{c}{n^{2-\varepsilon}} \right) \\ &= d(F_n(p), F_n(q)) + O(1/n^{1-\varepsilon}) < \operatorname{Diam}(\mathcal{D}) + O(1/n^{1-\varepsilon}). \end{aligned}$$

Hence, the diameter of $\mathcal{M}_n$ is bounded uniformly in n, $\operatorname{Diam}(\mathcal{M}_n) \leqslant K$.

A similar reasoning now applies: since, by lemma 5.2, distances in $\mathcal{M}_{n,i}$ and $\mathcal{N}_{n,i}$ differ only by $O(n^{-2+\varepsilon})$, any vertex in X_n is at a distance greater than r/n from an opposite edge; hence, any curve of length less than K intersects at most $16\pi n K/r\delta$ triangles. □

Thus, the distortion of F_n is $O(n)O(n^{-2+\varepsilon}) = O(n^{-1+\varepsilon})$, which completes the proof of proposition 5.1. See the proof of [7, theorem 3.1] for a similar argument. □

Corollary 5.4. *Let $(\mathcal{M}_n, d_n)$ be the sequence of compact metric spaces defined in § 3. Then, $(\mathcal{M}_n, d_n)$ GH converges to $(\mathcal{N}, d)$.*

Proof. The GH distance is a measure of distortions between metric spaces, and is a metric on isometry classes of compact metric spaces [10, ch. 10]. A sufficient and necessary condition for a sequence of metric spaces (Z_n, d_n) to converge in the GH sense to a metric space (Z, d) is that there exist bijections

$$T_n \colon A_n \to B_n,$$

where $A_n \subset Z_n$ and $B_n \subset Z$ are finite δ_n nets, $\delta_n \to 0$, and the distortion of T_n,

$$\operatorname{dis} T_n = \max_{x,y \in A_n} |d_n(x, y) - d(T_n(x), T_n(y))|$$

tends to zero.

In our case, restrict F_n to the vertices of the skeleton X_n. This forms a bijection between two finite $O(n^{-1})$ nets of $\mathcal{M}_n$ and $\mathcal{N}$, respectively. Since the distortion of a mapping does not grow under restriction, GH convergence is implied immediately by proposition 5.1. □

Note that alternative mappings between the vertices of X_n and Y_n can be used to prove GH convergence. In particular, one can use an analogue of the construction shown in [7, appendix A] to obtain a $O(1/n)$ distortion, i.e. a higher rate of convergence.

6. Convergence of Weitzenböck manifolds

Proposition 4.3, corollary 4.4 and proposition 5.1 cover the asymptotic surjectivity, mean asymptotic rigidity and vanishing distortion properties in definition 1.1, respectively. Therefore, to complete the proof of theorem 1.2 it remains to prove weak convergence of the connections, and extend the entire analysis to non-simply-connected domains.

PROPOSITION 6.1. *Let E be a fixed ∇-parallel orthonormal frame field on $\mathcal{N}$. Then there exist ∇_n-parallel orthonormal frame fields E_n on $\mathcal{M}_n$ such that, for every $p \in [1, \infty)$,*

$$\int_{F_n(\mathcal{M}_n)} |(F_n)_\star E^n - E|^p \, \mathrm{dVol}_{\mathfrak{g}} = O\left(\frac{1}{n^{1-\varepsilon}}\right). \tag{6.1}$$

Proof. Let $E = \{e_1, e_2\}$ be a fixed ∇-parallel orthonormal frame field on $\mathcal{N}$. Without loss of generality, assume E is oriented. Given $n \in \mathbb{N}$ we construct an oriented ∇_n-orthonormal parallel frame field $E^n = \{e_1^n, e_2^n\}$. For that, we only need to specify e_1^n at a single point. Take a point $p \in Y_n$, and let $q \in Y_n$ be another point that lies on the same edge of a triangle. There exists a unique unit vector v_p in $T_p\mathcal{N}$ such that $q = \exp_p^{\nabla}(tv_p)$ for some $t \in (0, \bar{L}/n]$. Likewise, there exists a unique unit vector $w_{F_n^{-1}(p)}$ in $T_{F_n^{-1}(p)}\mathcal{M}_n$, such that $F_n^{-1}(q) = F_n^{-1}(p) + tw_{F_n^{-1}(p)}$ for some $t \in (0, \bar{L}/n]$. e_1^n is the unique unit vector that satisfies

$$\mathfrak{g}_n(e_1^n, w_{F_n^{-1}(p)}) = \mathfrak{g}(e_1, v_p).$$

We claim that this choice implies that the above identity holds for every point $p' \in Y_n$, because parallel transport along a geodesic of a metric connection preserves both the length of the vector and its angle with the geodesic (recall that X_n and Y_n are unions of ∇_n- and ∇-geodesics, respectively).

Now let $p \in \mathcal{M}_n \setminus K_n$, and denote by q_p the vertex of X_n in its connected component. Denote by γ_p the ∇_n-geodesic connecting q_p and p and let $\sigma_p = F_n(\gamma_p)$ be the ∇-geodesic connecting $F_n(q_p) \in Y_n$ and $F_n(p)$. By the same argument about parallel transport along geodesics, the angle between e_i^n and γ_p at p equals the angle between e_i and σ_p at $F_n(p)$. Since

$$\mathrm{dist}(\mathrm{d}F_n, \mathrm{SO}(\mathfrak{g}_n, \mathfrak{g})) = O\left(\frac{1}{n}\right) \quad \text{on } \mathcal{M}_n \setminus K_n,$$

and since $\mathrm{d}F_n$ maps the unit tangent of γ_p to the unit tangent of σ_p, it follows that

$$|(F_n)_\star E^n - E| = O\left(\frac{1}{n}\right) \quad \text{on } F_n(\mathcal{M}_n \setminus K_n).$$

Moreover, $\mathrm{d}F_n$ is uniformly bounded, and thus so is $|(F_n)_\star E^n - E|$. Since $\mathrm{Vol}(K_n) = O(n^{-1+\varepsilon})$, (6.1) follows. $\square$

Finally, to extend the entire analysis to domains of finite genus, one has to partition the domain along geodesics into simply connected components, approximate each domain separately up to the lines of partitions (which is why they need to be geodesics) and glue the defective components together.

Appendix A. Riemannian and connection curvatures in two dimensions

In this appendix we prove proposition 2.1:

> let $(\mathcal{N}, \mathfrak{g})$ be a two-dimensional Riemannian manifold. Let ∇ be a metrically consistent connection defined as in (2.1) by a vector field V. Let K and $K^{\mathfrak{g}}$ denote the connection and the Riemannian Gaussian curvatures. Then,
> $$K\,\mathrm{dVol}_{\mathfrak{g}} = K^{\mathfrak{g}}\,\mathrm{dVol}_{\mathfrak{g}} - d\star V^\flat.$$

A tedious, but straightforward calculation shows that

$$\begin{aligned} R(X,Y)Z = R^{\mathfrak{g}}(X,Y)Z & \\ &+ \mathfrak{g}(Y,Z)\nabla^{\mathfrak{g}}_X V - \mathfrak{g}(X,Z)\nabla^{\mathfrak{g}}_Y V \\ &- \mathfrak{g}(\nabla^{\mathfrak{g}}_X V, Z)Y + \mathfrak{g}(\nabla^{\mathfrak{g}}_Y V, Z)X \\ &+ \mathfrak{g}(Y,Z)\mathfrak{g}(X,V)V - \mathfrak{g}(X,Z)\mathfrak{g}(Y,V)V \\ &- \mathfrak{g}(Y,Z)|V|^2 X + \mathfrak{g}(V,Z)\mathfrak{g}(V,Y)X \\ &+ \mathfrak{g}(X,Z)|V|^2 Y - \mathfrak{g}(V,Z)\mathfrak{g}(V,X)Y. \end{aligned}$$

In two dimensions, for an orthonormal frame (e_1, e_2),

$$\mathfrak{g}(R(e_1,e_2)e_1,e_2) = \mathfrak{g}(R^{\mathfrak{g}}(e_1,e_2)e_1,e_2) - \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_2}V, e_2) - \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_1}V, e_1),$$

that is,

$$K = K^{\mathfrak{g}} - \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_2}V, e_2) - \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_1}V, e_1).$$

Write $V = fe_1 + ge_2$ for some functions f, g, and let $(\vartheta^1, \vartheta^2)$ be the co-frame of (e_1, e_2). Then

$$\begin{aligned} \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_1}V, e_1) + \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_2}V, e_2) &= \vartheta^1(\nabla^{\mathfrak{g}}_{e_1}V) + \vartheta^2(\nabla^{\mathfrak{g}}_{e_2}V) \\ &= \vartheta^1(\nabla^{\mathfrak{g}}_{e_1}(fe_1 + ge_2)) + \vartheta^2(\nabla^{\mathfrak{g}}_{e_2}(fe_1 + ge_2)) \\ &= \mathrm{d}f(e_1) + g\vartheta^1(\nabla^{\mathfrak{g}}_{e_1}e_2) + f\vartheta^2(\nabla^{\mathfrak{g}}_{e_2}e_1) + \mathrm{d}g(e_2) \\ &= \mathrm{d}f(e_1) + g\omega^1_2(e_1) + f\omega^2_1(e_2) + \mathrm{d}g(e_2), \end{aligned}$$

where $\omega^1_2 = -\omega^1_2$ is the 1-form defining the connection. On the other hand, we have $V^\flat = f\vartheta^1 + g\vartheta^2$; hence, $\star V^\flat = f\vartheta^2 - g\vartheta^1$, and therefore

$$d\star V^\flat = \mathrm{d}f \wedge \vartheta^2 - \mathrm{d}g \wedge \vartheta^1 + f\,\mathrm{d}\vartheta^2 - g\,\mathrm{d}\vartheta^1.$$

Using Cartan's first structural equation, we obtain

$$d\star V^\flat(e_1,e_2) = \mathrm{d}f(e_1) + \mathrm{d}g(e_2) + f\omega^2_1(e_2) + g\omega^1_2(e_1) = \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_1}V, e_1) + \mathfrak{g}(\nabla^{\mathfrak{g}}_{e_2}V, e_2),$$

i.e.

$$K\,\mathrm{dVol}_{\mathfrak{g}} = K^{\mathfrak{g}}\,\mathrm{dVol}_{\mathfrak{g}} - d\star V^\flat.$$

Appendix B. Gauss–Bonnet theorem for metric connections

THEOREM B.1. *Let ∇ be a metric connection on a two-dimensional oriented Riemannian manifold $(\mathcal{M}, \mathfrak{g})$. Let γ be a closed piecewise-smooth closed curve encircling a domain Γ. Denote by $\{t_i\}_{i=1}^{k}$ the vertices of γ and by ϵ_i the exterior angles. Then,*

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} k_\nabla \,\mathrm{d}t + \sum_{i=1}^{k}\epsilon_i = 2\pi - \int_\Gamma K \,\mathrm{dVol},$$

where $t_0 = t_k$, k_∇ is the signed geodesic curvature of γ with respect to ∇, K is the Gaussian curvature of ∇ and dVol is the area form. In particular, if γ is piecewise geodesic and ∇ is flat, then the sum of the exterior angles is 2π.

Proof. The proof is basically an adaptation of the proof in [9, p. 164]. Let (e_1, e_2) be a local oriented orthonormal frame. Denote by $\theta(t)$ the unique angle between $\dot\gamma(t)$ and $e_1(\gamma(t))$, with $\theta(t_0) \in (-\pi, \pi]$, which is continuous at the smooth parts of γ and has jumps ϵ_i at the vertices.

By the fundamental theorem of calculus we have

$$\sum_i \epsilon_i + \sum_i \int_{t_{i-1}}^{t_i} \dot\theta(t)\,\mathrm{d}t = \theta(b) - \theta(a) = 2\pi. \tag{B 1}$$

At all smooth points of γ we have

$$\left.\begin{aligned} \dot\gamma(t) &= \cos\theta(t)e_1(t) + \sin\theta(t)e_2(t),\\ N(t) &= -\sin\theta(t)e_1(t) + \cos\theta(t)e_2(t). \end{aligned}\right\} \tag{B 2}$$

Differentiating (B 2) with respect to $\nabla_{\dot\gamma}$, we obtain

$$\begin{aligned} \frac{\mathrm{D}\dot\gamma}{\mathrm{d}t} &= -\sin\theta(t)\dot\theta(t)e_1 + \cos\theta(t)(-\omega(\dot\gamma(t))e_2) + \cos\theta(t)\dot\theta(t)e_2 + \sin\theta(t)\omega(\dot\gamma(t))e_1\\ &= \dot\theta(t)N(t) + \omega(\dot\gamma(t))(-\cos\theta(t)e_2 + \sin\theta(t)e_1), \end{aligned}$$

where ω is the 1-form defined by $\nabla_X e_1 = \omega(X)e_2$ (and by the metricity of the connection $\nabla_X e_2 = -\omega(X)e_1$).

By taking an inner product with $N(t)$, we obtain

$$k_\nabla(t) = \left\langle \frac{\mathrm{D}\dot\gamma(t)}{\mathrm{d}t}, N(t)\right\rangle_{\gamma(t)} = \dot\theta(t) - \omega(\dot\gamma(t)). \tag{B 3}$$

Inserting (B 3) into (B 1) and using Stokes's theorem, we obtain

$$2\pi = \sum_{i=1}^{k}\epsilon_i + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} k_\nabla\,\mathrm{d}t + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\omega = \sum_{i=1}^{k}\epsilon_i + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} k_\nabla\,\mathrm{d}t + \int_\Gamma \mathrm{d}\omega.$$

In two dimensions, $\mathrm{d}\omega = K\,\mathrm{dVol}$, which completes the proof. □

Appendix C. Proof of proposition 4.2

In this section we prove a general result concerning short curves in two-dimensional Riemannian manifolds, possibly endowed with a non-symmetric connection:

> let $\gamma\colon [0,\ell] \to \mathcal{N}$, $\ell \leqslant 3\bar{L}/n$, be a curve in arc-length parametrization with geodesic curvature $k(t)$ (the geodesic curvature is with respect to the connection ∇). Let $\sigma\colon [0,\ell] \to \mathbb{R}^2$ be a curve in arc-length parametrization with geodesic curvature $k(t)$ (with respect to the Euclidean connection). Then, there exists a constant $\Delta > 0$, which depends only on $\bar{L}$, $\bar{\mathcal{K}}$, $\underline{\mathcal{K}}$ and Λ such that, for every $t \in [0,\ell]$,
>
> $$|d(\gamma(t),\gamma(0)) - |\sigma(t)-\sigma(0)|| \leqslant \frac{\Delta}{n^2}.$$

Proof. We represent the curve γ using a semi-geodesic parametrization (r,θ) as in the proof of proposition 2.2, where the origin is the starting point of γ; thus,

$$r(t) = d(\gamma(t),\gamma(0)).$$

For a metric prescribed by a function $\varphi(r,\theta)$, with a connection prescribed by a vector field (V^r, V^θ), the parametric equations of a curve with geodesic curvature $k = k(t)$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{r} = \frac{\varphi_r}{\varphi}(\varphi\dot{\theta})^2 - (V^r\varphi\dot{\theta} + \varphi V^\theta \dot{r})\varphi\dot{\theta} - k\varphi\dot{\theta},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi\dot{\theta}) = -\frac{\varphi_r}{\varphi}\dot{r}\varphi\dot{\theta} + (V^r\varphi\dot{\theta} + \varphi V^\theta \dot{r})\dot{r} + k\dot{r}.$$

Using the fact that $(\varphi\dot{\theta})^2 = 1 - \dot{r}^2$, we rewrite these as follows:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(r\dot{r}) + (k+\bar{\Delta})r\varphi\dot{\theta} = 1, \\ \frac{\mathrm{d}}{\mathrm{d}t}(r\varphi\dot{\theta}) - (k+\bar{\Delta})r\dot{r} = 0, \end{aligned}\right\} \tag{C 1}$$

where

$$\bar{\Delta} = (V^r\varphi\dot{\theta} + \varphi V^\theta\dot{r}) - \left(\frac{\varphi_r}{\varphi} - \frac{1}{r}\right)\varphi\dot{\theta} \equiv \Delta_1 + \Delta_2.$$

We have the following estimates:

$$|\Delta_1| \leqslant \Lambda, \qquad |\Delta_2| \leqslant \sup\left|\frac{\varphi_r r - \varphi}{r\varphi}\right| = \sup\left|\frac{\varphi_{rr} r}{\varphi + r\varphi_r}\right| \leqslant \frac{3\bar{L}}{n}\max(|\underline{\mathcal{K}}|, |\bar{\mathcal{K}}|).$$

Defining $z = r(\dot{r} + \imath\varphi\dot{\theta})$, (C 1) takes the complex form:

$$\dot{z} - \imath(k+\bar{\Delta})z = 1.$$

Integrating, we obtain

$$(r\dot{r})(t) = \int_0^t \cos(\xi(t) - \xi(s))\,\mathrm{d}s,$$

where

$$\xi(t) = \xi_0(t) + \xi_1(t),$$

with

$$\xi_0(t) = \int_0^t k(s)\,\mathrm{d}s \quad \text{and} \quad \xi_1(t) = \int_0^t \bar{\Delta}(s)\,\mathrm{d}s = O(n^{-1}).$$

Integrating a second time yields

$$r^2(t) = 2\int_0^t \int_0^s \cos(\xi(s) - \xi(s'))\,\mathrm{d}s'\,\mathrm{d}s = \int_0^t \int_0^t \cos(\xi(s) - \xi(s'))\,\mathrm{d}s'\,\mathrm{d}s,$$

where the second equality follows by symmetry. Using the formula for the cosine of a difference, we get

$$r^2(t) = \left(\int_0^t \cos\xi(s)\,\mathrm{d}s\right)^2 + \left(\int_0^t \sin\xi(s)\,\mathrm{d}s\right)^2.$$

The Euclidean case is retrieved by setting $\xi_1 = 0$, namely,

$$r_{\mathrm{E}}^2(t) = |\sigma(t) - \sigma(0)| = \left(\int_0^t \cos\xi_0(s)\,\mathrm{d}s\right)^2 + \left(\int_0^t \sin\xi_0(s)\,\mathrm{d}s\right)^2.$$

Substituting $\xi = \xi_0 + \xi_1$, it takes simple manipulations to get

$$\begin{aligned} r^2(t) = r_{\mathrm{E}}^2(t) &+ 4\left(\int_0^t \sin\frac{\xi_1}{2}\sin\left(\frac{\xi_1}{2} - \xi_0\right)\mathrm{d}s\right)^2 + 4\left(\int_0^t \sin\frac{\xi_1}{2}\cos\left(\frac{\xi_1}{2} - \xi_0\right)\mathrm{d}s\right)^2 \\ &- 2\left(\int_0^t \cos\xi_0(s)\,\mathrm{d}s\right)\left(\int_0^t \sin\frac{\xi_1}{2}\sin\left(\frac{\xi_1}{2} - \xi_0\right)\mathrm{d}s\right) \\ &+ 2\left(\int_0^t \sin\xi_0(s)\,\mathrm{d}s\right)\left(\int_0^t \sin\frac{\xi_1}{2}\cos\left(\frac{\xi_1}{2} - \xi_0\right)\mathrm{d}s\right). \end{aligned}$$

Since

$$\left|\int_0^t \cos\xi_0\,\mathrm{d}s\right|, \left|\int_0^t \sin\xi_0\,\mathrm{d}s\right| \leqslant r_{\mathrm{E}}(t)$$

and both ξ_1 and t are $O(1/n)$, it follows that

$$|r^2(t) - r_{\mathrm{E}}^2(t)| \leqslant \frac{\Delta}{n^2} r_{\mathrm{E}} + O\left(\frac{1}{n^4}\right),$$

which implies

$$|r(t) - r_{\mathrm{E}}(t)| \leqslant \frac{\Delta}{n^2}.$$

□

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References

1 I. Agricola and C. Thier. The geodesics of metric connections with vectorial torsion. *Annals Global Analysis Geom.* **26** (2004), 321–332.

2 M. Berger. *A panoramic view of Riemannian geometry* (Springer, 2002).

3 B. A. Bilby, R. Bullough and E. Smith. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. R. Soc. Lond.* A **231** (1955), 263–273.

4 B. V. Dekster. The length of a curve in a space of curvature $\leqslant k$. *Proc. Am. Math. Soc.* **79** (1980), 271–278.

5 W. P. A. Klingenberg. *Riemannian geometry*, 2nd edn (Berlin: Walter de Gruyter, 1995).

6 E. Kröner. Contiuum theory of defects. In *Physics of defects*, Les Houches Summer School Proceedings (ed. R. Balian, M. Kleman and J.-P. Poirier) (Amsterdam: North-Holland, 1981).

7 R. Kupferman and C. Maor. The emergence of torsion in the continuum limit of distributed dislocations. *J. Geom. Mech.* **7** (2015), 361–387.

8 R. Kupferman, M. Moshe and J. P. Solomon. Metric description of defects in amorphous materials. *Arch. Rat. Mech. Analysis* **216** (2015), 1009–1047.

9 J. M. Lee. *Riemannian manifolds: an introduction to curvature*, Graduate Texts in Mathematics (Springer, 1997).

10 P. Petersen. *Riemannian geometry*, 2nd edn, Graduate Texts in Mathematics (Springer, 2006).