Introduction to Mathematical Computation

Assignment #10

Exercise 10.1 Show that the set of polynomials,

$$\phi_0(x) = \frac{1}{\sqrt{\pi}} \qquad \phi_k(x) = \frac{2}{\sqrt{\pi}} T_k(x), \quad k = 1, 2, \ldots,$$

where $T_k(x)$ are the Chebyshev polynomials, form an orthonormal basis on the segment $[-1, 1]$ with respect to the inner product,

$$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, \frac{dx}{\sqrt{1 - x^2}}.$$

Derive an expression for the best approximation of continuous functions in the interval $[-1, 1]$ with respect to the norm

$$\|f\|^2 = \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1 - x^2}}\, dx,$$

where the approximating function is a polynomial of degree less or equal n.

Exercise 10.2 Consider the space $C[-1, 1]$ endowed with inner product

$$(f, g) = \int_{-1}^{i} f(x) g(x)\, dx.$$

Use the Gram-Schmidt orthonormalization procedure to construct a basis for span $\{1, x, x^2, x^3\}$.

Exercise 10.3 Let X be an inner product space, and G a subspace spanned by the orthonormal vectors $\{g_1, g_2, \ldots, g_n\}$. For every $f \in X$ denote by Pf the best L^2-approximation of f by an element of G. Find an explicit formula for $\|f - Pf\|$.

Exercise 10.4 Suppose that we want to approximate an even function f by a polynomial $p_n \in \Pi_n$, using the norm

$$\|f\| = \left(\int_{-1}^{1} f^2(x)\, dx \right)^{1/2}.$$

Prove that p_n is also even.