Introduction to Mathematical Computation

Assignment #7

Exercise 7.1 Write an algorithm (i.e., a list of intructions in some pseudo-code) that calculates the solution to the linear system, $Ax = b$, by Gauss-Seidel's iterative procedure. The algorithm receives as input the matrix A and the vector b, and returns the solution x. Try to make the algorithm efficient.

Exercise 7.2 Show that the Jacobi iteration converges for 2-by-2 symmetric positive-definite systems.

Hint Suppose that the matrix to be inverted is

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

First, express the positive-definiteness of A as a condition on a, b, c. Then, proceed to write the matrix $(I - Q^{-1}A)$, where Q is the splitting matrix corresponding to the Jacobi iterative procedure. It remains to find a norm in which $\|I - Q^{-1}A\| < 1$ or compute the spectral radius.

Exercise 7.3 Will Jacobi's iterative method converge for

$$\begin{pmatrix} 10 & 2 & 3 \\ 4 & 50 & 6 \\ 7 & 8 & 90 \end{pmatrix}.$$

Exercise 7.4 Explain why at least one eigenvalue of the Gauss-Seidel iterative matrix must be zero.

Exercise 7.5 Show that if A is strictly diagonally dominant then the Gauss-Seidel iteration converges.

Exercise 7.6 What is the explicit form of the iteration matrix $G = (I - Q^{-1}A)$ in the Gauss-Seidel method when

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

Exercise 7.7 (Computer exercise) Solve the system

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

using both the Jacobi and the Gauss-Seidel iterations. Plot a graph of the norm of the errors as function of the number of iterations. Use the same graph for both methods for comparison.