Introduction to Mathematical Computation

Assignment #3

Exercise 3.1 Let p be a positive number. What is the value of the following expression:

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}.$$

By that, I mean the sequence $x_0 = p$, $x_{k+1} = \sqrt{p + x_k}$. (Interpret this as a fixed-point problem.)

Exercise 3.2 Show that the function

$$F(x) = 2 + x - \tan^{-1} x$$

satisfies $|F'(x)| < 1$. Show then that $F(x)$ doesn't have fixed points. Why doesn't this contradict the contractive mapping theorem?

Exercise 3.3 Bailey's iteration for calculating $\sqrt{a}$ is obtained by the iterative scheme:

$$x_{n+1} = g(x_n) \qquad g(x) = \frac{x(x^2 + 3a)}{3x^2 + a}.$$

Show that this iteration is of order at least three.

Exercise 3.4 (Here is an exercise which tests whether you *really* understand what root finding is about.) One wants to solve the equation $x + \ln x = 0$, whose root is $x \sim 0.5$, using one or more of the following iterative methods:

$$(i) \quad x_{k+1} = -\ln x_k \qquad (ii) \quad x_{k+1} = e^{-x_k} \qquad (iii) \quad x_{k+1} = \frac{x_k + e^{-x_k}}{2}.$$

① Which of the three methods *can* be used?

② Which method *should* be used?

③ Give an even better iterative formula; explain.

Exercise 3.5 Your dog chewed your calculator and damaged the division key! To compute reciprocals (i.e., one-over a given number R) without division, we can solve $x = 1/R$ by finding a root of a certain function f with Newton's method. Design such an algorithm (that, of course, does not rely on division).

Exercise 3.6 Prove that if r is a root of multiplicity k (i.e., $f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0$ but $f^{(k)}(r) \neq 0$), then the quadratic convergence of Newton's method will be restored by making the following modification to the method:

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}.$$

Exercise 3.7 Similarly to Newton's method (in one variable), derive a method for solving $f(x)$ given the functions $f(x)$, $f'(x)$ and $f''(x)$. What is the rate of convergence?

Exercise 3.8 What special properties must a function f have if Newton's method applied to f converges cubically?

Exercise 3.9 Go the the following site and enjoy the nice pictures:

`http://aleph0.clarku.edu/~djoyce/newton/newton.html`

(Read the explanations, of course....)

Exercise 3.10 (Computer exercise) Use Newton's method to solve the system of equations

$$\begin{aligned} xy^2 + x^2y + x^4 &= 3 \\ x^3y^5 - 2x^5y - x^2 &= -2. \end{aligned}$$

Start with various initial values and try to characterize the "basin of convergence" (the set of initial conditions for which the iterations converge).

Now, Matlab has a built-in root finder `fsolve()`. Try to solve the same problem using this functions, and evaluate whether it performs better or worse than your own program in terms of both speed and robustness.