Introduction to Mathematical Computation

Assignment #5

Exercise 5.1 A matrix is called ***normal*** if it has a complete set of orthogonal eigenvectors. Show that for normal matrices,

$$\|A\|_2 = \operatorname{spr} A.$$

Exercise 5.2 The ***spectrum*** $\Sigma(A)$ of a matrix A is the set of its eigenvalues. The ϵ***-pseudospectrum*** of A, which we denote by $\Sigma_\epsilon(A)$, is defined as the set of complex numbers z, for which there exists a matrix δA such that $\|\delta A\|_2 \le \epsilon$ and z is an eigenvalue of $A + \delta A$. In mathematical notation,

$$\Sigma_\epsilon(A) = \{z \in \mathbb{C} : \ \exists\, \delta A, \ \|\delta A\|_2 \le \epsilon, \ z \in \Sigma(A + \delta A)\}.$$

Show that

$$\Sigma_\epsilon(A) = \left\{z \in \mathbb{C} : \ \|(zI - A)^{-1}\|_2 \ge 1/\epsilon\right\}.$$

Exercise 5.3 Let $Ax = b$ and $(A + \delta A)\hat{x} = (b + \delta b)$. We showed in class that $\delta x = \hat{x} - x$ satisfies the inequality,

$$\|\delta x\|_2 \le \|A^{-1}\|_2 \left(\|\delta A\|_2 \|\hat{x}\|_2 + \|\delta b\|_2\right).$$

Show that this is not just an upper bound: that for sufficiently small $\|\delta A\|_2$ there exist non-zero δA, δb such that the above in an equality. (**Hint**: follow the lines of the proof that links the reciprocal of the condition number to the distance to the nearest ill-posed problem.)