Introduction to Mathematical Computation

Assignment #4

Exercise 4.1 Show that the p-norms do indeed satisfy the properties of a norm.

Exercise 4.2 Prove the following inequalities for vector norms:

$$\begin{aligned}
\|x\|_2 \le \|x\|_1 &\quad \le \sqrt{n}\|x\|_2 \\
\|x\|_\infty \le \|x\|_2 &\quad \le \sqrt{n}\|x\|_\infty \\
\|x\|_\infty \le \|x\|_1 &\quad \le n\,\|x\|_\infty .
\end{aligned}$$

Exercise 4.3 Show that for every invertible matrix A and norm $\|\cdot\|$,

$$\|A\|\|A^{-1}\| \ge 1.$$

Exercise 4.4 Prove that the matrix norm subordinate to the vector norm $\|\cdot\|_1$ is

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.$$

Exercise 4.5 ① Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and S be an $n\times n$ non-singular matrix. Define $\|x\|' = \|Sx\|$, and prove that $\|\cdot\|'$ is a norm on $\mathbb{R}^n$.

② Let $\|\cdot\|$ be the matrix norm subordinate to the above vector norm. Define $\|A\|' = \|SAS^{-1}\|$, and prove that $\|\cdot\|'$ is the matrix norm subordinate to the corresponding vector norm.

Exercise 4.6 True or false: if $\|\cdot\|$ is a matrix norm subordinate to a vector norm, so is $\|\cdot\|' = \frac{1}{2}\|\cdot\|$ (the question is not just whether $\|\cdot\|'$ satisfies the definition of a norm; the question is whether there exists a vector norm, for which $\|\cdot\|'$ is the subordinate matrix norm!).

Exercise 4.7 Show that $\operatorname{spr} A < 1$ if and only if

$$\lim_{k\to\infty} A^k x = 0, \qquad \forall x.$$

Exercise 4.8 True or false: the spectral radius $\operatorname{spr} A$ is a matrix norm.

Exercise 4.9 (Computer exercise) Construct a "random" 6×6 matrix A. Then plot the 1,2, and infinity norms of $\|A^n\|^{1/n}$ as function of n with the maximum n large enough so that the three curves are sufficiently close to the expected limit.