Introduction to Mathematical Computation

Assignment #2

Exercise 2.1 Find a positive root of

$$x^2 - 4x\sin x + (2\sin x)^2 = 0$$

accurate to two significant digits. *Use a hand calculator!*

Exercise 2.2 The two following sequences constitute iterative procedures to approximate the number $\sqrt{2}$:

$$x_{n+1} = x_n - \frac{1}{2}(x_n^2 - 2), \qquad x_0 = 2,$$

and

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \qquad x_0 = 2.$$

① Calculate the first six elements of both sequences.

② Calculate (numerically) the error, $e_n = x_n - \sqrt{2}$, and try to estimate the order of convergence.

③ Estimate the order of convergence by Taylor expansion.

Exercise 2.3 Let a sequence x_n be defined inductively by

$$x_{n+1} = F(x_n).$$

Suppose that $x_n \to x$ as $n \to \infty$ and that $F'(x) = 0$. Show that $x_{n+2} - x_{n+1} = o(x_{n+1} - x_n)$. (Hint: assume that F is continuously differentiable and use the mean value theorem.)

Exercise 2.4 Analyze the following iterative method,

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)},$$

designed for the calculation of the roots of $f(x)$ (this method is known as Steffensen's method). Prove that this method converges quadratically (order 2) under certain assumptions.

Exercise 2.5 Kepler's equation in astronomy is $x = y - \epsilon \sin y$, with $0 < \epsilon < 1$. Show that for every $x \in [0, \pi]$, there is a y satisfying this equation. (Hint: Interpret this as a fixed-point problem.)

Exercise 2.6 (Computer exercise) Write a Matlab function which gets for input the name of a real-valued function f, an initial value x_0, a maximum number of iterations M, and a tolerance ϵ. Let your function then perform iterations based on Newton's method for finding roots of f, until either the maximum of number iterations has been exceeded, or the convergence criterion $|f(x)| \leq \epsilon$ has been reached. Experiment your program on the function $f(x) = \tan^{-1} x$, whose only root is $x = 0$. Try to characterize those initial values x_0 for which the iteration method converges.