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Non-Euclidean plates and shells

Abstract An elastic theory of even non-Euclidean plates was recently proposed to describe a class of thin elastic bodies that exhibit residual stress in the absence of external forces or constraints. This plate theory was derived from a three-dimension theory of so-called incompatible elasticity; the terms non-Euclidean and incompatible refer to the existence of a reference metric that cannot be immersed in a three-dimensional Euclidean space. In this paper we generalize the reduced two-dimensional theory to account for structural inhomogeneities across the thickness, leading to intrinsic curvatures (non-Euclidean shells), as well as for inhomogenieties in the thickness of the thin body (uneven non-Euclidean plates). Two examples are discussed.

1 Introduction

In two recent publications [5,6], we proposed a theory for elastic bodies that do not have a stress-free rest configuration, even in the absence of external constraints (i.e., exhibit residual stress). Such bodies are ubiquitous in biological systems, as well as in many manufactured materials (see for example [12,10,15]). Our

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model departs from the principles of hyper-elasticity [14], and assumes that the local energy density vanishes if and only if the metric g (the right Cauchy-Green deformation tensor) equals a "reference metric" $\bar{g}$. This reference metric is a body property, which characterizes the intrinsic distances between material points; in plants, for example, it is determined in the course of the growth process. A three dimensional metric can be immersed in the physical (Euclidean) space only if it is flat, namely, if the Riemann curvature tensor is identically zero. A lack of stress-free configuration occurs when the reference metric is non-immersible. In such case, the body is necessarily in a state of frustration, as it cannot satisfy the reference metric everywhere simultaneously. In [5] the energy density was assumed to be quadratic in the deviation of the actual metric form the reference metric, resulting in a nonlinear elastic theory, which we named "three-dimensional incompatible elasticity". In the particular case where the reference metric $\bar{g}$ is flat, our theory coincides with standard nonlinear elasticity theories [3].

An important sub-class within the class of elastic bodies that exhibit residual stress consists of thin, sheet-like bodies. Because of the metric incompatibility, such bodies do not fall within the category of neither plates nor shells. The concept of even non-Euclidean plates was introduced in [5], defined as thin elastic bodies whose reference metric is invariant along the thin dimension, and whose thickness is uniform. A reduced two-dimensional theory was then derived, based on the Kirchhoff assumptions, resulting in an elastic energy that depends on the configuration of the mid-plane. This energy functional resembles the Koiter energy [11], with a stretching term which is linear in the plate thickness, and a bending term which is cubic in the plate thickness. The notable difference with the Koiter theory is that the stretching term measures the metric deviation from a non-flat two-dimensional metric. The frustration results from the competition between the stretching term, which favors bent configurations, and the bending term, which favors flat configurations.

The elastic response of bodies undergoing differential growth has been considered previously, and recently in [2,8,4]. In [8] the deformation tensor is multiplicative decomposed into a growth process and an elastic relaxation. In [4] the thin-plate limit is considered, leading to a Föppl-von Kármán-like plate theory.

In this paper we extend the treatment to thin elastic bodies of more general structure, namely, beyond the even plate structure. In the more general case, where the metric varies along the thin dimension, a natural curvature arises as in shells. These are however non-Euclidean shells as the strain is still measured with respect to an incompatible metric. Relying again on the Kirchhoff closure assumptions, we derive a reduced two-dimensional model for thin elastic structures, which we name "non-Euclidean shells".

Two examples are provided. The first, motivated by experiments with thermo-responsive gels [10], falls into the category of non-even plates. The second, motivated by the snapping mechanism of the Venus flytrap [7], corresponds to a non-Euclidean shell.

2 Three-dimensional "incompatible" elasticity

Let Ω be a bounded domain in $\mathbb{R}^3$; we denote points in Ω by $x = (x^1, x^2, x^3)$. A configuration of a body whose domain of parametrization is Ω is a function

$f : \Omega \to \mathbb{R}^3$. We require f to be differentiable, with a square integrable derivative, and have a fixed orientation, say, $\det \nabla f > 0$. The metric on Ω induced by the mapping f is a tensor $g(x)$ with entries

$$g_{ij}(x) = [(\nabla f)^T(\nabla f)]_{ij} = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, \quad i, j = 1,2,3.$$

In the context of elasticity it is also known as the right Cauchy-Green deformation tensor.

The principles of hyper-elasticity [14] state that to each configuration f corresponds a stored elastic energy of the form,

$$E[f] = \int_\Omega W(x, f(x), \nabla f(x))\, dV, \tag{1}$$

where W is the energy density and dV is an infinitesimal volume element. The energy density is usually divided into a strain ("internal") contribution, which only depends on x and $\nabla f(x)$, and a forcing ("external") contirbution which depends on x and $f(x)$. In this paper we are concerned with unconstrained bodies, hence we omit the explicit dependence of the energy density on the configuration, i.e., $W = W(x, \nabla f(x))$. This energy density is subject to a number of physical requirements, such as to satisfy the material objectivity property (rotational invariance),

$$W(x,A) = W(x,QA), \quad \text{for all } Q \in \mathrm{SO}(3) \text{ and } A \in \mathbb{R}^{3\times 3}.$$

It is easily seen that the material objectivity condition translates into the requirement that W be expressible as a function of the metric tensor, namely, $W = W(x, g(x))$.

The postulate underlying (linear and nonlinear) elasticity theories is that in the absence of external forcing, the identity map $f(x) = x$ corresponds to a state of zero elastic energy (in nonlinear elasticity, the energy density is zero if and only if the deformation tensor ∇f is a rotation). In other words, the intrinsic geometry of the body is Euclidean. Our approach for modeling bodies that exhibit residual stress is to assume that their intrinsic structure is prescribed by a metric tensor $\bar{g}(x)$ (the reference metric) that is not necessarily Euclidean. The Green-St. Venant strain tensor is proportional to the deviation of the actual metric from the reference metric,

$$\varepsilon(x) = \frac{1}{2}(g(x) - \bar{g}(x)). \tag{2}$$

Since $\bar{g}(x)$ is viewed as the intrinsic metric at the point x, we assume that for given x, the energy density $W(x, g(x))$ vanishes if and only if $g(x) = \bar{g}(x)$. If we further assume that $W(x,g)$ is at least twice differentiable with respect to g in the vicinity of $\bar{g}(x)$, then we expect

$$W(x,g) = \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + O(\varepsilon^4), \tag{3}$$

where $A^{ijkl} = A^{ijkl}(x)$ is an elastic tensor that depends on local material properties, but does not depend on the configuration. Here and below the Einstein summation convention is used. A simple calculation shows that for an isotropic material,

$$A^{ijkl} = \lambda \bar{g}^{ij} \bar{g}^{kl} + \mu \left(\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk} \right), \tag{4}$$

where $\lambda,\mu > 0$ are Lamé coefficients and $\bar{g}^{ij}$ are the entries of the reciprocal reference metric $\bar{g}^{-1}$. Combining eqs. (1)–(4), the elastic energy functional takes the form [5]

$$E[f] = \int_{\Omega} W(x, g(x))\, dV, \tag{5}$$

where $dV = \sqrt{|\bar{g}|}\, dx^1 dx^2 dx^3$, and

$$W(x,g) = \frac{\lambda}{2}\varepsilon_i^i \varepsilon_j^j + \mu \varepsilon_j^i \varepsilon_i^j + O(\varepsilon^4). \tag{6}$$

The raising and lowering of indices is with respect to the reference metric $\bar{g}$; for example,

$$\varepsilon_j^i = \bar{g}^{ik}\varepsilon_{kj}.$$

(The fact that the volume element is determined by the reference metric conforms with most models, as the Riemannian manifold with metric $\bar{g}$ is the Lagrangian system of coordinates; the only novelty is the fact that it is non-flat.) Omitting terms of order higher than quadratic in the strain, we obtain a nonlinear elastic theory applicable in the range of small strains.

The remarkable property of the energy functional (5), which distinguishes it from standard models, is that it has no stress-free rest configuration if the reference metric is not flat. The energy density can vanish locally, but cannot vanish everywhere simultaneously. The equilibrium state can be defined by the metric $g(x)$ that minimizes the weighted L^2 distance, (5), from the reference metric $\bar{g}$, under the constraint that $g(x)$ be immersible in $\mathbb{R}^3$.

For this elastic problem to be well-posed, more restrictions have to be imposed on the energy density $W(x,g)$. A substantial amount of literature exists on this matter. It is well-known, for example, that if W is a polyconvex function of ∇f, then a minimizer exists [1,3]. Such conditions are important to ensure the well-posedness of the model in the presence of large strains. For small strains, we may assume that the energy density is quadratic in the strain.

One can further define the second Piola-Kirchhoff stress tensor,

$$s^{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = A^{ijkl}\varepsilon_{kl}. \tag{7}$$

The Euler-Lagrange equations that correspond to the above variational formulation are

$$\begin{aligned} \bar{\nabla}_j s^{ij} + (\Gamma^i_{jk} - \bar{\Gamma}^i_{jk}) s^{jk} &= 0 \quad \text{in } \Omega \\ s^{ij} n_j &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

where Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols corresponding to the metrics g and $\bar{g}$, respectively, and $\bar{\nabla}_j$ is the covariant derivative associated with the reference metric (see [5] for more details).

3 Reduced two-dimensional models

We now turn our attention to thin bodies of the above class. Naturally, one would like to take advantage of the nearly two-dimensional nature of the geometry, and derive a model of reduced dimension, which only takes into account surface properties of the configuration (the surface could be, for example, the mid-surface of the body). Such a reduced theory for non-Euclidean plates was developed in [5], based on the Kirchhoff closure assumptions [9]. Formally, an even non-Euclidean plate was defined as a body for which there exists a parametrization in which

$$\Omega = S \times [-t/2, t/2],$$

where $S \subset \mathbb{R}^2$, and

$$\bar{g}(x^1, x^2, x^3) = \begin{pmatrix} \bar{a}(x^1, x^2) & 0 \\ 0 & 1 \end{pmatrix}, \tag{9}$$

where $\bar{a} \in \mathbb{R}^{2\times 2}$ is a two-dimensional reference metric. Such a body is considered thin if the thickness t is much smaller than any other characteristic lenghscale. It is easily seen that the three-dimensional metric $\bar{g}$ is immersible in $\mathbb{R}^3$ if and only if the two-dimensional metric $\bar{a}$ has zero Gaussian curvature ($\bar{g}$ is two-dimensional to the extent that its Riemann curvature tensor depends on a single scalar curvature). The "plate nature" of such a body is reflected by the fact that the internal structure is independent of the thin direction. That is, such a body can be viewed as a continuous stack of identical surfaces.

It should be noted that every three-dimensional metric can, provided that the body is sufficiently thin, be brought to a semi-geodesic form,

$$\bar{g}(x^1, x^2, x^3) = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} & 0 \\ \bar{g}_{21} & \bar{g}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the entries $\bar{g}_{11}$, $\bar{g}_{12}$, $\bar{g}_{21}$ and $\bar{g}_{22}$ are functions of (x^1, x^2, x^3), by first choosing a surface S with an arbitrary parameterization, and then issuing geodesic lines perpendicular to the surface to parametrize points off the surface.

Henceforth, we use lowercase Latin characters to denote indices of three-dimensional tensors, e.g., $i, j = 1,2,3$, and Greek characters to denote indices of two-dimensional tensors, e.g., $\alpha, \beta = 1,2$. Thus, the semi-geodesic parametrization satisfies

$$\bar{g}_{\alpha 3} = 0 \quad \text{and} \quad \bar{g}_{33} = 1.$$

To proceed, we adopt a generalization of the Kirchhoff assumptions. We first note that the boundary conditions in (8) imply that

$$s^{i3}(x) = 0,$$

when $x^3 = \pm t/2$. The key assumption in the Kirchhoff theory is that throughout the body, s^{i3} is at most of order t^2. In essence, this means that stress derivatives remain bounded as $t \to 0$. Explicitly, substituting (2) and (4) into (7)

$$s^{i3} = \lambda \bar{g}^{i3} \bar{g}^{kl} (g_{kl} - \bar{g}_{kl}) + 2\mu \bar{g}^{ik} (g_{k3} - \bar{g}_{k3}) = O(t^2). \tag{10}$$

For $i=\alpha=1,2$, using the fact that $\bar{g}^{\alpha 3}=0$ we obtain $\bar{g}^{\alpha\beta}g_{\beta 3}=O(t^2)$, which in turns yields

$$g_{\beta 3}=O(t^2). \tag{11}$$

Thus, the actual metric is of the form

$$g(x^1,x^2,x^3)=\begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}+O(t^2),$$

where the entries $g_{\alpha\beta}$ and g_{33} are functions of (x^1,x^2,x^3). Setting $i=3$ in (10) we get

$$s^{33}=\lambda\bar{g}^{\alpha\beta}(g_{\alpha\beta}-\bar{g}_{\alpha\beta})+(\lambda+2\mu)(g_{33}-1)=O(t^2),$$

from which we obtain that

$$g_{33}=1-\frac{\lambda}{\lambda+2\mu}\bar{g}^{\alpha\beta}(g_{\alpha\beta}-\bar{g}_{\alpha\beta})+O(t^2), \tag{12}$$

Substituting (11) and (12) into the energy density (6), we obtain

$$W=\frac{1}{2}\bar{\mathcal{A}}^{\alpha\beta\gamma\delta}(g_{\alpha\beta}-\bar{g}_{\alpha\beta})(g_{\alpha\beta}-\bar{g}_{\alpha\beta})+O(t^2), \tag{13}$$

where

$$\bar{\mathcal{A}}^{\alpha\beta\gamma\delta}=\frac{Y}{4(1-\nu^2)}\left[\nu\bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta}+\frac{1}{2}(1-\nu)(\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}+\bar{g}^{\alpha\delta}\bar{g}^{\beta\gamma})\right],$$

and the Young modulus Y and the Poisson ratio ν are defined through the relations,

$$\frac{\lambda}{\lambda+2\mu}=\frac{\nu}{1-\nu} \quad \text{and} \quad Y=2\mu(1+\nu).$$

So far, the fact that the body is thin was only used in the ansatz $s^{i3}=O(t^2)$. We now expand the $\alpha\beta$ components of both the reference metric and actual metric about the mid-surface,

$$\begin{aligned} \bar{g}_{\alpha\beta}(x^1,x^2,x^3)&=\bar{a}_{\alpha\beta}(x^1,x^2)+\bar{a}'_{\alpha\beta}(x^1,x^2)x^3+O(t^2) \\ g_{\alpha\beta}(x^1,x^2,x^3)&=a_{\alpha\beta}(x^1,x^2)+a'_{\alpha\beta}(x^1,x^2)x^3+O(t^2). \end{aligned} \tag{14}$$

We also introduce the two dimensional strain,

$$e_{\alpha\beta}=\frac{1}{2}(a_{\alpha\beta}-\bar{a}_{\alpha\beta}),$$

and note that

$$\bar{g}^{\alpha\beta}=\bar{a}^{\alpha\beta}-\bar{a}^{\alpha\gamma}\bar{a}^{\beta\delta}\bar{a}'_{\gamma\delta}x^3+O(t^2),$$

where $\bar{a}^{\alpha\beta}$ are the entries of $\bar{a}^{-1}$. It follows that

$$\bar{\mathcal{A}}^{\alpha\beta\gamma\delta}(x^1,x^2,x^3)=\mathcal{A}^{\alpha\beta\gamma\delta}(x^1,x^2)+x^3\,\mathcal{B}^{\alpha\beta\gamma\delta}(x^1,x^2)+O(t^2), \tag{15}$$

where

$$\mathcal{A}^{\alpha\beta\gamma\delta} = \frac{Y}{4(1-\nu^2)}\left[\nu\bar{a}^{\alpha\beta}\bar{a}^{\gamma\delta} + \frac{1}{2}(1-\nu)(\bar{a}^{\alpha\gamma}\bar{a}^{\beta\delta} + \bar{a}^{\alpha\delta}\bar{a}^{\beta\gamma})\right],$$

and $\mathcal{B}(x^1x,^2)$ comprises terms cubic in $\bar{a}^{-1}$ and linear in a'.

To obtain a reduced two-dimensional energy density, we substitute the expansions (14) and (15) into the three-dimensional energy density (13), an integrate over the x^3 coordinate. Odd powers of x^3 cancel, and we remain with the following reduced energy functional,

$$E = \int_S \mathcal{W}(x^1,x^2)\sqrt{|\bar{a}|}dx^1dx^2, \tag{16}$$

where the two-dimensional energy density is

$$\begin{aligned}\mathcal{W}(x^1,x^2) &= \frac{t}{2}\mathcal{A}^{\alpha\beta\gamma\delta}(a_{\alpha\beta}-\bar{a}_{\alpha\beta})(a_{\gamma\delta}-\bar{a}_{\gamma\delta})\\ &+\frac{t^3}{48}\mathcal{A}^{\alpha\beta\gamma\delta}(a'_{\alpha\beta}-\bar{a}'_{\alpha\beta})(a'_{\gamma\delta}-\bar{a}'_{\gamma\delta})\\ &+O(t^4,t^3|e|).\end{aligned} \tag{17}$$

As shown in [5,6], the two dimensional strain $e_{\alpha\beta}$ tends to zero as $t \to 0$, which is why we consider terms of order $t^3|e|$ as small compared to terms of order t^3.

The tensor $a_{\alpha\beta}$ is the two-dimensional metric of the mid-surface. As is well known, a two-dimensional metric does not uniquely determine the three-dimensional configuration of a surface. A surface is uniquely determined by its first and second fundamental forms, i.e, by its metric and curvatures; if $r(x^1,x^2)$ is a surface in $\mathbb{R}^3$, then its second fundamental form is given by the tensor

$$b_{\alpha\beta} = \partial_\alpha\partial_\beta r\cdot\mathcal{N},$$

where $\partial_\alpha = \partial/\partial x^\alpha$, and $\mathcal{N}$ is the unit vector normal to the surface.

To connect between the tensor a' and the second fundamental form, we expend the configuration in powers of x^3,

$$f(x^1,x^2,x^3) = r(x^1,x^2) + x^3\,r'(x^1,x^2) + O(t^2).$$

Differentiating once, we obtain the first fundamental form,

$$\begin{aligned}g_{\alpha\beta} &= \partial_\alpha r\cdot\partial_\beta r + x^3\,(\partial_\alpha r\cdot\partial_\beta r' + \partial_\beta r\cdot\partial_\alpha r') + O(t^2)\\ g_{\alpha 3} &= \partial_\alpha r\cdot r' + O(t)\\ g_{33} &= r'\cdot r' + O(t).\end{aligned}$$

From the fact that $g_{\alpha 3} = O(t^2)$ follows that r' is perpendicular to $\partial_\alpha r$, i.e., it is proportional to the unit normal $\mathcal{N}$,

$$r'(x^1,x^2) = r'(x^1,x^2)\,\mathcal{N}(x^1,x^2).$$

From the expansion (14) of $g_{\alpha\beta}$ follows that

$$\partial_\alpha r\cdot\partial_\beta r = a_{\alpha\beta},$$

and

$$r'(\partial_\alpha r\cdot\partial_\beta \mathcal{N}+\partial_\beta r\cdot\partial_\alpha \mathcal{N})=a'_{\alpha\beta}.$$

However,

$$\partial_\alpha r\cdot\partial_\beta \mathcal{N}=\partial_\beta\,(\partial_\alpha r\cdot\mathcal{N})-\partial_\beta\partial_\alpha r\cdot\mathcal{N}=-b_{\alpha\beta},$$

hence,

$$a'_{\alpha\beta}=-2r'\,b_{\alpha\beta}.$$

Finally, from the expansion (14) of g_{33} follows that

$$(r')^2=1-\frac{\nu}{1-\nu}\bar{a}^{\alpha\beta}(a_{\beta\alpha}-\bar{a}_{\beta\alpha})=1+O(e), \tag{18}$$

that is,

$$a'_{\alpha\beta}=-2\,b_{\alpha\beta}+O(e).$$

Substituting back into (17),

$$\mathcal{W}(x^1,x^2)=\mathcal{W}_S(x^1,x^2)+\mathcal{W}_B(x^1,x^2)+O(t^4,t^3|e|), \tag{19}$$

where

$$\mathcal{W}_S(x^1,x^2)=\frac{t}{2}\mathcal{A}^{\alpha\beta\gamma\delta}(a_{\alpha\beta}-\bar{a}_{\alpha\beta})(a_{\gamma\delta}-\bar{a}_{\gamma\delta})$$
$$\mathcal{W}_B(x^1,x^2)=\frac{t^3}{24}\mathcal{A}^{\alpha\beta\gamma\delta}(b_{\alpha\beta}-\bar{b}_{\alpha\beta})(b_{\gamma\delta}-\bar{b}_{\gamma\delta}),$$

where $-2\bar{b}_{\alpha\beta}=\bar{a}'_{\alpha\beta}$, and $\mathcal{W}_S$ and $\mathcal{W}_B$ are the respective densities of the stretching and bending energies.

Comments

1. The reduced energy functional (16) with density (19) is very similar to Koiter's shell model [11]. There is a stretching term, which is linear in the thickness of the body, and the bending term, which is cubic in the thickness. The tensors $\bar{a}$ and $\bar{b}$ are the reference first and second forms; the energy density vanishes if and only if $a=\bar{a}$ and $b=\bar{b}$. This is however possible only if the tensors $\bar{a}$ and $\bar{b}$ satisfy the Gauss-Codazzi-Mainardi equations [13]. The notable difference with Koiter's theory is that these two requirements may be incompatible.
2. We have used the fact that the plate thickness is uniform (i.e., that the plate is even) when identifying the normal stresses with s^{i3}. This remains approximately correct as long as thickness variations are moderate. An example of a non-even plate is described in the next section.
3. The fact that $O(e)$ term were neglected compared to $O(1)$ terms is due to the fact that for any isometric immersion with finite bending energy, $E=O(t^3)$, from which follows that at equilibrium the stretching energy is at most $O(t)$, namely,

$$\int_S A^{\alpha\beta\gamma\delta}(a_{\alpha\beta}-\bar{a}_{\alpha\beta})(a_{\gamma\delta}-\bar{a}_{\gamma\delta})\sqrt{|\bar{a}|}dx^1dx^2\le Ct^2. \tag{20}$$

The weakness of this argument is that (20) only implies that $e=O(t)$ in the mean-square, and not pointwise. Thus, establishing our theory on firmer mathematical grounds requires a more delicate analysis.

4 Examples

4.1 Non-Euclidean plates: thermo-responsive gels

Recent experiments in thermally responsive gel discs [10] provide a convenient testing ground for elastic bodies that fall into the category of non-Euclidean plates. In these experiments, initially flat stress-free objects shrink differentially upon heating, according to a pre-determined chemical concentration in their composition. The shrinking is homogeneous across the thickness, but inhomogeneous in the lateral directions. Such bodies are therefore plates, because their reference metric is constant along the thin direction.

Specifically, if we denote by $\eta(x^1,x^2)$ the degree of shrinking, then the reference metric has an isothermal (or conformal) structure,

$$\bar{g}_{ij} = \eta^2(x^1,x^2)\,\delta_{ij},$$

with $x \in S \times [-t/2, t/2]$. This reference metric is not exactly of the form (9) as $\bar{g}_{33} \neq 1$.

One could repeat the analysis of the previous section, deriving a reduced two dimensional model for a reference metric of this type. From the condition that $s^{i3} = 0$ follows that $g_{\alpha 3} = O(t^2)$ and

$$g_{33} = \eta^2 - \frac{\lambda}{\lambda + 2\mu}(g_{11} + g_{22} - 2\eta^2) + O(t^2).$$

The rest of the analysis remains more or less verbatim, up to (18), where we obtain that

$$(r')^2 = \eta^2 + O(e),$$

which implies that

$$a'_{\alpha\beta} = -2\eta b_{\alpha\beta} + O(e).$$

Thus, we end up with a reduced energy density (16) with a density of the form (19), where

$$\mathcal{W}_S(x^1,x^2) = \frac{\eta t}{2}\mathcal{A}^{\alpha\beta\gamma\delta}(a_{\alpha\beta} - \bar{a}_{\alpha\beta})(a_{\gamma\delta} - \bar{a}_{\gamma\delta})$$

$$\mathcal{W}_B(x^1,x^2) = \frac{(\eta t)^3}{24}\mathcal{A}^{\alpha\beta\gamma\delta} b_{\alpha\beta} b_{\gamma\delta}$$

(a factor of η emerges as $\sqrt{|\bar{g}|} = \eta\sqrt{|\bar{a}|}$. Thus, the only difference with the plate model derived in [5] is that $t \mapsto \eta t$, i.e., the thickness is no longer spatially uniform.

4.2 Non-Euclidean shells: the Venus flytrap

Let the domain of parametrization of the mid-surface be $S = [0,R] \times [0,2\pi)$ (with periodicity with the x^2 direction), and consider a particular surface given by

$$r(x^1,x^2) = \frac{1}{\sqrt{K}}\left(\sin\sqrt{K}x^1 \cos x^2, \sin\sqrt{K}x^1 \sin x^2, \cos\sqrt{K}x^1\right), \tag{21}$$

which is a spherical cap of Gaussian curvature $K > 0$. It is easily verified that the first and second fundamental forms of this surface are

$$a^{\text{cap}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{K}\sin^2\sqrt{K}x^1 \end{pmatrix},$$

and

$$b^{\text{cap}} = -\begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}}\sin^2\sqrt{K}x^1 \end{pmatrix}.$$

If we set $\bar{a} = a^{\text{cap}}$ and $\bar{b} = b^{\text{cap}}$ then a stress-free rest configuration exists, given by (21). A stress-free rest configuration exists also if we rather set $\bar{b} = -b^{\text{cap}}$, in which case the equilibrium solution is a spherical cap with opposite orientation. In either case, we are within the realm of ordinary, Euclidean shells.

Suppose next, that the reference metric and curvatures are given by

$$\bar{a} = a^{\text{cap}} \qquad \text{and} \qquad \bar{b} = \theta\, b^{\text{cap}},$$

where θ is a constant. Unless $\theta = \pm 1$, $\bar{a}$ and $\bar{b}$ do not satisfy the Gauss-Codazzi-Mainardi compatibility conditions, and the system is a non-Euclidean shells. For sufficiently thin shells, we expect the equilibrium configuration to be very close to an isometric immersion (the maximum deviation scales like $O(t)$, and is concentrated in a layer of size $O(t^{1/2})$ near the boundary [6]. As pointed up above, there are two such isometries that retain the axisymmetry of the geometry. The one that is more stable is the one that has a lower bending content, i.e., it depends on the sign of the constant θ.

Let $\theta > 0$. Then, provided that the body is sufficiently thin, the global equilibrium configuration is close to the isometric immersion (21). Suppose then that θ is continuously varied, and eventually changes sign. Despite the fact that (21) is no longer (close to) the absolute energy minimizer, it remains a local minimum, i.e., it is meta-stable. The extent of meta-stability depends both on the thickness (the thinner the shell is, the more stable is the meta-stable state), and on the parameter θ of the reference curvature. It is natural to expect that at a critical value of $\theta < 0$ (which depends on t) the solution (21) becomes unstable, leading to a transition to the second solution (of opposite mean curvature). For thin shells, we expect this transition to be fast.

Such a mechanism is believed to govern the motion of the well-known Venus flytrap, a carnivorous plant which captures insects by closing a pair of lobes. The closing (or snapping) of the lobes is triggered when an insect stimulates the surface of the lobes. Recent work [7] analyzes the snapping mechanism in terms of geometric and elastic properties of the lobes. In essence, the trapping mechanism is based on a bistability of the type discussed above. The stimulus triggers changes in the geometry of the lobe from being concave to being convex. The rapidity of the trap is due to the fact that it is an elastic response to a loss of stability. In this context, we note that for the trap to be efficient, the thickness of the lobes has to be within some intermediate range. A too thick lobe would imply a bending dominated response, i.e., a continuous tracking of the (slow) geometrical changes. A too thin lobe, in the other hand, would render the meta-stable energy wells too deep for allowing transitions. This should not be viewed as a thorough study of the Venus flytrap; we only point out that the snapping mechanism could be interpreted within the framework of non-Euclidean shells.

5 Conclusions

We have presented here a theory for thin elastic bodies that do not possess a stress-free rest state due to a geometric incompatibility. We have named such bodies non-Euclidean plates and shell. The essential ingredient is a three-dimensional elasticity theory in which the strain is measured with respect to a reference metric, which is a material property. A lack of stress-free rest configuration occurs when this reference metric is non-flat. The reduction to quasi-two-dimensional theories is done using the Kirchhoff assumption whereby stresses along the thin axis are vanishingly small in the limit of a thin body. We obtain a Koiter-like energy functional, with a stretching energy in which the strain is relative to a two-dimensional reference metric. Incompatibility occurs when the first and second reference forms do not satisfy the Gauss-Codazzi-Mainardi equations. For uneven plates, we recover the model in [5] with, however, a non-uniform thickness. The potential uses of our model in both physical and biological sciences were demonstrated by analyzing two model systems.

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