A Riemannian approach to the membrane limit of non-Euclidean elasticity

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Abstract Non-Euclidean, or incompatible elasticity is an elastic theory for pre-stressed materials, which is based on a modeling of the elastic body as a Riemannian manifold. In this paper we derive a dimensionally-reduced model of the so-called membrane limit of a thin incompatible body. By generalizing classical dimension reduction techniques to the Riemannian setting, we are able to prove a general theorem that applies to an elastic body of arbitrary dimension, arbitrary slender dimension, and arbitrary metric. The limiting model implies the minimization of an integral functional defined over immersions of a limiting submanifold in Euclidean space. The limiting energy only depends on the first derivative of the immersion, and for frame-indifferent models, only on the resulting pullback metric induced on the submanifold, i.e., there are no bending contributions.

Keywords Riemannian manifolds · Nonlinear elasticity · Incompatible elasticity · Membranes · Gamma-convergence

1 Introduction

In recent years there has been a renewed interest in the elastic properties of bodies that have an intrinsically non-Euclidean geometry. The original interest in such systems stemmed from the study of crystalline defects, in which case the intrinsic geometry exhibits singularities; see Bilby and co-workers [5,6], Kondo [18], Wang [30], and Kröner [19]. The motivation for the recent interest in non-Euclidean bodies arises

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from the study of growing tissues [11,3,2], thermal expansion [27], and other mechanisms of differential expansion of shrinkage [16]; in all these examples the intrinsic geometry can be assumed to be smooth.

Mathematically, we model an elastic body as a three-dimensional Riemannian manifold, $(\mathcal{M}, \mathfrak{g})$, equipped with an energy function that assigns an energy to every configuration $f : \mathcal{M} \to \mathbb{R}^3$ of the manifold into the ambient Euclidean space, $(\mathbb{R}^3, \mathfrak{e})$. This energy is a measure of the strain, i.e., of the deviation of the pullback metric $f^\star \mathfrak{e}$ from the intrinsic metric $\mathfrak{g}$. The body is said to be non-Euclidean if the intrinsic metric has non-zero Riemannian curvature, in which case it cannot be immersed isometrically in three-dimensional Euclidean space. The elastostatic problem consists of finding the configuration f that minimizes the elastic energy given possibly boundary conditions and external forces.

A central theme in material sciences is the derivation of dimensionally-reduced models, which are applicable to elastic bodies that display one or more slender axes. In such models the elastic body is viewed as a lower-dimensional limit of thin bodies (which can be viewed as the mid-surface). The derivation of dimensionally reduced models goes back to Euler, D. Bernoulli, Cauchy, and Kirchhoff [15], and in the last century, to name just a few, to von Karman [29], E. and F. Cosserat, Love [24], and Koiter [17].

Dimensionally-reduced models are commonly classified according to two main criteria: the dimension of the limiting manifold (which may be either 1 or 2) and the energy scaling of the reduced energy functional. Plates and shells are examples of two-dimensional reduced models in which the limiting manifold can be embedded in $\mathbb{R}^3$ smoothly enough so that the main energy contribution comes from the second fundamental form (bending effects). Membranes are examples of two-dimensional reduced models in which the main energy contribution is from metric deviations of the two-dimensional pullback metric from the metric of the limiting manifold (stretching effects). Rods are examples of one-dimensional reduced models.

Until about 20 years ago, dimension reduction analyses were based essentially on formal asymptotic methods and uncontrolled ansatzes. The rigorous derivation of dimensionally-reduced models was first achieved in the Euclidean case, where the bodies have a natural rest configuration with respect to which deviations can be measured. The membrane limit was derived by Le Dret and Raoult [21,22], and generalized by Braides et al. [7] and Babadjian and Francfort [4], whereas the plate and shells limits were derived by James et al. [13] and [14]. The rod limit was derived by Mora and Müller [26]. For non-Euclidean bodies the limiting plate theory was derived by Lewicka and Pakzad [23], whereas Kupferman and Solomon [20] proved a general theorem that yields plate, shell and rod limits in non-Euclidean cases. All the above mentioned work relies on Γ-convergence techniques [10].

In this work we derive the membrane limit of non-Euclidean elasticity. A typical application of such limit would be the study of a thin plant tissue under stretching conditions. We consider here pure displacement problems without body forces; the inclusion of external forces and/or surface traction is not expected to involve any complications [21].

We now describe our main results; precise definitions and formulations are given in the next section. We denote by Ω_h a family of n-dimensional submanifolds of an

n-dimensional manifold $(\mathcal{M}, \mathfrak{g})$ that converge to an $(n-k)$-dimensional manifold $\mathcal{S}$; here h is the thickness of the domains. With every configuration $f_h : \Omega_h \to \mathbb{R}^n$ (which is regular enough and satisfies the boundary conditions) we associate an energy

$$I_h(f_h) = \fint_{\Omega_h} W(df_h) \, d\mathrm{vol}_{\mathfrak{g}},$$

where $\fint$ denotes volume average, and W is an h-independent energy density satisfying some regularity, growth and coercivity conditions, as well as a homogeneity condition. Considering pure displacement problems, we prove that I_h Γ-converges as $h \to 0$ to a functional that assigns, to regular enough configurations $F : \mathcal{S} \to \mathbb{R}^n$ that satisfy the boundary conditions, an energy

$$I(F) = \fint_{\mathcal{S}} QW_0(dF) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}},$$

where QW_0 is the quasiconvex envelope of a projection of the restriction of W to $\mathcal{S}$. Moreover, every sequence f_h of (possibly approximate) minimizers of I_h has a subsequence that converges to a minimizer of I.

The basic tools are the analytic techniques developed in [21] along with the geometric framework developed in [20]. The main difference between our analysis and that in [21] is that the current analysis applies to an arbitrary Riemannian setting and to arbitrary dimensions. As such, it does not distinguish a priori between "plate-like", "shell-like" or "rod-like" bodies, and neither between Euclidean and non-Euclidean geometries.

Moreover, the Riemannian setting reveals the geometric content of classical notions in elasticity and analysis. It requires the revision and the generalization of the notions of quasiconvexity, measurable selection theorems, and Carathéodory functions. In additon, the material science notion of homogeneity needs to be reinterpreted, which leads to new insights into its geometric meaning. Finally, the geometric analysis "toolbox" constructed in [20] is expanded to treat different function spaces and more general energy densities.

2 Problem statement and main results

2.1 Modeling of slender bodies

We start by presenting the general geometric framework. Let $\mathcal{M}$ be a smooth n-dimensional oriented manifold; let $\mathcal{S} \subset \mathcal{M}$ be a smooth m-dimensional compact oriented submanifold with Lipschitz-continuous boundary; let k denote the codimension of $\mathcal{S}$ in $\mathcal{M}$. We endow $\mathcal{M}$ with a metric $\mathfrak{g}$, and denote the induced metric on $\mathcal{S}$ by $\mathfrak{g}|_{\mathcal{S}}$.

We view $T\mathcal{S}$ as a sub-bundle of $T\mathcal{M}|_{\mathcal{S}}$, and denote its orthogonal complement, the normal bundle of $\mathcal{S}$ in $\mathcal{M}$, by $N\mathcal{S}$, so that

$$T\mathcal{M}|_{\mathcal{S}} \cong T\mathcal{S} \oplus N\mathcal{S}.$$

Let h be a continuous positive parameter, and define a family of tubular neighborhoods of $\mathcal{S}$ by

$$\Omega_h = \{ p \in \mathcal{M} : \mathrm{dist}(p, \mathcal{S}) < h \} \subset \mathcal{M}.$$
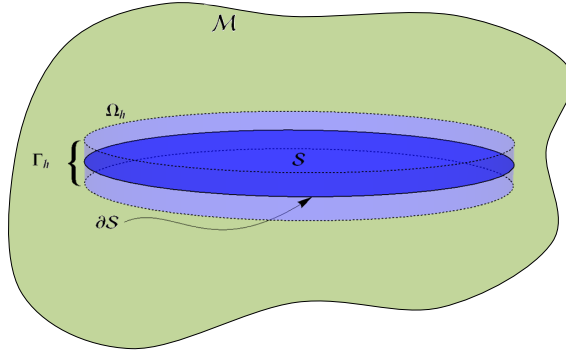
Fig. 1 Schematic illustration of the manifolds $\mathcal{M}$, Ω_h, $\mathcal{S}$, Γ_h and $\partial\mathcal{S}$.

These tubular neighborhoods inherit the metric $\mathfrak{g}$. Our smoothness and compactness assumptions on $\mathcal{S}$ imply that for small enough h (say $h \in (0, h_0]$ for some $h_0 > 0$) the exponential map,

$$\exp : \{(p, \xi) \in N\mathcal{S} : |\xi| < h\} \to \Omega_h$$

is a diffeomorphism between an open subset of $N\mathcal{S}$ and Ω_h. Therefore, we have a structure of fiber bundle $\pi : \Omega_h \to \mathcal{S}$, with the fiber being a k-dimensional ball, and the projection π is obtained by moving along the geodesic from a point $p \in \Omega_h$ to its nearest neighbor in $\mathcal{S}$.

2.2 Configurations and boundary conditions

We view Ω_h as a family of (shrinking) bodies. A *configuration* of Ω_h is a map $f_h : \Omega_h \to \mathbb{R}^n$ from the so-called material manifold $(\Omega_h, \mathfrak{g})$ to the physical space $(\mathbb{R}^n, \mathfrak{e})$, where $\mathfrak{e}$ is the Euclidean metric.

In the elastic context, we consider a *pure displacement* problem, where the boundary conditions are imposed on the "outer ring" of Ω_h,

$$\Gamma_h = \{\xi \in \Omega_h : \pi(\xi) \in \partial\mathcal{S}\}.$$

A sketch of the manifolds $\mathcal{M}$, Ω_h, $\mathcal{S}$ and the boundary manifolds Γ_h and $\partial\mathcal{S}$ are shown in Figure 1.

We impose the boundary conditions by specifying a mapping of $\partial\mathcal{S}$ into $\mathbb{R}^n$, and extending it linearly to Γ_h via a mapping of normal vectors. Specifically, let F_{bc} be a mapping $\partial\mathcal{S} \to \mathbb{R}^n$ and $\mathfrak{q}_{bc}^{\perp}$ be a section of $(N\mathcal{S}^* \otimes \mathbb{R}^n)|_{\partial\mathcal{S}}$ (an assumption on the regularity of these mappings will be imposed later). A mapping $f_h : \Omega_h \to \mathbb{R}^n$ satisfies the boundary conditions if

$$f_h(\xi) = F_{bc}(\pi(\xi)) + (\mathfrak{q}_{bc}^{\perp})_{\pi(\xi)}(\xi) \qquad \xi \in \Gamma_h, \tag{2.1}$$

where we identify $\xi \in \Gamma_h$ with its image under the diffeomorphism of Ω_h to an open set in $\mathcal{NS}$.

The condition (2.1) can be written in a more compact form. For a section $\mathfrak{q}^\perp \in \Gamma(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n)$, the pullback $\pi^*\mathfrak{q}^\perp$ is a section in $\Gamma(\Omega_h; \pi^*\mathcal{NS}^* \otimes \mathbb{R}^n)$, and

$$(\pi^*\mathfrak{q}^\perp)_\xi(\eta) = \mathfrak{q}^\perp_{\pi(\xi)}(\eta),$$

where $\xi \in \Omega_h$ and $\eta \in (\pi^*\mathcal{NS})_\xi \cong \mathcal{NS}_{\pi(\xi)}$. With these identifications we can write $(\mathfrak{q}^\perp_{bc})_{\pi(\xi)}(\xi) = (\pi^*\mathfrak{q}^\perp_{bc})_\xi(\xi)$. Introducing the tautological section $\lambda \in \Gamma(\Omega_h; \pi^*\mathcal{NS})$ defined by

$$\lambda_\xi = \xi \in (\pi^*\mathcal{NS})_\xi,$$

we can write (2.1) as follows:

$$f|_{\Gamma_h} = F_{bc} \circ \pi + \pi^*\mathfrak{q}^\perp_{bc} \circ \lambda. \tag{2.2}$$

2.3 The energy functional

The assumption whereby the bodies Ω_h are hyper-elasticity means that to each admissible (in a sense to be made precise below) configuration f_h corresponds an elastic energy of the form

$$\fint_{\Omega_h} W(df_h)\, d\mathrm{vol}_{\mathfrak{g}}, \tag{2.3}$$

where

$$W : T^*\Omega_{h_0} \otimes \mathbb{R}^n \to \mathbb{R}.$$

is an *elastic energy density*; note that W is independent of h.

For $q \in T^*\Omega_{h_0} \otimes \mathbb{R}^n$, we denote by $|q|$ the norm that is inherited from both $\mathfrak{g}$ and $\mathfrak{e}$. We assume W to be continuous, and that there exists a $p \in (1, \infty)$ such that:

1. Growth condition: $|W(q)| \leq C(1 + |q|^p)$,
2. Coercivity: $W(q) \geq \alpha|q|^p - \beta$,
3. Lipschitz property: for $q, q' \in T^*_x\Omega_{h_0} \otimes \mathbb{R}^n$,

$$|W(q) - W(q')| \leq C(1 + |q|^{p-1} + |q'|^{p-1})|q - q'|,$$

4. Homogeneity over fibers, which will be defined in the next section. When $(\Omega_h, \mathfrak{g})$ is Euclidean, this condition amounts to $W : \Omega_h \times \mathbb{R}^{n \times n} \to \mathbb{R}$ being in fact a mapping $\mathcal{S} \times \mathbb{R}^{n \times n} \to \mathbb{R}$, i.e., the spatial dependence of the energy density only depends on the projection on the mid-surface.

Under these conditions, the total elastic energy (2.3) is defined for all $f_h \in W^{1,p}(\Omega_h; \mathbb{R}^n)$.

A prototypical energy density that satisfies these conditions for $p = 2$ is

$$W(\cdot) = \mathrm{dist}^2(\cdot, \mathrm{SO}(\Omega_{h_0}; \mathbb{R}^n)), \tag{2.4}$$

where $\mathrm{SO}(\Omega_{h_0}; \mathbb{R}^n)$ denotes the metric and orientation preserving transformations $T\Omega_{h_0} \to \mathbb{R}^n$. This energy density measures how far a configuration is from being a

local isometry. Note however that we do not assume W to satisfy frame-indifference or isotropy.

The space of admissible configurations is defined by requiring that (2.3) be well-defined, as well as the satisfaction of the boundary conditions (2.2). We denote:

$$W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n) = \{ f \in W^{1,p}(\Omega_h; \mathbb{R}^n) : f|_{\Gamma_h} = F_{bc} \circ \pi + \pi^* \mathfrak{q}_{bc}^\perp \circ \lambda \}.$$

We assume that F_{bc} and $\mathfrak{q}_{bc}^\perp$ are regular enough such that the spaces $W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ are not empty for small enough h. Note that each $W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ is an affine space with respect to the vector space $W_0^{1,p}(\Omega_h; \mathbb{R}^n)$.

For technical reasons it is convenient to extend the domain of the energy functional to configurations $L^p(\Omega_h; \mathbb{R}^n)$ that may not satisfy either regularity or boundary conditions as follows:

$$I_h(f) = \begin{cases} \fint_{\Omega_h} W(df) \, d\mathrm{vol}_{\mathfrak{g}} & f \in W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n) \\ \infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where $\fint$ denotes a volume average, namely

$$\fint_{\Omega_h} \alpha \, d\mathrm{vol}_{\mathfrak{g}} = \frac{\int_{\Omega_h} \alpha \, d\mathrm{vol}_{\mathfrak{g}}}{\int_{\Omega_h} d\mathrm{vol}_{\mathfrak{g}}}.$$

2.4 Main result

We now define an energy density for configurations of the mid-surface $F : \mathcal{S} \to \mathbb{R}^n$. The restriction $W|_{\mathcal{S}}$ is a map $T^*\Omega_h|_{\mathcal{S}} \otimes \mathbb{R}^n \to \mathbb{R}$, which we may identify with a map $(T^*\mathcal{S} \oplus N\mathcal{S}^*) \otimes \mathbb{R}^n \to \mathbb{R}$. We then define

$$W_0 : T^*\mathcal{S} \otimes \mathbb{R}^n \to \mathbb{R}$$

as follows:

$$W_0(q) = \min_{r \in (N\mathcal{S}^* \otimes \mathbb{R}^n)_{\pi(q)}} W|_{\mathcal{S}}(q \oplus r).$$

Note that the coercivity condition on W implies that the minimum is indeed attained. Let $QW_0 : T^*\mathcal{S} \otimes \mathbb{R}^n \to \mathbb{R}$ be the quasiconvex envelope of W_0 (for more details on quasiconvex functions in a Riemannian setting see next section). The growth condition imposed on W implies that W_0 and QW_0 satisfy similar conditions (see Lemma 3.10 and Corollary 3.4 below).

We are now ready to state our main result:

Theorem 2.1 *The sequence of functionals* $(I_h)_{h \le h_0}$ Γ*-converges in the strong* L^p *topology, as* $h \to 0$*, to a limit* $I : L^p(\mathcal{S}; \mathbb{R}^n) \to \mathbb{R}$ *defined by:*

$$I(F) = \begin{cases} \fint_{\mathcal{S}} QW_0(dF) \, d\mathit{vol}_{\mathfrak{g}|_{\mathcal{S}}} & F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n), \\ \infty & \text{otherwise,} \end{cases}$$

where $W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n) = \{ F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n) : F|_{\partial \mathcal{S}} = F_{bc} \}$.

Note that each I_h is defined over a different functional space, which requires a slight modification in the definition of Γ-convergence; see next section.

A classical corollary of Γ-convergence implies that I can be viewed an $(n-k)$-dimensional approximation to the n-dimensional elastic functional I_h for small h, in the following sense:

Corollary 2.1 *Let $f_h \in W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ be a sequence of (approximate) minimizers of I_h, that is,*

$$I_h(f_h) = \inf_{L^p(\Omega_h; \mathbb{R}^n)} I_h(\cdot) + r(h),$$

where $\lim_{h \to 0} r(h) = 0$. Then (f_h) is a relatively compact sequence (with respect to the strong L^p topology), and all its limits points are minimizers of I. Moreover,

$$\lim_{h \to 0} \inf_{L^p(\Omega_h; \mathbb{R}^n)} I_h(\cdot) = \min_{L^p(\mathcal{S}; \mathbb{R}^n)} I(\cdot).$$

3 Preliminaries

3.1 Geometric setting

3.1.1 Decomposition of $T\mathcal{M}|_{\mathcal{S}}$

As stated in the previous section, we view Ω_h as a restriction of $\mathcal{NS}$ via the exponential map, where $\mathcal{NS}$ is the normal bundle of $\mathcal{S}$ in $\mathcal{M}$; we denote by π the projection from $\mathcal{NS}$ or Ω_h into $\mathcal{S}$.

We define the projection operators

$$P^{\parallel} : T\mathcal{M}|_{\mathcal{S}} \to T\mathcal{S} \qquad \text{and} \qquad P^{\perp} : T\mathcal{M}|_{\mathcal{S}} \to \mathcal{NS},$$

and the corresponding inclusions

$$\iota^{\parallel} : T\mathcal{S} \hookrightarrow T\mathcal{M}|_{\mathcal{S}} \qquad \text{and} \qquad \iota^{\perp} : \mathcal{NS} \hookrightarrow T\mathcal{M}|_{\mathcal{S}}.$$

3.1.2 Pullback bundles

Let $E \to \mathcal{S}$ and $F \to \mathcal{NS}$ (or Ω_h) be vector bundles. The pullback $\pi^* E$ is a vector bundle over $\mathcal{NS}$, such that for $\xi \in \mathcal{NS}$, the fiber $(\pi^* E)_\xi$ is identified with the fiber $E_{\pi(\xi)}$. Let $\Phi : \pi^* E \to F$, i.e.,

$$\Phi_\xi : (\pi^* E)_\xi \to F_\xi.$$

Since $(\pi^* E)_\xi$ is canonically identified with $E_{\pi(\xi)}$, we can unambiguously apply Φ_ξ to elements of $E_{\pi(\xi)}$.

3.1.3 Connections and parallel transport

Let ∇ denote the Levi-Civita connection on $T\mathcal{M}$, and by abuse of notation, also on its restriction to $T\mathcal{M}|_{\mathcal{S}}$. The induced connection on $N\mathcal{S}$ is defined by

$$\nabla^\perp = P^\perp \circ \nabla \circ \iota^\perp.$$

Let $\xi \in \Omega_h$, and denote by Π_ξ the parallel transport with respect to ∇ from $T_{\pi(\xi)}\mathcal{M}$ to $T_\xi\mathcal{M}$ along the geodesic from $\pi(\xi)$ to ξ. That is, Π is a bundle map

$$\Pi : \pi^* T\mathcal{M}|_{\mathcal{S}} \to T\Omega_h,$$

that satisfies

$$\mathfrak{g}_\xi(\Pi_\xi u, \Pi_\xi v) = \mathfrak{g}_{\pi(\xi)}(u, v),$$

for every $\xi \in \Omega_h$ and $u, v \in T_{\pi(\xi)}\mathcal{M}$.

3.1.4 Homogeneity

With parallel transport defined, we can now define "homogeneity over fibers" of the energy density W,

Definition 3.1 W is **homogenous over fibers** if for every $q \in T_\xi^*\Omega_h \otimes \mathbb{R}^n$,

$$W_\xi(q) = W_{\pi(\xi)}(q \circ \Pi_\xi \circ \Pi_{\pi(\xi)}^{-1}),$$

($\Pi_\xi \circ \Pi_{\pi(\xi)}^{-1}$ is the parallel transport $T_{\pi(\xi)}\Omega_h \to T_\xi\Omega_h$ along the geodesic connecting $\pi(\xi)$ to ξ). Equivalently,

$$W = \pi^* W|_{\mathcal{S}} \circ \Pi^*.$$

In the classical (Euclidean) context, homogeneity of the energy density means that its dependence on the infinitesimal deformation does not depend on position. In a Riemannian setting, such a statement is problematic since there is no canonical identification of the tangent spaces at different points. A natural generalization of homogeneity is invariance under parallel transport. Note however that parallel transport is dependent on the trajectory between the end points, and therefore homogeneity requires an invariance that is independent on the trajectory. In a coordinate system, homogeneity means that the spatial dependence of W is only through the entries $\mathfrak{g}_{ij}$ of the Riemannian metric. The prototypical energy density (2.4) is an example of such density.

Homogeneity over fibers is a weaker property, which can be defined for tubular neighborhoods of a submanifold. It implies invariance under parallel transport along normal geodesics, while allowing inhomogeneity in the "spatial" directions. In the particular case of a Euclidean metric, homogeneity over fibers means that the energy density does not depend explicitly on the normal coordinate. As such, it is not an intrinsic material property, however it is a sufficient condition for our purposes.

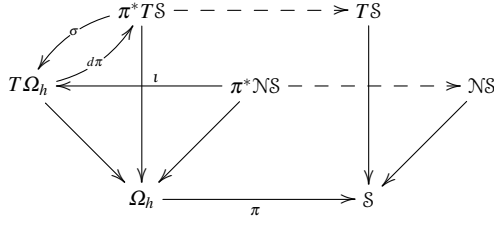An immediate consequence of homogeneity over fibers is the following:

Fig. 2 Commutative diagram for σ, ι and $d\pi$.

Lemma 3.1 *For every $q \in T^*\mathcal{M}|_{\mathcal{S}} \otimes \mathbb{R}^n$,*

$$W|_{\mathcal{S}}(q) \geq W_0(q \circ \iota^{\|}),$$

or equivalently,

$$W|_{\mathcal{S}} \geq W_0 \circ \iota^{\|^*}.$$

Moreover, if W is homogenous over fibers, then for every $q \in T^*_\xi \Omega_h \otimes \mathbb{R}^n$,

$$W_\xi(q) \geq (W_0)_{\pi(\xi)}(q \circ \Pi_\xi \circ \Pi^{-1}_{\pi(\xi)} \circ \iota^{\|}),$$

or equivalently,

$$W \geq \pi^*(W_0 \circ \iota^{\|^*}) \circ \Pi^*.$$

Proof: From the definition of W_0, for every $q \in T^*\mathcal{M}|_{\mathcal{S}} \otimes \mathbb{R}^n$,

$$W|_{\mathcal{S}}(q) \geq W_0(q \circ \iota^{\|}) = (W_0 \circ \iota^{\|^*})(q).$$

The second part of the lemma is immediate from this inequality and the definition of homogeneity over fibers. ∎

3.1.5 Approximating Π and $\mathfrak{g}$

We now construct another bundle isomorphism

$$\sigma \oplus \iota : \pi^*T\mathcal{S} \oplus \pi^*\mathcal{N}\mathcal{S} \cong \pi^*T\mathcal{M}|_{\mathcal{S}} \to T\Omega_h$$

(see Figure 2) that satisfies $d\pi \circ \iota = 0$, $d\pi \circ \sigma = \text{Id}$ and $\Pi - \sigma \oplus \iota = O(h)$; the relation between $\sigma \oplus \iota$ and π makes it simpler to analyze than Π, and our assumptions on W will imply that W is "almost" homogenous over fibers with respect to the parallel-transport-like map $\sigma \oplus \iota$. We will use this bundle isomorphism to define another metric, $\tilde{\mathfrak{g}}$, on Ω_h, such that $\sigma \oplus \iota$ is its parallel transport. The metric $\tilde{\mathfrak{g}}$ approximates $\mathfrak{g}$ in a sense that will be made precise. We will then repeatedly switch between the two metrics, thus exploiting the simpler structure of $\tilde{\mathfrak{g}}$.

Let $\iota : \pi^*\mathcal{N}\mathcal{S} \hookrightarrow T\mathcal{N}\mathcal{S}$ denote the canonical identification of the vector bundle $\mathcal{N}\mathcal{S}$ with its own vertical tangent space. Explicitly, for $\xi \in \mathcal{N}\mathcal{S}$ and $\eta \in (\pi^*\mathcal{N}\mathcal{S})_\xi$, there is

a canonical identification of η with an element of $(\mathcal{N}\mathcal{S})_{\pi(\xi)}$. We then define a curve $\gamma : I \to \mathcal{N}\mathcal{S}$,

$$\gamma(t) = \xi + \eta\, t,$$

and identify $\iota_\xi(\eta) = \dot{\gamma}(0)$. Clearly $d\pi : T\mathcal{N}\mathcal{S} \to \pi^*T\mathcal{S}$ and ι satisfy:

$$d\pi \circ \iota = 0.$$

To fully determine an isomorphism $T\mathcal{N}\mathcal{S} \cong \pi^*T\mathcal{S} \oplus \pi^*\mathcal{N}\mathcal{S}$ we need a map

$$\sigma : \pi^*T\mathcal{S} \to T\mathcal{N}\mathcal{S},$$

such that

$$d\pi \circ \sigma = \mathrm{Id}.$$

To this end we use the induced connection on $\mathcal{N}\mathcal{S}$. Define σ to be the unique map such that for any curve $\alpha : I \to S$, and any parallel normal field $\gamma \to \mathcal{N}\mathcal{S}$ along α, we have

$$\sigma_{\gamma(0)}(\dot{\alpha}) = \dot{\gamma}.$$

In other words, given $\dot{\alpha} \in T\mathcal{S}$ and $\xi \in \mathcal{N}\mathcal{S}_{\pi(\dot{\alpha})}$, $\sigma_\xi(\dot{\alpha})$ is the equivalence class of a curve $\gamma : I \to \mathcal{N}\mathcal{S}$ along α, that satisfies

$$\gamma(0) = \xi \qquad \text{and} \qquad \nabla^\perp_{\dot{\alpha}}\dot{\gamma} = 0.$$

Since γ is a curve along α, that is, $\pi \circ \gamma = \alpha$, we have, by differentiation, that indeed $d\pi \circ \sigma = \mathrm{Id}$.

Note that we defined the range of $\sigma \oplus \iota : \pi^*T\mathcal{M}|_{\mathcal{S}} \to T\mathcal{N}\mathcal{S}$ to be the total bundle $T\mathcal{N}\mathcal{S}$, which means that π is viewed as a projection $\pi : \mathcal{N}\mathcal{S} \to S$. Restricting π to Ω_h, we may view $\sigma \oplus \iota$ as a mapping $\pi^*T\mathcal{M}|_{\mathcal{S}} \to T\Omega_h$, similar to Π.

The following lemmas are concerned with the deviation of $\sigma \oplus \iota$ from Π and its consequences:

Lemma 3.2 *The restrictions of $\sigma \oplus \iota$ and Π to bundle maps over Ω_h (i.e. when π is viewed as a mapping $\Omega_h \to \mathcal{S}$) satisfy*

$$\sigma \oplus \iota - \Pi = O(h).$$

That is, there exists $C > 0$, independent of h, such that for every $v \in T\mathcal{M}|_{\mathcal{S}}$ and $\xi \in (\Omega_h)_{\pi(v)}$,

$$|(\sigma \oplus \iota - \Pi)_\xi(v)| \le Ch|v|$$

Proof: See Lemma 3.1 in [20]. ∎

An important corollary of Lemma 3.2 is that W is almost homogeneous over fibers with respect to $\sigma \oplus \iota$:

Corollary 3.1 *There exists $C > 0$ such that for every $q \in T^*\Omega_h \otimes \mathbb{R}^n$,*

$$|W(q) - \pi^*W|_{\mathcal{S}} \circ (\sigma \oplus \iota)^*(q)| \le Ch(1 + |q|^p).$$

Proof: By the homogeneity over fibers and the Lipschitz property of W:

$$|W(q) - \pi^*W|_{\mathcal{S}} \circ (\sigma \oplus \iota)^*(q)| = |\pi^*W|_{\mathcal{S}} \circ \Pi^*(q) - \pi^*W|_{\mathcal{S}} \circ (\sigma \oplus \iota)^*(q)|$$
$$= |\pi^*W|_{\mathcal{S}} (q \circ \Pi) - \pi^*W|_{\mathcal{S}} \circ (q \circ (\sigma \oplus \iota))|$$
$$\leq C(1 + |q \circ \Pi|^{p-1} + |q \circ (\sigma \oplus \iota)|^{p-1})|q \circ (\Pi - \sigma \oplus \iota)|$$
$$\leq C(1 + |q|^{p-1})|q| \cdot h \leq Ch(1 + |q|^p).$$

$\blacksquare$

Let $\tilde{\mathfrak{g}}$ denote the unique metric on Ω_h such that $\tilde{\mathfrak{g}}|_{\mathcal{S}} = \mathfrak{g}|_{\mathcal{S}}$ and $\sigma \oplus \iota$ is an isometry. The following corollary follows immediately from Lemma 3.2 (see [20] for details).

Corollary 3.2 *1.*
$$\tilde{\mathfrak{g}} - \mathfrak{g} = O(h),$$

that is, $|\tilde{\mathfrak{g}}(u,v) - \mathfrak{g}(u,v)| \leq Ch|u||v|$ for every $u,v \in T\Omega_h$.

2.
$$dvol_{\tilde{\mathfrak{g}}} - dvol_{\mathfrak{g}} = O(h).$$

3. For small enough h, the L^p (resp. $W^{1,p}$) norm on $(\Omega_h, \mathfrak{g})$ is equivalent to the L^p (resp. $W^{1,p}$) norm on $(\Omega_h, \tilde{\mathfrak{g}})$.

We now state some further properties of the metric $\tilde{\mathfrak{g}}$. We show that $dvol_{\tilde{\mathfrak{g}}}$ decomposes into a product $\eta \wedge \omega$ where η is related to the volume form on $\mathcal{S}$ and ω is a k-form. This decomposition will allow us to use repeatedly Fubini's theorem. Moreover, it will be shown to satisfy nice properties upon the rescaling of the tubular neighborhoods Ω_h. The definitions are given below; for full details and proofs see [20].

Let $E, F \to M$ be vector bundles and let $\chi : E \to F$ be a morphism of vector bundles. Denote by $\Lambda^a \chi : \Lambda^a E \to \Lambda^a F$ the associated vector bundle morphism between the ath exterior powers of E and F. Write

$$\rho = (\sigma \oplus \iota)^{-1*} \circ (\pi^* P^{\|})^* : \pi^* T^* \mathcal{S} \to T^* \Omega_h$$
$$\theta = (\sigma \oplus \iota)^{-1*} \circ (\pi^* P^{\perp})^* : \pi^* \mathcal{N} \mathcal{S}^* \to T^* \Omega_h.$$

Note that

$$\sigma^* \circ \rho = \text{Id}, \qquad \iota^* \circ \theta = \text{Id}, \qquad \sigma^* \circ \theta = 0, \qquad \iota^* \circ \rho = 0. \qquad (3.1)$$

Moreover, equations (3.1) uniquely characterize ρ and θ. Taking exterior powers, we have

$$\Lambda^i \rho : \Lambda^i \pi^* T^* \mathcal{S} \to \Lambda^i T^* \Omega_h, \qquad \Lambda^j \theta : \Lambda^j \pi^* \mathcal{N} \mathcal{S}^* \to \Lambda^j T^* \Omega_h,$$

and equations (3.1) then imply that

$$\bigoplus_{i+j=l} \Lambda^i \rho \wedge \Lambda^j \theta = \Lambda^l (\sigma \oplus \iota)^{-1*}.$$

Let $\tilde{\eta}$ be the unit norm section of $\Lambda^{n-k} T^* \mathcal{S}$ belonging to the orientation class, i.e. $\tilde{\eta} = dvol_{\mathfrak{g}|_{\mathcal{S}}}$. Let $\tilde{\omega}$ be the unit norm section of $\Lambda^k \mathcal{N} \mathcal{S}^*$ belonging to the orientation class determined by the orientations of $\mathcal{M}$ and $\mathcal{S}$. Define

$$\eta = \Lambda^{n-k} \rho \circ \pi^* \tilde{\eta}, \qquad \omega = \Lambda^k \theta \circ \pi^* \tilde{\omega}.$$

In particular, $\eta \in A^{n-k}(\Omega_h)$ and $\omega \in A^k(\Omega_h)$. It follows from the definition that

$$\eta \wedge \omega = d\mathrm{vol}_{\tilde{\mathfrak{g}}}. \tag{3.2}$$

Lemma 3.3 (properties of η and ω) *Denote by $\pi^\star$ and $\pi_\star$ the pullback and push-forward of forms on $\mathcal{S}$ and on Ω_h. Then*

$$\eta = \pi^\star d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} \qquad and \qquad \pi_\star(\omega) = v_k h^k,$$

where v_k is the volume of the k-dimensional unit ball.

Lemma 3.4 *We have*

$$\frac{\pi_\star d\mathrm{vol}_{\mathfrak{g}}}{|\Omega_h|} - \frac{d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}}{|\mathcal{S}|} = O(h), \qquad |\mathcal{S}|v_k h^k - |\Omega_h| = O(h^{1+k}). \tag{3.3}$$

3.1.6 Rescaling of tubular neighborhoods

There is a natural notion of strong convergence for functions defined over shrinking tubular neighborhoods; see next subsection. The notion of weak convergence turns out to be more subtle. To define it we will need to rescale the tubular neighborhoods (for more details and proofs, see [20]).

Define the rescaling operator $\mu_h : \Omega_{h_0} \to \Omega_{h_0 h}$ by

$$\mu_h(\xi) = h\xi.$$

Clearly $\pi \circ \mu_h = \pi$. We assume that h_0 is small enough such that part 3 in Corollary 3.2 holds.

Lemma 3.5 *1. For every $\xi \in \Omega_{h_0}$,*

$$d\mu_h \circ \sigma_\xi = \sigma_{h\xi}, \qquad d\mu_h \circ \iota_\xi = h\iota_{h\xi}.$$

2.

$$\mu_h^\star \omega = h^k \omega, \qquad \mu_h^\star \eta = \eta.$$

3. Let $f \in L^1(\Omega_{h_0 h})$. Then

$$\fint_{\Omega_{h_0 h}} f \, d\mathrm{vol}_{\mathfrak{g}} = \frac{1 + O(h)}{v_k h_0^k |\mathcal{S}|} \int_{\Omega_{h_0}} (f \circ \mu_h) \, \eta \wedge \omega.$$

3.2 Convergence in tubular neighborhoods

We start by defining strong convergence over shrinking domains:

Definition 3.2 Let $f_h \in L^p(\Omega_h; \mathbb{R}^n)$ and $F \in L^p(\mathcal{S}; \mathbb{R}^n)$. We say that $f_h \to F$ in the strong L^p topology if

$$\lim_{h \to 0} \fint_{\Omega_h} |f_h - F \circ \pi|^p \, d\mathrm{vol}_{\mathfrak{g}} = 0.$$

In other words, defining

$$\|f_h\|^p_{L^p(\Omega_h; \mathbb{R}^n)} = \fint_{\Omega_h} |f_h|^p \, d\mathrm{vol}_{\mathfrak{g}},$$

$f_h \to F$ if for every $\varepsilon > 0$

$$f_h \in B_\varepsilon(F \circ \pi)$$

for every small enough h.

We now discuss the relations between both strong and weak convergence in tubular neighborhoods and "standard" convergence of mappings rescaled to mappings over Ω_{h_0}. The following lemma establishes the equivalence between both notions for strong convergence:

Lemma 3.6 *Let $f_h \in L^p(\Omega_{h_0 h}; \mathbb{R}^n)$ and $F \in L^p(\mathcal{S}; \mathbb{R}^n)$. Then, $f_h \to F$ in L^p if and only if $f_h \circ \mu_h \to F \circ \pi$ in the strong $L^p(\Omega_{h_0}; \mathbb{R}^n)$ topology.*

Proof: It follows from Lemma 3.5 and the relation $\pi \circ \mu_h = \pi$ that

$$\fint_{\Omega_{h_0 h}} |f_h - F \circ \pi|^p \, d\mathrm{vol}_{\mathfrak{g}} = \frac{1 + O(h)}{v_k h_0^k |\mathcal{S}|} \int_{\Omega_{h_0}} (|f_h - F \circ \pi|^p \circ \mu_h) \, \omega \wedge \eta =$$

$$= \frac{1 + O(h)}{v_k h_0^k |\mathcal{S}|} \int_{\Omega_{h_0}} |f_h \circ \mu_h - F \circ \pi|^p \, \omega \wedge \eta.$$

Hence $f_h \to F$ if and only if the function $f_h \circ \mu_h \to F \circ \pi$ in $L^p(\Omega_{h_0}; \mathbb{R}^n)$ with respect to the metric $\tilde{\mathfrak{g}}$. Since L^p convergence in $(\Omega_{h_0}, \mathfrak{g})$ is equivalent to L^p convergence in $(\Omega_{h_0}, \tilde{\mathfrak{g}})$ (Corollary 3.2), the proof is complete. ∎

To address weak convergence we first need a technical lemma, which basically states that if the normal derivative of a mapping is zero, than the mapping does not depend on the normal coordinate.

Lemma 3.7 *Suppose that $f \in W^{1,p}(\Omega_h; \mathbb{R}^n)$ satisfies $df \circ \iota = 0$. Then, there exists an $F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n)$ such that $f = F \circ \pi$.*

Proof: Let $p \in \mathcal{S}$ be given and let $\gamma : I \to \pi^{-1}(p)$ be a curve in the fiber of Ω_h above p. Since $\pi \circ \gamma = p$, it follows that

$$d\pi(\dot{\gamma}) = 0.$$

Since $d\pi \circ \iota = 0$ and $\dim \operatorname{Im}\iota = \dim \ker(d\pi) = k$, we have that $\operatorname{Im}\iota = \ker(d\pi)$, therefore $\dot{\gamma} \in \operatorname{Im}\iota$, and since $df \circ \iota$ is follows that

$$df(\dot{\gamma}) = 0,$$

hence $f \circ \gamma = \text{const}$, and there exists an $F : \mathcal{S} \to \mathbb{R}^n$ such that $f = F \circ \pi$.

Note that F can be expressed as

$$F = \frac{\pi_\star(F \circ \pi \, \omega)}{\pi_\star(\omega)} = \frac{\pi_\star(f \, \omega)}{\pi_\star(\omega)} = \frac{1}{\nu_k h^k} \pi_\star(f \, \omega),$$

hence $F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n)$. ∎

The following lemma generalizes to sequences over tubular neighborhoods the classical fact that bounded sequences in a reflexive Banach space have a weakly compact subsequence.

Lemma 3.8 *Suppose that $f_h \in W^{1,p}(\Omega_{h_0 h}; \mathbb{R}^n)$ is a uniformly bounded sequence (each f_h with its respective volume-averaged norm). Then there exists a $F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n)$ and a subsequence f_{h_n} such that $f_{h_n} \circ \mu_{h_n} \rightharpoonup F \circ \pi$ in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$. In particular, $f_{h_n} \to F$ in the strong L^p topology.*

Proof: Part 3 of Lemma 3.5 implies that a sequence $y_h \in L^p(\Omega_{h_0 h})$ is uniformly bounded if and only if the sequence $y_h \circ \mu_h$ is uniformly bounded in $L^p(\Omega_{h_0})$. Therefore the boundedness of f_h in $W^{1,p}(\Omega_{h_0 h}; \mathbb{R}^n)$ implies that $f_h \circ \mu_h$ is uniformly bounded in $L^p(\Omega_{h_0}; \mathbb{R}^n)$ and that $\int_{\Omega_{h_0}} |df_h|^p \circ \mu_h \, \eta \wedge \omega$ is uniformly bounded.

For $f \in W^{1,p}(\Omega_{h_0 h}; \mathbb{R}^n)$ and $\xi \in \Omega_{h_0}$,

$$|df|^2_{\tilde{\mathfrak{g}}} \circ \mu_h(\xi) = \tilde{\mathfrak{g}}_{h\xi}(d_{h\xi}f, d_{h\xi}f) = \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (\sigma \oplus \iota)_{h\xi}, d_{h\xi}f \circ (\sigma \oplus \iota)_{h\xi}) =$$
$$= \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}, d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}) +$$
$$+ \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}, d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}).$$

The last equation follows from the definition of the inner product on the cotangent bundle and the fact that $(d_{h\xi}f \circ (0 \oplus \iota)_{h\xi})^\sharp \in N\mathcal{S}$ and that $(d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi})^\sharp \in T\mathcal{S}$. Since $\int_{\Omega_{h_0}} |df_h|^p \circ \mu_h \, \eta \wedge \omega$ is uniformly bounded, it follows that

$$\int_{\Omega_{h_0}} (\mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}, d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}))^{p/2} \eta \wedge \omega(\xi),$$

and

$$\int_{\Omega_{h_0}} (\mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}, d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}))^{p/2} \eta \wedge \omega(\xi)$$

are also uniformly bounded. On the other hand, part 1 of Lemma 3.5 implies that

$$|d(f \circ \mu_h)|^2_{\tilde{\mathfrak{g}}}(\xi) = \tilde{\mathfrak{g}}_\xi(d_{h\xi}f \circ d_\xi\mu, d_{h\xi}f \circ d_\xi\mu) =$$
$$= \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ d_\xi\mu \circ (\sigma \oplus \iota)_\xi, d_{h\xi}f \circ d_\xi\mu \circ (\sigma \oplus \iota)_\xi) =$$
$$= \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (\sigma \oplus h\iota)_{h\xi}, d_{h\xi}f \circ (\sigma \oplus h\iota)_{h\xi}) = \qquad (3.4)$$
$$= \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}, d_{h\xi}f \circ (\sigma \oplus 0)_{h\xi}) +$$
$$+ h^2 \mathfrak{g}_{\pi(\xi)}(d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}, d_{h\xi}f \circ (0 \oplus \iota)_{h\xi}),$$

and therefore $\int_{\Omega_{h_0}} |d(f_h \circ \mu_h)|_{\tilde{\mathfrak{g}}}^p \, \eta \wedge \omega$ is uniformly bounded, hence $f_h \circ \mu_h$ is a bounded sequence in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$. It follows that $f_h \circ \mu_h$ has a subsequence weakly convergent in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$ (recall that by Corollary 3.2 convergence with respect to $\mathfrak{g}$ is equivalent to convergence with respect to $\tilde{\mathfrak{g}}$); denote the limit by f. Equation (3.4) implies that $\int_{\Omega_{h_0}} |d(f_h \circ \mu_h) \circ \iota|^p \eta \wedge \omega = O(h^p)$, hence $df \circ \iota = 0$ a.e.

Applying Lemma 3.7, there exists an $F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n)$ such that $f = F \circ \pi$. Therefore, $f_{h_n} \circ \mu_{h_n} \rightharpoonup F \circ \pi$ in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$, and in particular $f_{h_n} \circ \mu_{h_n} \to F \circ \pi$ in $L^p(\Omega_{h_0}; \mathbb{R}^n)$. By Lemma 3.6, $f_{h_n} \to F$ in the strong L^p topology. ∎

3.3 Γ-convergence

The main theorem of this paper is concerned with Γ-convergence of functionals $I_h : L^p(\Omega_h; \mathbb{R}^n) \to \bar{\mathbb{R}}$ to a functional $I : L^p(\mathcal{S}; \mathbb{R}^n) \to \bar{\mathbb{R}}$. Since the standard definition of Γ-convergence requires the functionals to be defined on the same space, we need a definition of Γ-convergence over shrinking tubular neighborhoods, and establish its properties. The proof of properties satisfied by Γ-convergence over shrinking domains is essentially the same as for fixed domains, and will therefore be omitted; see [10] for details.

Definition 3.3 Let $I_h : L^p(\Omega_h; \mathbb{R}^n) \to \bar{\mathbb{R}}$ and $I : L^p(\mathcal{S}; \mathbb{R}^n) \to \bar{\mathbb{R}}$. We will say that I_h Γ-converges to $\tilde{I}$ in the strong L^p topology if

① Lower-semicontinuity: for every sequence $f_h \to F$,

$$I(F) \leq \liminf_{h \to 0} I_h(f_h).$$

② Recovery sequence: for every F there exists a sequence $f_h \to F$ such that

$$I(F) = \lim_{h \to 0} I_h(f_h).$$

An equivalent topological definition is given by the following lemma:

Lemma 3.9 $I_h \xrightarrow{\Gamma} I$ if and only if for every F,

$$\lim_{\varepsilon \to 0} \liminf_{h \to 0} \inf_{B_\varepsilon(F \circ \pi)} I_h(\cdot) = \lim_{\varepsilon \to 0} \limsup_{h \to 0} \inf_{B_\varepsilon(F \circ \pi)} I_h(\cdot) = I(F).$$

Proposition 3.1 (Urysohn's property) *If for every sequence $h_n \to 0$ there exists a subsequence h_{n_k} such that*

$$I_{h_{n_k}} \xrightarrow{\Gamma} I,$$

then $I_h \xrightarrow{\Gamma} I$.

Proposition 3.2 (Sequential compactness) *Every sequence of functionals $I_h : L^p(\Omega_h; \mathbb{R}^n) \to \bar{\mathbb{R}}$ has a Γ-converging partial limit $I : L^p(S; \mathbb{R}^n) \to \bar{\mathbb{R}}$.*

Propositions 3.1 and 3.2 immediately imply the following corollary:

Corollary 3.3 *$I_h \xrightarrow{\Gamma} I$ if and only if I is the limit of every Γ-convergent subsequence.*

3.4 Quasiconvexity

Definition 3.4 Let $(\mathcal{M}, \mathfrak{g})$ be a Riemannian manifold, and let $U : T^*\mathcal{M} \otimes \mathbb{R}^m \to \mathbb{R}$ be a fiber-wise locally integrable function. We say that U is **quasiconvex** if for every $p \in \mathcal{M}$, every $A \in T_p^*\mathcal{M} \otimes \mathbb{R}^m$, every open bounded set $D_p \subset T_p\mathcal{M}$, and every $\varphi \in C_0^\infty(D_p; \mathbb{R}^m)$,

$$U(A) \leq \fint_{D_p} U(A + d\phi \circ \kappa) \, \omega_p, \tag{3.5}$$

where $\kappa : \pi^* T_p\mathcal{M} \to TT_p\mathcal{M}$ is the canonical identification, and ω_p is the n-form

$$\omega_p = d\mathrm{vol}_\mathfrak{g}|_p \circ \Lambda^n \kappa^{-1}.$$

The integrand in (3.5) reads as follows: for $\xi \in D_p \subset T_p\mathcal{M}$ and $\eta \in T_p\mathcal{M}$,

$$[d\phi \circ \kappa(\xi)](\eta) = d_\xi \phi \circ \kappa_\xi(\eta) = d_\xi \phi([\xi + \eta t]) \in \mathbb{R}^m,$$

hence $U(A + d\phi \circ \kappa)$ is indeed a map from D_p to $\mathbb{R}^m$. As for the volume form ω_p, choosing a coordinate system $(x_1, \ldots, x_n)$ in $\mathcal{M}$, and denoting by $(\partial_1, \ldots, \partial_n)$ the corresponding coordinates on $T_p\mathcal{M}$, we have that ω_p is of the form

$$\omega_p = \sqrt{\det \mathfrak{g}_{ij}(p)} \, d_p x_1 \circ \kappa^{-1} \wedge \cdots \wedge d_p x_n \circ \kappa^{-1} = \sqrt{\det \mathfrak{g}_{ij}(p)} \, d\partial_1 \wedge \cdots \wedge d\partial_n.$$

The constant $\sqrt{\det \mathfrak{g}_{ij}(p)}$ cancels upon volume average, and therefore the above definition reduces, after choosing coordinates, to the classical definition of quasiconvexity, see e.g. [1] or [9] (it also shows that the definition is independent of the metric).

In the proof of the main theorem we will need the following two relations between quasiconvexity and lower-semicontinuity, which are extensions of classical results to the Riemannian setting (see [1]); the proofs are in Appendix B. In these theorems, we assume that $(\mathcal{M}, \mathfrak{g})$ is a Riemannian manifold of finite volume that can be covered by a finite number of charts (note that $\mathcal{S}$ and therefore Ω_h, satisfy this condition), and that $U : T^*\mathcal{M} \otimes \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function (see Appendix A) that satisfies the growth condition

$$-\beta \leq U(q) \leq C(1 + |q|^p) \tag{3.6}$$

for some $\beta, C > 0$ and $p \in (1, \infty)$. Both theorems also hold also if $W^{1,p}$ is replaced with $W_0^{1,p}$.

Theorem 3.1 *Under the above conditions, the functional $I_A : W^{1,p}(A; \mathbb{R}^m) \to \mathbb{R}$ defined by*

$$I_A(f) := \int_A U(df) \, dvol_\mathfrak{g},$$

where $A \subset \mathcal{M}$ is an open subset, is weakly sequential lower-semicontinuous for every A if and only if U is quasiconvex.

Theorem 3.2 *Under the above conditions, the weakly sequential lower-semicontinuous envelope of the functional $I_{\mathcal{M}} : W^{1,p}(\mathcal{M};\mathbb{R}^m) \to \mathbb{R}$ (as defined in the previous theorem) is $\Gamma I_{\mathcal{M}} : W^{1,p}(\mathcal{M};\mathbb{R}^m) \to \mathbb{R}$ given by*

$$\Gamma I_{\mathcal{M}}(f) := \int_{\mathcal{M}} QU(df)\, dvol_{\mathfrak{g}},$$

where $QU(q) = \sup\{V(q) : V \leq U \text{ is quasiconvex}\}$ is the quasiconvex envelope of U; moreover, QU is a Carathéodory quasiconvex function.

We now show that Theorem 3.2 applies to W_0:

Lemma 3.10 *W_0 is continuous (and in particular Carathéodory) and satisfies the same growth and coercivity conditions as W (possibly with different constants).*

Proof: The proof of the growth and coercivity condition is the same as in Proposition 1 in [21].

To prove the continuity, we prove that W_0 is both lower- and upper-semicontinuous. Let $q, q_i \in T^*\mathcal{S} \otimes \mathbb{R}^n$ such that $q_i \to q$, and let $r_i \in \mathcal{NS}^*_{\pi(q_i)} \otimes \mathbb{R}^n$ such that $W_0(q_i) = W|_{\mathcal{S}}(q_i \oplus r_i)$. Let q_i be a subsequence (not relabeled) such that $W_0(q_i)$ converges. The growth property of W_0 and the coercivity property of W imply that

$$\alpha |r_i|^p - \beta \leq W|_{\mathcal{S}}(q_i \oplus r_i) = W_0(q_i) \leq C(1 + |q_i|^p),$$

and since $|q_i|$ is a bounded sequence, so is $|r_i|$. Since $r_i \in (\mathcal{NS}^* \otimes \mathbb{R}^n)_{\pi(q_i)}$ and $\pi(q_i) \to \pi(q)$, there is a convergent subsequence $r_{i_k} \to r \in (\mathcal{NS}^* \otimes \mathbb{R}^n)_{\pi(q)}$. Therefore,

$$\lim_{i \to \infty} W_0(q_i) = \lim_{k \to \infty} W|_{\mathcal{S}}(q_{i_k} \oplus r_{i_k}) \to W|_{\mathcal{S}}(q \oplus r) \geq W_0(q).$$

Since this holds for every convergent subsequence of the original sequence $W_0(q_i)$, W_0 is lower-semicontinuous.

To prove upper-semicontinuity, let $q, q_i \in T^*\mathcal{S} \otimes \mathbb{R}^n$ such that $q_i \to q$, let $r \in \mathcal{NS}^*_{\pi(q)} \otimes \mathbb{R}^n$ such that $W_0(q) = W|_{\mathcal{S}}(q \oplus r)$, and let ρ be a section of $\mathcal{NS}^* \otimes \mathbb{R}^n$ such that $\rho(\pi(q)) = r$. Then, by the continuity of W,

$$W_0(q) = W|_{\mathcal{S}}(q \oplus r) = \lim_{i \to \infty} W|_{\mathcal{S}}(q_i \oplus \rho(\pi(q_i))) \geq \limsup_{i \to \infty} W_0(q_i),$$

hence W_0 is upper-semicontinuous.   ∎

It follows that Theorems 3.1 applies to QW_0:

Corollary 3.4 *QW_0 is a Carathéodory function and satisfies* (3.6).

Proof: From Lemma 3.10, $QW_0(q) \leq W_0(q) \leq C(1 + |q|^p)$ for some $C > 0$. Also, observe that the constant function $-\beta$ is a quasiconvex function not larger than W_0, hence $QW_0 \geq -\beta$. Lemma 3.10 also implies that Theorem 3.2 applies to W_0, and therefore QW_0 is a Carathéodory function.   ∎

4 Proof of the main results

We restate our main Theorem:

The sequence of functionals $(I_h)_{h \le h_0}$ define by (2.5) Γ-converges to $I : L^p(\mathcal{S}; \mathbb{R}^n) \to \bar{\mathbb{R}}$ defined by:

$$I(F) = \begin{cases} \fint_{\mathcal{S}} QW_0(dF)\, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} & F \in W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n), \\ \infty & \text{otherwise.} \end{cases}$$

To prove this theorem, we use Corollary 3.3. That is, we prove that I is the limit of every Γ-converging subsequence of I_h. Explicitly, we assume that I_h is a (not relabeled) Γ-convergent subsequence with limit $I_0 : L^p(\mathcal{S}; \mathbb{R}^n) \to \bar{\mathbb{R}}$, and show that $I_0 = I$. The proof is long hence we divide it into several steps: (1) if $F \notin W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n)$ then $I_0(F) = I(F)$; (2) if $F \in W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n)$ then $I_0(F) \ge I(F)$; and (3) if $F \in W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n)$ then $I_0(F) \le I(F)$. With a slight abuse of notation, we write Ω_h instead of $\Omega_{h_0 h}$ whenever rescaling arguments are concerned, as it does not cause confusion and makes the proof more readable.

Step 1: $I_0(F) = I(F)$ when F does not satisfy either the regularity or the boundary conditions

Proposition 4.1 *If $F \in L^p(\mathcal{S}; \mathbb{R}^n) \setminus W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n)$ then $I_0(F) = \infty = I(F)$.*

Proof: Let $F \in L^p(\mathcal{S}; \mathbb{R}^n)$ be such that $I_0(F) < \infty$; we will show that $F \in W^{1,p}_{bc}(\mathcal{S}; \mathbb{R}^n)$. Let $f_h \to F$ be a recovery sequence, namely,

$$I_h(f_h) \to I_0(F) < \infty.$$

We can assume that $I_h(f_h)$ is uniformly bounded by some constant $C < \infty$ for sufficiently small h, hence $f_h \in W^{1,p}_{bc}(\Omega_h; \mathbb{R}^n)$. Since $f_h \to F$ in L^p, it follows that $\|f_h\|_{L^p}$ is bounded uniformly in h. From the coercivity of W, we have

$$C \ge I_h(f_h) = \fint_{\Omega_h} W(df_h)\, d\mathrm{vol}_{\mathfrak{g}} \ge \alpha \fint_{\Omega_h} |df_h|^p\, d\mathrm{vol}_{\mathfrak{g}} - \beta,$$

hence $\|df_h\|_{L^p}$ is also uniformly bounded, hence f_h is uniformly bounded in $W^{1,p}_{bc}(\Omega_h; \mathbb{R}^n)$. By Lemma 3.8, there is a (not relabeled) subsequence such that $f_h \circ \mu_h \rightharpoonup F \circ \pi$ in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$. In particular, $F \circ \pi \in W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$, hence $F \in W^{1,p}(\mathcal{S}; \mathbb{R}^n)$.

It remains to show that $F|_{\partial \mathcal{S}} = F_{bc}$. Indeed, since $f_h \in W^{1,p}_{bc}(\Omega_h; \mathbb{R}^n)$, it is immediate that $f_h \circ \mu_h|_{\Gamma_{h_0}} = F_{bc} \circ \pi + h \pi^* \mathfrak{q}^{\perp}_{bc} \circ \lambda$. Therefore $f_h \circ \mu_h|_{\Gamma_{h_0}} \to F_{bc} \circ \pi$ uniformly in Γ_{h_0}. By the continuity of the trace operator as a mapping $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n) \to L^p(\Gamma_{h_0}; \mathbb{R}^n)$, we have that $F \circ \pi|_{\Gamma_{h_0}} = F_{bc} \circ \pi$, hence $F|_{\partial \mathcal{S}} = F_{bc}$. ∎

Step 2: $I_0(F) \geq I(F)$ when F satisfies both the regularity and the boundary conditions

Proposition 4.2 $I_0(F) \geq I(F)$ *for every* $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$.

Proof: Let $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$ be given. Let $f_h \to F$ be a recovery sequence, namely,

$$I_0(F) = \lim_{h \to 0} I_h(f_h).$$

If $I_0(F) = \infty$ then the claim is trivial. Otherwise, $I_h(f_h)$ is bounded for sufficiently small h, and therefore $f_h \in W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ and

$$I_h(f_h) = \fint_{\Omega_h} W(df_h) \, d\mathrm{vol}_{\mathfrak{g}}.$$

The coercivity of W implies that df_h is uniformly bounded in L^p, hence by the Poincaré inequality, f_h is uniformly bounded in $W^{1,p}(\Omega_h; \mathbb{R}^n)$ (note that we need here a version of the Poincaré inequality in which the function is prescribed on a subset of the boundary that has positive $(n-1)$-dimensional measure; see Theorem 6.1-8 in [8] for the Euclidean case; the non-Euclidean case is analogous).

By Lemma 3.8, Lemma 3.6 and the uniqueness of the limit,

$$f_h \circ \mu_h \rightharpoonup F \circ \pi \quad \text{in } W^{1,p}(\Omega_{h_0}; \mathbb{R}^n).$$

By Corollary 3.1, Lemma 3.1, and the definition of QW_0:

$$\begin{aligned}
I_h(f_h) &= \fint_{\Omega_h} W(df_h) \, d\mathrm{vol}_{\mathfrak{g}} \\
&= \fint_{\Omega_h} \pi^* W|_{\mathcal{S}} \circ (\sigma \oplus \iota)^* (df_h) \, d\mathrm{vol}_{\mathfrak{g}} + O(h) \\
&\geq \fint_{\Omega_h} \pi^* W_0 \circ \pi^* \iota^{\|^*} \circ (\sigma \oplus \iota)^* (df_h) \, d\mathrm{vol}_{\mathfrak{g}} + O(h) \\
&\geq \fint_{\Omega_h} \pi^* QW_0 \circ \pi^* \iota^{\|^*} \circ (\sigma \oplus \iota)^* (df_h) \, d\mathrm{vol}_{\mathfrak{g}} + O(h) \\
&= \fint_{\Omega_h} \pi^* QW_0 \circ \sigma^* (df_h) \, d\mathrm{vol}_{\mathfrak{g}} + O(h).
\end{aligned}$$

The growth condition of QW_0 implies that $\pi^* QW_0 \circ \sigma^*(df_h) \in L^1(\Omega_h)$, hence from the third part of Lemma 3.5 we have that

$$I_h(f_h) \geq \frac{1 + O(h)}{v_k h_0^k |\mathcal{S}|} \int_{\Omega_{h_0}} (\pi^* QW_0 \circ \sigma^*(df_h) \circ \mu_h) \, \eta \wedge \omega + O(h). \tag{4.1}$$

We next show that

$$\pi^* QW_0 \circ \sigma^*(df_h) \circ \mu_h = \pi^* QW_0 \circ \sigma^*(d(f_h \circ \mu_h)). \tag{4.2}$$

Indeed, on the one hand, for $\xi \in \Omega$:

$$\begin{aligned}
\pi^* QW_0 \circ \sigma^*(df_h) \circ \mu_h(\xi) &= \pi^* QW_0(df_h \circ \sigma)(h\xi) \\
&= (QW_0)_{\pi(h\xi)}(d_{h\xi} f_h \circ \sigma_{h\xi}),
\end{aligned}$$

whereas from the first part of Lemma 3.5.

$$\pi^*QW_0 \circ \sigma^*(d(f_h \circ \mu_h))(\xi) = (QW_0)_{\pi(\xi)}(d_\xi(f_h \circ \mu_h) \circ \sigma_\xi)$$
$$= (QW_0)_{\pi(\xi)}(d_{h\xi}f_h \circ d_\xi\mu_h \circ \sigma_\xi)$$
$$= (QW_0)_{\pi(\xi)}(d_{h\xi}f_h \circ \sigma_{h\xi}).$$

Substituting (4.2) into (4.1):

$$I_h(f_h) \geq \frac{1 + O(h)}{v_k h_0^k |\mathbb{S}|} \int_{\Omega_{h_0}} \pi^*QW_0 \circ \sigma^*(d(f_h \circ \mu_h))\, \eta \wedge \omega + O(h).$$

Since $I_0(F) = \lim_{h \to 0} I_h(f_h)$, it follows that

$$I_0(F) \geq \liminf_{h \to 0} J(f_h \circ \mu_h), \tag{4.3}$$

where $J : W^{1,p}(\Omega_{h_0}; \mathbb{R}^n) \to \mathbb{R}$ is defined by

$$J(f) := \frac{1}{v_k h_0^k |\mathbb{S}|} \int_{\Omega_{h_0}} \pi^*QW_0 \circ \sigma^*(df)\, \eta \wedge \omega.$$

Lemma 4.1 below shows that the quasiconvexity of QW_0 implies the quasiconvexity of $\pi^*QW_0 \circ \sigma^* : T^*\Omega_{h_0} \otimes \mathbb{R}^n \to \mathbb{R}$. By Corollary 3.4, QW_0 is a Carathéodory that satisfies (3.6), and therefore the same holds for $\pi^*QW_0 \circ \sigma^*$, hence, by Theorem 3.1, J is sequentially weak lower-semicontinuous in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$. Since $f_h \circ \mu_h \rightharpoonup F \circ \pi$ in $W^{1,p}(\Omega_{h_0}; \mathbb{R}^n)$, it follows that

$$I_0(F) \geq J(F \circ \pi).$$

It remain to calculate $J(F \circ \pi)$,

$$J(F \circ \pi) = \frac{1}{v_k h_0^k |\mathbb{S}|} \int_{\Omega_{h_0}} \pi^*QW_0 \circ \sigma^*(d(F \circ \pi))\, \eta \wedge \omega =$$
$$= \frac{1}{v_k h_0^k |\mathbb{S}|} \int_{\Omega_{h_0}} \pi^*QW_0(dF)\, \eta \wedge \omega =$$
$$= \frac{1}{v_k h_0^k |\mathbb{S}|} \int_{\Omega_{h_0}} \pi^\star(QW_0(dF)\, d\mathrm{vol}_{\mathfrak{g}|_\mathbb{S}}) \wedge \omega =$$
$$= \frac{1}{v_k h_0^k |\mathbb{S}|} \int_{\mathbb{S}} \pi_\star(\pi^\star(QW_0(dF)\, d\mathrm{vol}_{\mathfrak{g}|_\mathbb{S}}) \wedge \omega) =$$
$$= \fint QW_0(dF)\, d\mathrm{vol}_{\mathfrak{g}|_\mathbb{S}} = I(F).$$

where in the passage from the first to the second line we used the fact that $d\pi \circ \sigma = Id_{\pi^*T\mathbb{S}}$; in the passage from the second to the third line we used the identity $(\pi^*QW_0)(dF) = \pi^*(QW_0(dF))$, Lemma 3.3 and Fubini's theorem; in the passage form the third to the fourth line we used Fubini's theorem along with the definition of

the push-forward operator $\pi_\star$; in the passage from the fourth to the fifth line we used the property of the push-forward operator,

$$\pi_\star(\pi^\star\alpha \wedge \beta) = \alpha \wedge \pi_\star(\beta),$$

and Lemma 3.3. This concludes the proof. ∎

The following lemma, which is used in the proof of proposition 4.2, is a generalization of a property proved in [21] (part of the proof of Prop. 6). It states that a function that is quasiconvex on $\mathcal{S}$ can be extended to a quasiconvex function on Ω_h, via a combination of pullback and projection.

Lemma 4.1 *Let $U : T^*\mathcal{S} \otimes \mathbb{R}^n \to \mathbb{R}$ be quasiconvex. Then, $\pi^*U \circ \sigma^* : T^*\Omega_h \otimes \mathbb{R}^n \to \mathbb{R}$ is quasiconvex.*

Proof: Fix h, $\xi \in \Omega_h$ and $A \in T^*_\xi\Omega_h \otimes \mathbb{R}^n$ and let $D_\xi \subset T_\xi\Omega_h$ be some bounded set (which we will choose later). We need to prove that for every $\varphi \in C_0^\infty(D_\xi; \mathbb{R}^n)$,

$$(\pi^*U \circ \sigma^*)_\xi(A) \le \fint_{D_\xi} (\pi^*U \circ \sigma^*)_\xi(A + d\varphi \circ \kappa)\omega_\xi$$

where κ and ω_ξ are as in Definition 3.4.

Denote

$$T_\xi\Omega_h^\| := \mathrm{Im}\sigma_\xi = \sigma_\xi(T_{\pi(\xi)}\mathcal{S}),$$
$$T_\xi\Omega_h^\perp := \mathrm{Im}\iota_\xi = \iota_\xi(\mathcal{N}\mathcal{S}_{\pi(\xi)}).$$

Obviously, $T_\xi\Omega_h = T_\xi\Omega_h^\| \oplus T_\xi\Omega_h^\perp$, and denote by $P^\perp : T_\xi\Omega_h \to T_\xi\Omega_h^\perp$ and $P^\| : T_\xi\Omega_h \to T_\xi\Omega_h^\|$ the projections.

Observe that $\omega_\xi = \omega^\| \wedge \omega^\perp$, where $\omega^\|$ and $\omega^\perp$ are respectively, $(n-k)$ and k forms on $T_\xi\Omega_h^\|$ and $T_\xi\Omega_h^\perp$ (to simplify the notation, we identify $\omega^\|$ with $P^{\|\star}\omega^\|$ and $\omega^\perp$ with $P^{\perp\star}\omega^\perp$). Indeed, we can choose local coordinates on Ω_h at ξ such that the induced coordinates on $T_\xi\Omega_h$ are $(x_1,\ldots,x_{n-k},z_1,\ldots,z_k)$, where $(x_1,\ldots,x_{n-k})$ and $(z_1,\ldots,z_k)$ are bases for $T_\xi\Omega_h^\|$ and $T_\xi\Omega_h^\perp$, respectively. In these coordinates,

$$\omega_\xi = d\mathrm{vol}_\mathfrak{g}|_\xi \circ \Lambda^n\kappa^{-1} = a\,dx_1 \wedge \ldots \wedge dx_{n-k} \wedge dz_1 \wedge \ldots \wedge dz_k,$$

where a is some (constant) number. Now define $\omega^\| := a\,dx_1 \wedge \ldots \wedge dx_{n-k}$ and $\omega^\perp := dz_1 \wedge \ldots \wedge dz_k$.

Choose D_ξ such that $D_\xi = D^\| \times D^\perp$, where $D^\|$ and $D^\perp$ are bounded subsets of $T_\xi\Omega_h^\|$ and $T_\xi\Omega_h^\perp$, respectively. Restricting $P^\perp$ to D_ξ, we have

$$\int_{D_\xi} (\pi^*U \circ \sigma^*)_\xi(A + d\varphi \circ \kappa)\omega_\xi = \int_{D^\perp} P^\perp_\star((\pi^*U \circ \sigma^*)_\xi(A + d\varphi \circ \kappa)\omega^\| \wedge \omega^\perp)$$

$$= \int_{D^\perp} P^\perp_\star((\pi^*U \circ \sigma^*)_\xi(A + d\varphi \circ \kappa)\omega^\|) \wedge \omega^\perp$$

$$= \int_{D^\perp} P^\perp_\star((\pi^*U)_\xi(A \circ \sigma_\xi + d\varphi \circ \kappa \circ \sigma_\xi)\omega^\|) \wedge \omega^\perp.$$

$$(4.4)$$

We now analyze the integral $P^\perp{}_\star((\pi^*U)_\xi\,(A\circ\sigma_\xi+d\varphi\circ\kappa\circ\sigma_\xi)\omega^\|)$. Let (x,z) be a point in D_ξ, where $x\in D^\|$ and $z\in D^\perp$. Define $\varphi_z:D^\|\to\mathbb{R}^n$ by $\varphi_z(\cdot):=\varphi(\cdot,z)$. Let $\alpha\in D^\|$. We then have

$$d_{(x,z)}\varphi\circ\kappa_{(x,z)}(\alpha)=d_{(x,z)}\varphi([(x,z)+\alpha t])=\frac{d}{dt}\varphi((x,z)+\alpha t)$$

$$=\frac{d}{dt}\varphi_z(x+\alpha t)=d_x\varphi_z([x+\alpha t])=d_x\varphi_z\circ\kappa_x(\alpha),$$

which implies that

$$d_{(x,z)}\varphi\circ\kappa_{(x,z)}\circ\sigma_\xi=d_x\varphi_z\circ\kappa_x\circ\sigma_\xi, \tag{4.5}$$

since the image of σ_ξ is in $T_\xi\Omega_h^\|$. Since σ_ξ is a linear mapping,

$$d_y\sigma_\xi=\kappa_{\sigma_\xi(y)}\circ\sigma_\xi\circ(\kappa_y^{\mathcal{S}})^{-1},$$

where $y\in T_{\pi(\xi)}\mathcal{S}$ and $\kappa_y^{\mathcal{S}}:T_{\pi(\xi)}\mathcal{S}\to T_yT_{\pi(\xi)}\mathcal{S}$ is the canonical identification. Therefore, equation (4.5) implies that

$$d_{(x,z)}\varphi\circ\kappa_{(x,z)}\circ\sigma_\xi=d_{\sigma_\xi^{-1}(x)}\left(\varphi_z\circ\sigma_\xi\right)\circ\kappa_{\sigma_\xi^{-1}(x)}^{\mathcal{S}}.$$

Fixing z, we have that on the fiber $D^\|\times\{z\}$,

$$(\pi^*U)_\xi\,(A\circ\sigma_\xi+d\varphi\circ\kappa\circ\sigma_\xi)=(\sigma_\xi^{-1})^*\left(U_{\pi(\xi)}\left(A\circ\sigma_\xi+d\left(\varphi_z\circ\sigma_\xi\right)\circ\kappa^{\mathcal{S}}\right)\right).$$

Denoting by p the mapping $\sigma_\xi^{-1}(D^\|)\to\{z\}$, we then have, using the change of variables formula $P^\perp{}_\star\left(\sigma_\xi^{-1}\right)^\star=p_\star$,

$$P^\perp{}_\star((\pi^*U)_\xi\,(A\circ\sigma_\xi+d\varphi\circ\kappa\circ\sigma_\xi)\omega^\|)$$

$$=P^\perp{}_\star\left((\sigma_\xi^{-1})^*\left(U_{\pi(\xi)}\left(A\circ\sigma_\xi+d\left(\varphi_z\circ\sigma_\xi\right)\circ\kappa^{\mathcal{S}}\right)\right)(\sigma_\xi^{-1})^*(\sigma_\xi^{-1})_\star\omega^\|\right) \tag{4.6}$$

$$=p_\star\left(U_{\pi(\xi)}\left(A\circ\sigma_\xi+d\left(\varphi_z\circ\sigma_\xi\right)\circ\kappa^{\mathcal{S}}\right)(\sigma_\xi^{-1})_\star\omega^\|\right).$$

Since up to a constant, $(\sigma_\xi^{-1})_\star\omega^\|$ is just the form $d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}|_{\pi(\xi)}\circ\Lambda^{n-k}(\kappa^{\mathcal{S}})^{-1}$, the quasiconvexity of U implies that

$$p_\star\left(U_{\pi(\xi)}\left(A\circ\sigma_\xi+d\left(\varphi_z\circ\sigma_\xi\right)\circ\kappa^{\mathcal{S}}\right)(\sigma_\xi^{-1})_\star\omega^\|\right)$$
$$\geq U_{\pi(\xi)}\left(A\circ\sigma_\xi\right)p_\star\left((\sigma_\xi^{-1})_\star\omega^\|\right)=(\pi^*U\circ\sigma^*)_\xi\,(A)P^\perp{}_\star\left(\omega^\|\right), \tag{4.7}$$

where in the last step we used again the change of variables formula.

Combining equations (4.6)-(4.7), and inserting them into equation (4.4), we have

$$\int_{D_\xi}(\pi^*U\circ\sigma^*)_\xi\,(A+d\varphi\circ\kappa)\omega_\xi\geq(\pi^*U\circ\sigma^*)_\xi\,(A)\int_{D^\perp}P^\perp{}_\star\left(\omega^\|\right)\wedge\omega^\perp$$

$$=(\pi^*U\circ\sigma^*)_\xi\,(A)\int_{D_\xi}\omega_\xi,$$

which completes the proof. ∎

Step 3: $I_0(F) \leq I(F)$ when F satisfies both the regularity and the boundary conditions

Proposition 4.3 $I_0(F) \leq I(F)$ for every $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$.

Proof: Let $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$ and consider a sequence $f_h \in W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ defined by

$$f_h = F \circ \pi + \pi^* \mathfrak{q}^\perp \circ \lambda,$$

where $\mathfrak{q}^\perp \in \{r \in \Gamma(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n) : r|_{\partial \mathcal{S}} = \mathfrak{q}_{bc}^\perp\}$ (a simple argument using a partition of unity of $\mathcal{S}$ shows that this set is non-empty). That is, for every $\xi \in \Omega_h$,

$$f_h(\xi) = F(\pi(\xi)) + \mathfrak{q}_{\pi(\xi)}^\perp(\xi).$$

It is easy to see that $f_h \to F$ in L^p, hence by the lower-semicontinuity property of the Γ-limit:

$$I_0(F) \leq \liminf_{h \to 0} I_h(f_h) = \liminf_{h \to 0} \fint_{\Omega_h} W(df_h) \, d\mathrm{vol}_{\mathfrak{g}}. \tag{4.8}$$

From Proposition 6.2 in [20], we have that

$$|df_h \circ \Pi - \pi^*(dF \oplus \mathfrak{q}^\perp)| \leq Ch(1 + |df_h|).$$

It follows from the Lipschitz and the homogeneity over fibers properties of W that

$$\begin{aligned}
\fint_{\Omega_h} W(df_h) \, d\mathrm{vol}_{\mathfrak{g}} &= \fint_{\Omega_h} \pi^* W|_{\mathcal{S}} (df_h \circ \Pi) \, d\mathrm{vol}_{\mathfrak{g}} \\
&= \fint_{\Omega_h} \pi^* W|_{\mathcal{S}} (\pi^*(dF \oplus \mathfrak{q}^\perp)) \, d\mathrm{vol}_{\mathfrak{g}} + O(h) \\
&= \int_{\mathcal{S}} W|_{\mathcal{S}} (dF \oplus \mathfrak{q}^\perp) \frac{\pi_\star d\mathrm{vol}_{\mathfrak{g}}}{|\Omega_h|} + O(h),
\end{aligned}$$

where in the last step we used Fubini's theorem to first integrate over the fibers. Using Lemma 3.4 to evaluate $\pi_\star d\mathrm{vol}_{\mathfrak{g}}/|\Omega_h|$, Equation (4.8) reduces to

$$I_0(F) \leq \fint_{\mathcal{S}} W|_{\mathcal{S}} (dF \oplus \mathfrak{q}^\perp) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}.$$

This inequality holds for every $\mathfrak{q}^\perp \in \{r \in \Gamma(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n) : r|_{\partial \mathcal{S}} = \mathfrak{q}_{bc}^\perp\}$. Since $\{r \in \Gamma(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n) : r|_{\partial \mathcal{S}} = \mathfrak{q}_{bc}^\perp\}$ is dense in $L^p(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n)$, it follows from the Lipschitz property of W that

$$I_0(F) \leq \inf_{\mathfrak{q}^\perp \in L^p(\mathcal{S}; \mathcal{NS}^* \otimes \mathbb{R}^n)} \fint_{\mathcal{S}} W|_{\mathcal{S}} (dF \oplus \mathfrak{q}^\perp) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}. \tag{4.9}$$

By the definition of W_0, there exists a function $\mathfrak{q}_0^\perp : \mathcal{S} \to \mathcal{NS}^* \otimes \mathbb{R}^n$ such that

$$W|_{\mathcal{S}} (dF \oplus \mathfrak{q}_0^\perp) = W_0(dF). \tag{4.10}$$

Thus, it seems that we can bound the infimum on the right hand side of (4.9) with $\fint_{\mathcal{S}} W_0(dF) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}$. There is however one caveat: there is not a priori guarantee that

$\mathfrak{q}_0^\perp$ is measurable. Lemma 4.2 below proves that there exists $\mathfrak{q}_0^\perp \in L^p(\mathcal{S}; \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n)$ that satisfies (4.10), hence for every $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$:

$$I_0(F) \leq \fint_{\mathcal{S}} W_0(dF) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} \equiv G(F). \tag{4.11}$$

We introduce the following notation: for a function $H : W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n) \to \mathbb{R}$ we set $\widetilde{H} : L^p(\mathcal{S}; \mathbb{R}^n) \to \mathbb{R}$ to be

$$\widetilde{H}(F) = \begin{cases} H(F) & F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n) \\ \infty & \text{otherwise.} \end{cases}$$

Equation (4.11) implies that $I_0(F) \leq \widetilde{G}(F)$ for every $F \in L^p(\mathcal{S}; \mathbb{R}^n)$. Since, moreover, I_0 is a Γ-limit, it is lower-semicontinuous with respect to the strong L^p-topology, and therefore $I_0 \leq \Gamma\widetilde{G}$, where $\Gamma\widetilde{G}$ denoted the lower-semicontinuous envelope (with respect to the same topology) of $\widetilde{G}$.

Next denote by $\Gamma_w G$ the sequential lower-semicontinuous envelope of G with respect to the *weak* topology in $W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$. Lemma 5 in [21] implies that $\widetilde{\Gamma_w G} = \Gamma\widetilde{G}$, hence $I_0 \leq \widetilde{\Gamma_w G}$; in particular, for $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$, $I_0(F) \leq \Gamma_w G(F)$. Finally, it follows from [1] that

$$\Gamma_w G(F) = \fint_{\mathcal{S}} QW_0(dF) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} = I(F),$$

which completes the proof.   ∎

Lemma 4.2 *There exists* $\mathfrak{q}_0^\perp \in L^p(\mathcal{S}; \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n)$ *that satisfies*

$$W|_{\mathcal{S}} \left(dF \oplus \mathfrak{q}_0^\perp \right) = W_0(dF).$$

Proof: We first prove that there exists a measurable $\mathfrak{q}_0^\perp : \mathcal{S} \to \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n$ such the equality above holds, and then prove that it is in L^p.

Define $W^\perp = W|_{\mathcal{S}}(dF \oplus \cdot) : \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n \to \mathbb{R}$. We are looking for a measurable section $\mathfrak{q}_0^\perp$ that minimizes $W^\perp$ on every fiber. The measurable selection theorem (Theorem A.1) deals with the existence of such measurable sections. However, while $W^\perp$ satisfies the regularity assumptions in the theorem (it is a Carathéodory function as (i) it is fiber-wise continuous due to the continuity of $W|_{\mathcal{S}}$, and (ii) for every smooth section $\eta \in \Gamma(\mathcal{S}; \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n)$, $W^\perp \circ \eta$ is measurable as a composition of a continuous and a measurable function (see Appendix A)), the measurable selection theorem cannot be applied directly to $W^\perp$, as the fiber is a vector space and hence not compact. We therefore proceed by using the coercivity and growth properties of W to overcome this problem of non-compactness.

The coercivity and growth properties imply that for every $x \in \mathcal{S}$ and $\xi \in \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n$,

$$\alpha|\xi|^p - \beta < W^\perp(\xi)$$

and

$$\inf_{\eta \in (\mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n)_x} W^\perp(\eta) < C(1 + |d_x F|^p).$$

Hence, if we denote by $\mathcal{S}_N$ the set $\{x \in \mathcal{S} : |d_xF|^p < N\}$, the measurable selection theorem (Corollary A.1) can be applied to $W^{\perp}|_{\mathcal{S}_N}$, since the minimizer on every fiber lies in a ball of radius $\left(\frac{C(1+N)+\beta}{\alpha}\right)^{1/p}$. Denote this measurable minimizer by $\mathfrak{q}_{0,N}^{\perp}$, and construct $\mathfrak{q}_0^{\perp}$ by

$$\mathfrak{q}_0^{\perp}(x) = \mathfrak{q}_{0,N}^{\perp}(x), \qquad N = \min\{M \in \mathbb{N} : x \in \mathcal{S}_M\}.$$

Since, up to a null set, $\cup_N \mathcal{S}_N = \mathcal{S}$, $\mathfrak{q}_0^{\perp}$ is well defined on almost every point in $\mathcal{S}$. It is obviously measurable, since for every N, $\mathfrak{q}_{0,N}^{\perp}$ is measurable.

Finally, $\mathfrak{q}_0^{\perp} \in L^p(\mathcal{S}; \mathcal{N}\mathcal{S}^* \otimes \mathbb{R}^n)$ since

$$\alpha \int_{\mathcal{S}} |\mathfrak{q}_0^{\perp}|^p \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} - \beta \cdot \mathrm{Vol}(\mathcal{S}) \leq \int_{\mathcal{S}} W|_{\mathcal{S}}(dF \oplus \mathfrak{q}_0^{\perp}) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}$$

$$\leq \int_{\mathcal{S}} W(dF \oplus 0) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}}$$

$$\leq C \int_{\mathcal{S}} (1 + |dF|^p) \, d\mathrm{vol}_{\mathfrak{g}|_{\mathcal{S}}} < \infty.$$

$\blacksquare$

We have thus completed the proof of Theorem 2.1. We finish this section by a proof of the main corollary:

Let $f_h \in W_{bc}^{1,p}(\Omega_h; \mathbb{R}^n)$ be a sequence of (approximate) minimizers of I_h. Then (f_h) is a relatively compact sequence (with respect to the strong L^p topology), and all its limits points are minimizers of I. Moreover,

$$\lim_{h \to 0} \inf_{L^p(\Omega_h; \mathbb{R}^n)} I_h(\cdot) = \min_{L^p(\mathcal{S}; \mathbb{R}^n)} I(\cdot).$$

Proof: Let f_h be a sequence of approximate minimizers of I_h. We first prove that it is relatively compact, i.e., that every subsequence (not relabeled) of f_h has a subsequence that converges in L^p.

Let $g \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$ be arbitrary and let $g_h \in L^p(\Omega_h; \mathbb{R}^n)$ be a recovery sequence for g. Then,

$$\inf_{L^p} I_h(\cdot) \leq I_h(g_h) \xrightarrow[h \to 0]{} I(g) < \infty,$$

due to the growth property of QW_0. This shows that $\inf_{L^p} I_h(\cdot)$ is bounded.

It follows that $I_h(f_h)$ is bounded, hence by coercivity df_h is uniformly bounded in L^p, and together with the Poincaré inequality, f_h is uniformly bounded in $W^{1,p}$. Lemma 3.8 implies the existence of a subsequence $f_h \to F$ in L^p, proving the relative compactness of f_h.

We now prove that F is a minimizer of I. Let $g \in L^p(\mathcal{S}; \mathbb{R}^n)$ be an arbitrary function, and let $g_h \in L^p(\Omega_h; \mathbb{R}^n)$ be a recovery sequence for g. Therefore,

$$I(g) = \lim_{h \to 0} I_h(g_h) \geq \liminf_{h \to 0} \inf_{L^p} I_h(\cdot) = \lim_{h \to 0} I_h(f_h) \geq I(F),$$

where the last inequality follows from the lower-semicontinuity property of Γ-convergence. Since g is arbitrary, F is a minimizer of I. Moreover, by choosing $g = F$ we conclude that

$$I(F) = \liminf_{h \to 0} \inf_{L^p} I_h(\cdot).$$

$\blacksquare$

5 Discussion

This paper generalizes the work of Le Dret and Raoult [21,22] to a general Riemannian setting and general dimension and co-dimension, hence is applicable to slender pre-stressed bodies. We now emphasize the main differences between the present analysis and the prior work that was derived in the Euclidean setting; these can be partitioned into analytical issues and modeling issues.

Analytical issues Thin bodies are modeled as a family of tubular neighborhoods Ω_h of a Riemannian manifold $(\mathcal{M}, \mathfrak{g})$, that converges to a lower-dimensional submanifold $\mathcal{S}$. Accordingly, we defined a notion of convergences of functions $L^p(\Omega_h; \mathbb{R}^n)$ to functions $L^p(\mathcal{S}; \mathbb{R}^n)$. The fact that configurations for different h are defined over different functional spaces requires only minor adaptations in the Γ-convergence approach, since the latter is not affected by the Riemannian structure. Other analytic notions, however, such as quasiconvexity and measure theoretic issues require a more detailed attention.

The Γ-limit of a sequence of functionals is lower-semicontinuous. As weak lower-semicontinuity of an integral functional is closely related to the quasiconvexity of the integrand, we had to properly define the notion of quasiconvexity of functions over manifolds (Definition 3.4) and show that the classical results in [1] remains valid in this settings (Appendix B). As in the Euclidean case, the limit energy density QW_0 is the quasiconvex envelope of W_0, which is a projection of the original energy density W to the limiting submanifold.

For the energy functional to be well-defined, the density has to be sufficiently regular. We assume W to be continuous and show that this also implies the continuity of W_0 (Lemma 3.10). The more challenging step is to show that QW_0 is sufficiently regular, and specifically, a Carathéodory function. To do so, we need to properly define Carathéodory functions over fiber bundles (see Appendix A), and show that the quasiconvex envelope of a Carathéodory function over fiber bundles is again a Carathéodory function (Theorem 3.2 and Corollary 3.4). This notion of Carathéodory functions is also needed for the theorems regarding quasiconvexity and lower-semicontinuity mentioned above.

Finally, a generalization of the related notion of normal integrand to functions over fiber bundles enables us to prove a generalization of a measurable selection theorem, which shows the existence of a measurable minimizing section (Theorem A.1 and Corollary A.1), which is needed in the proof of Lemma 4.2.

Modeling issues and properties of the limit energy density The main assumption on the energy density W is its homogeneity over fibers (other than that, there are only regularity assumptions). In particular, we do not assume frame-indifference or isotropy. However, if the energy density does satisfy either of them, so does the limit energy density QW_0; the proofs are essentially the same as in Theorems 9 and 13 in [21]. Since our setting is not necessarily Euclidean, isotropy here means invariance of the energy density under orientation preserving isometries of the relevant manifold ($(\mathcal{M}, \mathfrak{g})$ in the case of W, $(\mathcal{S}, \mathfrak{g}|_{\mathcal{S}})$ in the case of QW_0). If W satisfies frame indifference, the frame indifference of QW_0 implies that the limit functional I depends on an immersion $F \in W_{bc}^{1,p}(\mathcal{S}; \mathbb{R}^n)$, only through the pullback metric on $\mathcal{S}$ induced by F, that is $F^\star \mathfrak{e}$. In other words, as expected, the only contribution to the membrane energy is from stretching of the limiting submanifold, in contrast to bending dominated limits, such as in [20].

The absence of bending contributions to the energy holds even without assuming frame indifference, as the limiting energy functional depends only on first derivatives of an immersion of the limiting submanifold $\mathcal{S}$. Moreover, the membrane energy does not "know" whether it is a limit of a Euclidean or a non-Euclidean problem, in the following sense. Since $\dim \mathcal{S} < n$, the Nash-Kuiper embedding theorem implies that $\mathcal{S}$ can be C^1-isometrically embedded in the Euclidean space $\mathbb{R}^n$. Consider $(\mathcal{S}, \mathfrak{g}|_{\mathcal{S}})$ as a sub-manifold of $\mathbb{R}^n$, and let Ω_h' be its tubular neighborhoods; they are Euclidean shells. We may then define an energy density W' on Ω_h', with $(W')_0 = W_0$. The limiting membrane model for $\mathcal{S}$ as a submanifold of $(\mathbb{R}^n, \mathfrak{e})$ with energy density W' is the same as the one obtained for $\mathcal{S}$ as a submanifold of $(\mathcal{M}, \mathfrak{g})$ with energy density W.

The homogeneity over fibers assumption enables us to relate between energy densities on the limit submanifold $\mathcal{S}$ and on the tubular neighborhoods Ω_h, and is therefore essential to the proof of Proposition 4.2 (it can be slightly weakened as long as Corollary 3.1 holds for some positive power of h). Indeed, in [7] and [4] it is shown that for Euclidean plate membranes the limit energy density is substantially different (and more complicated) when the inhomogeneity is in the normal directions (when choosing coordinates, homogeneity over fibers allows inhomogeneity only in the tangent directions). Note also that while homogeneity or inhomogeneity is usually stated in a specific coordinate system, our definition does not depend on the coordinate system, hence it reveals the geometric essence of this notion. Moreover, our coordinate-free approach can also be used to treat slender bodies of "varying thickness", considered in [7], since their notion of varying thickness is coordinate-dependent and can be viewed as tubular neighborhoods of constant "thickness" when $\mathcal{M}$ is endowed with an appropriate metric.

The stronger and more common notion of homogeneity (which was assumed in [21, 22]), implies that the energy density at one point determines it everywhere. This notion can also be generalized to the non-Euclidean case via parallel transport. Note, however, that unlike the Euclidean case, parallel transport is generally path-dependent. Thus, while in the Euclidean case every energy density at a point extends to a homogeneous energy density over the entire manifold, this is not so in the non-Euclidean case. This issue does not arise in our analysis, as homogeneity over fibers implies a specific choice of non-intersecting paths (the fibers) along which parallel transport is calculated.

A Measurability issues on Riemannian manifolds

The main result in this section is a generalization of a measurable selection theorem to fiber bundles over manifolds. First, we give some basic definitions.

A.1 Normal integrands and Carathéodory functions

Definition A.1 Let $\mathcal{M}$ be an m-dimensional differentiable manifold. A function $f : \mathcal{M} \to \bar{\mathbb{R}}$ is *measurable* if for every chart $X : U \subset \mathbb{R}^m \to \mathcal{M}$, $f \circ X : U \to \bar{\mathbb{R}}$ is (Lebesgue) measurable. Similarly, if $\mathcal{N}$ is an n-dimensional differentiable manifold, a function $f : \mathcal{M} \to \mathcal{N}$ is measurable if for every chart $Y : V \subset \mathbb{R}^n \to \mathcal{N}$, $Y^{-1} \circ f \circ X$ is (Lebesgue) measurable on its domain of definition.

A standard argument shows that it is enough to check measurability over an atlas. This definition gives a natural notion of Lebesgue-induced measurable subsets of a differentiable manifold (which obviously contain the Borel σ-algebra); for short, we will call these subsets Lebesgue measurable. We may therefore extend our definition of measurability to functions $f : A \to \mathcal{N}$, where A is a (measurable) subset of $\mathcal{M}$. Again, similar to the Euclidean case, if $\{A_i\}$ is a countable measurable partition of $\mathcal{M}$, and $f_i : A_i \to \mathcal{N}$ are measurable for every i, then $f = \cup f_i : \mathcal{M} \to \mathcal{N}$ is a measurable function.

A Riemannian metric on the differentiable manifold $\mathcal{M}$ induces a measure on this Lebesgue σ-algebra. As in the Euclidean case, the corresponding L^p space coincides with the completion of the smooth functions with respect to the L^p norm.

We now define the notions of normal integrands and Carathéodory functions. Let $F \subset \mathbb{R}^l$ be a Borel set, and let $\pi : E \to \mathcal{M}$ be a fiber bundle with fiber F. we define a "hybrid" σ-algebra on E as the σ-algebra generated by the Borel σ-algebra on F and the Lebesgue σ-algebra on $\mathcal{M}$.

Definition A.2 Let $\pi : E \to \mathcal{M}$ be as above. A function $f : E \to \bar{\mathbb{R}}$ is a *normal integrand* if

1. f is measurable with respect to the hybrid σ-algebra.
2. for almost every $p \in \mathcal{M}$, $f|_{E_p}$ is lower-semicontinuous.

A normal integrand satisfies the following property: if ρ is a measurable section of E, then $f \circ \rho : \mathcal{M} \to \bar{\mathbb{R}}$ is measurable. Note that the definition of a normal integrand can be extended to the case where $\pi : E \to A$ is a vector bundle over some measurable subset $A \subset \mathcal{M}$.

Definition A.3 Let $\pi : E \to \mathcal{M}$ be as above. A function $f : E \to \bar{\mathbb{R}}$ is a *Carathéodory function* if

1. for every smooth section ρ of E, $f \circ \rho : \mathcal{M} \to \bar{\mathbb{R}}$ is measurable.
2. for almost every $p \in \mathcal{M}$, $f|_{E_p}$ is continuous.

These definitions coincide with the classical definitions of normal integrands and Carathéodory functions over the trivial bundles $\mathbb{R}^n \times F$ (see e.g. [12] or [28]). A standard argument, based on a local trivialization of the bundle, shows that a function is a normal integrand (resp. Carathéodory function) if and only if it is locally a normal integrand (resp. Carathéodory function) in the classical sense. It can therefore be deduced that a function f is Carathéodory if and only if both f and $(-f)$ are normal integrands.

A.2 A measurable selection theorem

Theorem A.1 *Let $F \subset R^l$ be a compact set, and let $\pi : E \to \mathcal{M}$ be a fiber-bundle with fiber F. Let $f : E \to \bar{\mathbb{R}}$ be a normal integrand. Then there exists a measurable section $\rho : \mathcal{M} \to E$ such that for almost every $p \in \mathcal{M}$*

$$f \circ \rho(p) = \min_{\xi \in E_p} \{f(\xi)\}.$$

Proof: First, we cover $\mathcal{M}$ with a countable atlas $X_i : U_i \to \mathcal{M}$, where U_i are open sets in $\mathbb{R}^n$, and let $Y_i : U_i \times F \to E$ be a corresponding atlas of E. By the measurable selection theorem for trivial bundles (see e.g. [12]), there exists a measurable $v_i : U_i \to F$ such that

$$f \circ Y_i(x, v_i(x)) = \min_{a \in F} f \circ Y_i(x, a) \qquad \forall x \in U_i$$

Define $\rho_i(p) = Y_i(X_i^{-1}(p), v_i \circ X_i^{-1}(p))$. This is a measurable section of $E|_{X_i(U_i)}$ that satisfies

$$f \circ \rho_i(p) = \min_{\xi \in E_p} \{f(\xi)\} \qquad \forall p \in X_i(U_i).$$

Define next a section $\rho : \mathcal{M} \to E$ by

$$\rho(p) = \rho_N(p), \qquad N = \min\{i \in \mathbb{N} : p \in U_i\}.$$

ρ is obviously measurable and satisfies $f \circ \rho(p) = \min_{\xi \in E_p}\{f(\xi)\}$ for almost every $p \in \mathcal{M}$. ■

Corollary A.1 *The theorem also holds for a normal integrand $f' : E' \to \bar{\mathbb{R}}$, where $\pi' : E' \to A$ is a fiber bundle with a compact fiber F over a base A which is a measurable subset of $\mathcal{M}$.*

Proof: Extend f' to $f : E \to \bar{\mathbb{R}}$ by defining it to be $-\infty$ on $E \setminus E'$. f is a normal integrand, hence we can apply the theorem to f, and then restrict the resulting $\rho : \mathcal{M} \to E$ to A. ■

B On quasiconvexity and lower-semicontinuity

In this section we prove Theorems 3.1–3.2 that generalize classical results on the relation between quasiconvexity and lower-semicontinuity in the Riemannian setting. The proofs are basically applications of the main results of [1]. Let $(\mathcal{M}, \mathfrak{g})$ be a Riemannian manifold of finite volume that can be covered by a finite number of charts, let $U : T^*\mathcal{M} \otimes \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function that satisfies the growth condition $-\beta \leq U(q) \leq C(1 + |q|^p)$. The growth condition and the finite number of charts assumption imply that when choosing coordinates, U satisfies the growth assumptions (II.2) and (II.6) in [1] (actually, $U + \beta$ satisfies the conditions, but since we assumed that $\mathcal{M}$ has finite volume, the addition of a constant to the integrand is immaterial). First we prove Theorem 3.1:

> Under the above conditions , the functional $I_A : W^{1,p}(A; \mathbb{R}^m) \to \mathbb{R}$ defined by
>
> $$I_A(f) := \int_A U(df)\, dvol_{\mathfrak{g}},$$
>
> where $A \subset \mathcal{M}$ is an open subset, is weakly sequential lower-semicontinuous for every A if and only if U is quasiconvex.

Proof: First assume that I_A is weakly sequential lower-semicontinuous in $W^{1,p}(A; \mathbb{R}^m)$ for some open $A \subset \mathcal{M}$ contained in a coordinate neighborhood. Using coordinates, we apply Theorem [II.2] in [1], and get that U is quasiconvex in $T^*A \otimes \mathbb{R}^m$. Since quasiconvexity is a fiber-wise condition, if $U|_A$ is quasiconvex for every open $A \subset \mathcal{M}$, then U is quasiconvex (as in Definition 3.4).

Now assume that U is quasiconvex. Let $f, f_n \in W^{1,p}(\mathcal{M}; \mathbb{R}^m)$ such that $f_n \rightharpoonup f$. We can partition $\mathcal{M}$, up to a null-set, into a finite number of disjoint open sets $\{A_i\}$ such that for every i, A_i is contained in a coordinate neighborhood. Since U is a Carathéodory quasiconvex function and satisfies the growth condition (3.6), it follows from Theorem [II.4] in [1] that I_{A_i} is weakly sequential lower-semicontinuous in $W^{1,p}(\mathcal{M}; \mathbb{R}^m)$. Therefore,

$$\liminf_{n \to \infty} I_{\mathcal{M}}(f_n) = \sum_i \liminf_{n \to \infty} I_{A_i}(f_n|_{A_i}) \geq \sum_i I_{A_i}(f|_{A_i}) = I_{\mathcal{M}}(f),$$

which shows that $I_{\mathcal{M}}$ is weakly sequential lower-semicontinuous, and therefore I_A is weakly sequential lower-semicontinuous for every open $A \subset \mathcal{M}$. ■

Next we prove Theorem 3.2:

Under the above conditions, the weakly sequential lower-semicontinuous envelope of the functional $I_{\mathcal{M}} : W^{1,p}(\mathcal{M};\mathbb{R}^m) \to \mathbb{R}$ is $\Gamma I_{\mathcal{M}} : W^{1,p}(\mathcal{M};\mathbb{R}^m) \to \mathbb{R}$ given by

$$\Gamma I_{\mathcal{M}}(f) := \int_{\mathcal{M}} QU(df)\, dvol_{\mathfrak{g}},$$

where $QU(q) = \sup\{V(q) : V \leq U \text{ is quasiconvex}\}$ is the quasiconvex envelope of U; moreover QU is a Carathéodory quasiconvex function.

Proof: Let $A \subset \mathcal{M}$ be a coordinate neighborhood. We apply Statement [III.7] in [1] to get that the weakly sequential lower-semicontinuous envelope of I_A is

$$\Gamma I_A(f) := \int_A Q(U|_A)(df)\, d\mathrm{vol}_{\mathfrak{g}}.$$

Because quasiconvexity is a fiberwise condition, the restriction of the quasiconvex envelope is the quasiconvex envelope of the restriction, and therefore we can replace $Q(U|_A)$ with $(QU)|_A$. Theorem [III.6] in [1] implies that $(QU)|_A$ is a Carathéodory function, and since being a Carathéodory function is a local condition (see Appendix A), we have that QU is indeed Carathéodory. It is also quasiconvex, since generally the supremum of quasiconvex functions is quasiconvex (again, this is a fiberwise argument, and therefore it is true since it holds in the Euclidean case). By Theorem 3.1, $\Gamma I_{\mathcal{M}}$ is weakly sequential lower-semicontinuous, hence it is bounded by the sequentially lower-semicontinuous envelope of $I_{\mathcal{M}}$, denoted by $\widetilde{I}_{\mathcal{M}}$.

Next, for a given $f \in W^{1,\infty}(A;\mathbb{R}^m) \subset W^{1,p}(A;\mathbb{R}^m)$, Statement [III.7] also implies that for every $\varepsilon > 0$ there exists a sequence $f_n \in f + W_0^{1,\infty}(A;\mathbb{R}^m)$ such that $f_n - f \overset{*}{\rightharpoonup} 0$ in $W_0^{1,\infty}(A;\mathbb{R}^m)$ and $\liminf_n I_A(f_n) \leq \Gamma I_A(f) + \varepsilon$. This follows from the fact that in the notation of [1],

$$\Gamma I_A(f) = \lim_{r \to \infty} F_0(r,f,A) = \lim_{r \to \infty} \left[\inf\{ \liminf_{n \to \infty} I_A(f_k) : f_n - f \overset{*}{\rightharpoonup} 0 \text{ in } W_0^{1,\infty}(A;\mathbb{R}^m) \text{ and } |du_n|_\infty \leq r \} \right].$$

See also Theorem 3.8 in [25].

For the reverse inequality, let $\{A_i\}$ be a finite partition (up to a null-set) of $\mathcal{M}$, such that for every i, A_i is an open set contained in a coordinate neighborhood, and let $f \in W^{1,\infty}(\mathcal{M};\mathbb{R}^m)$. Fix $\varepsilon > 0$. For every i, let $f_i^n \in f|_{A_i} + W_0^{1,\infty}(A_i;\mathbb{R}^m)$ be a sequence such that $f_i^n - f|_{A_i} \overset{*}{\rightharpoonup}_n 0$ in $W_0^{1,\infty}(A_i;\mathbb{R}^m)$ and

$$\liminf_{n \to \infty} I_{A_i}(f_i^n) \leq \Gamma I_A(f|_{A_i}) + \frac{\varepsilon}{2^i}.$$

Obviously, $f^n := \cup_i f_i^n \overset{*}{\rightharpoonup} f$ in $W^{1,\infty}(\mathcal{M};\mathbb{R}^m)$, and therefore

$$\widetilde{I}_{\mathcal{M}}(f) \leq \liminf_{n \to \infty} I_{\mathcal{M}}(f^n) = \liminf_{n \to \infty} \sum_i I_{A_i}(f_i^n) \leq \sum_i \Gamma I_A(f|_{A_i}) + \varepsilon = \Gamma I_{\mathcal{M}}(f) + \varepsilon,$$

and since ε is arbitrary, it shows that $\widetilde{I}_{\mathcal{M}}(f) \leq \Gamma I_{\mathcal{M}}(f)$ for every $f \in W^{1,\infty}(\mathcal{M};\mathbb{R}^m)$. To extend this to $W^{1,p}(\mathcal{M};\mathbb{R}^m)$, observe that $W^{1,\infty}(\mathcal{M};\mathbb{R}^m)$ is dense in $W^{1,p}(\mathcal{M};\mathbb{R}^m)$ with respect to the strong $W^{1,p}$ topology, and that $\Gamma I_{\mathcal{M}}$ is continuous in $W^{1,p}(\mathcal{M};\mathbb{R}^m)$ with respect to this topology (see Proposition B.1 below). Hence, given $f \in W^{1,p}(\mathcal{M};\mathbb{R}^m)$, let $f_n \in W^{1,\infty}(\mathcal{M};\mathbb{R}^m)$ such that $f_n \to f$, and we obtain that

$$\widetilde{I}_{\mathcal{M}}(f) \leq \liminf_{n \to \infty} \widetilde{I}_{\mathcal{M}}(f_n) = \liminf_{n \to \infty} \Gamma I_{\mathcal{M}}(f_n) = \Gamma I_{\mathcal{M}}(f),$$

which completes the proof. ∎

The strong continuity of $\Gamma I_{\mathcal{M}}$ follows from the following generalization of the Carathéodory continuity theorem:

Proposition B.1 *Let $\mathcal{M}$ and U satisfy the assumptions as above. Then the functional $I_{\mathcal{M}} : W^{1,p}(\mathcal{M};\mathbb{R}^m) \to \mathbb{R}$ is continuous with respect to the strong $W^{1,p}$ topology.*

Proof: This is an immediate consequence of the analogous Euclidean proposition (see e.g. [10], Example 1.22). Indeed, let A_i be an open partition (up to a null set) of $\mathcal{M}$, such that for every i, A_i is contained in a coordinate neighborhood. For every i, $U|_{A_i}$ is a Carathéodory function that satisfies (3.6), since U does. Hence, by the Carathéodory continuity theorem, I_{A_i} is strongly continuous in $W^{1,p}(A_i; \mathbb{R}^m)$. Let $f, f_n \in W^{1,p}(\mathcal{M}; \mathbb{R}^m)$ such that $f_n \to f$. In particular, for every i, $f_n|_{A_i} \to f|_{A_i}$ in $W^{1,p}(A_i; \mathbb{R}^m)$. Therefore,

$$\lim_{n \to \infty} I_{\mathcal{M}}(f_n) = \lim_{n \to \infty} \sum_i I_{A_i}(f_n|_{A_i}) = \sum_i I_{A_i}(f|_{A_i}) = I_{\mathcal{M}}(f).$$

■

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