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Long-time limit for a class of quadratic infinite-dimensional dynamical systems inspired by models of viscoelastic fluids

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Abstract

We study a class of quadratic, infinite-dimensional dynamical systems, inspired by models for viscoelastic fluids. We prove that these equations define a semi-flow on the cone of positive, essentially bounded functions. As time tends to infinity, the solutions tend to an equilibrium manifold in the L^2-norm. Convergence to a particular function on the equilibrium manifold is only proved under additional assumptions. We discuss several possible generalizations.

MSC: 35F25; 37C70; 35Q72

Keywords: Quadratic differential systems; Viscoelastic toy model; Global attractor; Equilibrium manifold

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1. Introduction

This paper is concerned with evolution equations of the form

$$\frac{dy}{dt} = y\mathcal{P}(a - y), \qquad y(x, 0) = y_0(x), \tag{1}$$

where $y(\cdot, t)$ is an unknown and $a(\cdot)$ is a given real-valued function, both defined on a measure space (Ω, μ) with finite mass ($\mu(\Omega) < \infty$). The operator $\mathcal{P}$ is an orthogonal projection on the Hilbert space $L^2 = L^2(\Omega, \mu)$. We will use the standard notations $(\cdot,\cdot)$ for the inner product in L^2, and $\|\cdot\|_p$ for the $L^p = L^p(\Omega, \mu)$ norms. We denote by $L^{\infty,+}$ the cone of strictly positive functions in $L^\infty = L^\infty(\Omega, \mu)$:

$$L^{\infty,+} = \Big\{ u \in L^\infty \colon \operatorname*{ess\,inf}_{x\in\Omega} u(x) > 0 \Big\}.$$

Eq. (1) is subject to the following assumptions:

Assumption 1.

(i) The operator $\mathcal{P} : L^2 \to L^2$ is an orthogonal projection, satisfying $\mathcal{P}(L^\infty) \subset L^\infty$. Its null space, $\mathcal{N}(\mathcal{P})$, is one-dimensional, spanned by an essentially positive function $n \in L^{\infty,+}$, which we take to be normalized, $(n, n) = 1$.

(ii) The function $a(x) \in L^\infty$. Without loss of generality, we can assume that

$$\mathcal{P}(a) = a.$$

The system (1) is a toy model inspired by models of viscoelastic fluids. Specifically, the Oldroyd-B model for incompressible viscoelastic fluids in the creeping flow regime consists of a Stokes system,

$$-\nabla p + \nu_s \Delta \boldsymbol{u} + G \operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{a}) = 0, \qquad \operatorname{div} \boldsymbol{u} = 0, \tag{2}$$

coupled to the Maxwell constitutive equation [1]

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\sigma} = (\nabla \boldsymbol{u})^T \boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \boldsymbol{u}) + \frac{1}{\lambda}(\boldsymbol{\sigma} - I). \tag{3}$$

Here $\boldsymbol{u}(x, t)$ is the velocity field, $p(x, t)$ is the pressure, ν_s is the solvent viscosity, and G is the elastic modulus. The matrix-valued function $\boldsymbol{\sigma}(x, t)$ is the so-called conformation tensor, a quantity associated with the microscopic structure of the polymers, which is closely related to the stress-tensor; it is a symmetric positive-definite tensor field. The tensor field $\boldsymbol{a} = \boldsymbol{a}(x)$ is an external forcing (without loss of generality a vector-valued force field can be written as the divergence of some tensor field). Finally, λ is the elastic relaxation time.

The system (2), (3) poses both analytical and numerical challenges. As explained in [3], it is of interest to obtain better insight into the nonlinear feedback mechanism between the velocity field $\boldsymbol{u}$ and the conformation tensor $\boldsymbol{\sigma}$. To this end, we replace the system (2), (3), by a closely related toy model in which the advection and relaxation of the conformation tensor have been discarded. (Note that advection term does not increase the L^∞ norm, but may well cause the increase of higher order Sobolev norms. Yet, it was shown in [2] that global in time existence hinges precisely on the finiteness of the L^∞ norm, which motivates the omission of the advection term which is therefore not directly responsible for potential blowups.) This yields a model of the form

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = (\nabla \boldsymbol{u})^T \boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla \boldsymbol{u}), \tag{4}$$

coupled to the Stokes system (2).

The Stokes system induced a linear mapping $(\boldsymbol{\sigma} - \boldsymbol{a}) \mapsto \nabla \boldsymbol{u}$, which can be explicitly written by means of a Green function,

$$\nabla \boldsymbol{u}(x) = \int_\Omega G_\Omega(x, y) \cdot \operatorname{div}\big(\boldsymbol{\sigma}(y) - \boldsymbol{a}(y)\big)\, dy \equiv -\mathcal{P}(\boldsymbol{\sigma} - \boldsymbol{a})(x),$$

where Ω is the domain (which may be bounded on not) and G_Ω is the corresponding Stokes kernel (i.e., the Green function of the Stokes elliptic system). It can easily be shown that the linear mapping $\mathcal{P}$ is, in fact, an orthogonal projection (see [3]). Thus, (4) takes the form

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \big[\mathcal{P}(\boldsymbol{a} - \boldsymbol{\sigma})\big]^T \boldsymbol{\sigma} + \boldsymbol{\sigma}\mathcal{P}(\boldsymbol{a} - \boldsymbol{\sigma}). \tag{5}$$

The system (1) is a one-dimensional scalar toy model, that mimics the dynamics (5).

Eq. (1) can also be viewed as an infinite-dimensional generalization of a Lotka–Volterra system [4]. In Section 2 we prove that (1) defines a global (in time) semi-flow on the cone of positive functions $L^{\infty,+}$ (Theorem 2.2 in Section 2). We then proceed to analyze the long-time behavior of this system. It is clear that every function y satisfying $\mathcal{P}(y) = \mathcal{P}(a)$ is an equilibrium solution of (1), and these are the only equilibria in $L^{\infty,+}$. Our main theorem asserts that the equilibrium set

$$\mathcal{M} := \big\{y \in L^{\infty,+}\colon \mathcal{P}(y) = a\big\}$$

is the global attractor for all initial data $y_0 \in L^{\infty,+}$ (Theorem 3.1 in Section 3). The convergence of $y(\cdot, t)$ to the manifold $\mathcal{M}$, as $t \to \infty$, is in the L^2-norm. The theorem does not guarantee uniform convergence, nor does it guarantee that $y(\cdot, t)$ converges to a specific equilibrium in $\mathcal{M}$. For this to happen, additional assumptions are made; various situations are considered in Section 4. We conclude this paper with a discussion about open questions and various generalizations.

2. Global existence

We start by establishing the well-posedness of Eq. (1) under Assumption 1. The first step is to show existence and uniqueness of solutions for short times:

Theorem 2.1 *(Local-in-time existence and uniqueness). Let $y_0 \in L^\infty$ be given. Then there exist times $T_1, T_2 > 0$, depending on y_0 only, such that* (1) *has a unique solution $y \in C^1((-T_1, T_2); L^\infty)$.*

Proof. Note first that due to Assumption 1 the operator $\mathcal{P}$ has the explicit form

$$\mathcal{P}(z) = z - (z, n)n.$$

It is a bounded linear operator $L^\infty \to L^\infty$ since

$$\|\mathcal{P}(z)\|_\infty \leqslant \big(1 + \mu(\Omega)\|n\|_\infty^2\big)\|z\|_\infty.$$

We rewrite (1) as

$$\frac{dy}{dt} = ya - y\mathcal{P}(y) \equiv F(y).$$

The short-time existence and uniqueness of solutions follows from Picard's theorem over Banach spaces, provided that F is a locally Lipschitz continuous mapping $L^\infty \to L^\infty$. This is indeed the case as $\mathcal{P}$ is a bounded operator, hence it is locally Lipschitz, and the product of locally Lipschitz functions is again locally Lipschitz. □

We then show that solutions that are initially positive remain so at all times:

Proposition 2.1 *(Positivity). Let* $y \in C^1((-T_1, T_2); L^\infty)$ *be a solution of* (1)*, with initial condition* $y_0 \in L^{\infty,+}$*. Then* $y(x,t)$ *remains positive, i.e.,* $\operatorname{ess\,inf}_{x\in\Omega} y(x,t) > 0$*, for all* $t \in (-T_1, T_2)$*. In other words, the cone* $L^{\infty,+}$ *is an invariant set for the dynamics.*

Proof. The positivity follows readily from the fact that the unique solution of (1) solves the integral equation

$$y(\cdot,t) = y_0 \exp\left(\int_0^t \mathcal{P}\big(a - y(\cdot,s)\big)\,ds\right). \qquad \square \tag{6}$$

The next step is to show that the solution with initial data in $L^{\infty,+}$, as long as it exists, is bounded, uniformly in time, in L^∞, by a constant that only depends on the initial data. The proof relies on the fact that the dynamics (1) subject to Assumption 1 preserve the natural order among functions. To simplify notations, we define $\mathcal{Q} := \mathcal{I} - \mathcal{P}$ to be the orthogonal complement of the projection $\mathcal{P}$, namely, $\mathcal{Q}y = (n, y)n$.

Lemma 2.1. *Let* $y \in L^\infty$ *be a non-negative function,* $y \geqslant 0$*. Then,*

$$\operatorname*{ess\,inf}_{\Omega} \mathcal{Q}y(\cdot) \geqslant 0,$$

with equality if and only if $y = 0$.

Proof. The non-negativity of y and the positivity of n imply that

$$\operatorname*{ess\,inf}_{\Omega} \mathcal{Q}y(\cdot) = (n, y) \cdot \operatorname*{ess\,inf}_{\Omega} n(\cdot) \geqslant 0.$$

Since $\operatorname{ess\,inf} n(x) > 0$ equality occurs if and only if $(n, y) = 0$, i.e., if and only if $y = 0$. □

Proposition 2.2 *(Comparison principle). Let* $y, z \in C^1([0,T); L^{\infty,+})$ *be two solutions of* (1) *with initial data* $y_0, z_0 \in L^{\infty,+}$*. If* $y_0 \geqslant z_0$ *a.e. in* Ω *then*

$$y(\cdot,t) \geqslant z(\cdot,t) \tag{7}$$

a.e. in Ω *for all* $0 \leqslant t < T$.

Proof. Let t_0 be the supremum of all values of $t \geqslant 0$ for which the statement holds, i.e., $y(x,t) \geqslant z(x,t)$ a.e. in Ω for all $0 \leqslant t \leqslant t_0$ (it is possible that $t_0 = 0$). If $t_0 = \infty$, there is nothing to prove. If $t_0 < \infty$, then by definition

$$y(\cdot,t_0) \geqslant z(\cdot,t_0). \tag{8}$$

It follows, by Lemma 2.1 that

$$C := \operatorname*{ess\,inf}_{\Omega} \mathcal{Q}\big(y(\cdot,t_0) - z(\cdot,t_0)\big) > 0.$$

We now define the following sets

$$\Omega_+ := \left\{x \in \Omega\colon\ y(x,t_0) - z(x,t_0) > \frac{C}{2}\right\},$$
$$\Omega_- := \left\{x \in \Omega\colon\ y(x,t_0) - z(x,t_0) \leqslant \frac{C}{2}\right\}.$$

By the continuity of the mappings $t \mapsto y(\cdot,t)$ and $t \mapsto z(\cdot,t)$ from $[0,T)$ to L^∞, there exists a time interval $\delta_1 > 0$ such that

$$y(x,t) > z(x,t) \quad \text{for all } t \in [t_0, t_0+\delta_1),\ \text{for a.e. } x \in \Omega_+. \tag{9}$$

We then turn our attention to the set Ω_-, where

$$\begin{aligned}\operatorname*{ess\,sup}_{\Omega_-} \mathcal{P}\big(y(\cdot,t_0) - z(\cdot,t_0)\big) &\leqslant \operatorname*{ess\,sup}_{\Omega_-}\big[y(\cdot,t_0) - z(\cdot,t_0)\big] \\ &\quad - \operatorname*{ess\,inf}_{\Omega_-} \mathcal{Q}\big(y(\cdot,t_0) - z(\cdot,t_0)\big) \leqslant \frac{C}{2} - C. \end{aligned} \tag{10}$$

By the differentiability of the mappings $t \mapsto \log y(\cdot,t)$ and $t \mapsto \log z(\cdot,t)$ from $[0,T)$ to L^∞, there exists for every $\epsilon > 0$ a time interval $\delta_2 > 0$, such that for all $t \in [t_0, t_0+\delta_2)$,

$$\big\|\log y(\cdot,t) - \log y(\cdot,t_0) - (t-t_0)\mathcal{P}\big(a - y(\cdot,t_0)\big)\big\|_\infty < \epsilon(t-t_0),$$
$$\big\|\log z(\cdot,t) - \log z(\cdot,t_0) - (t-t_0)\mathcal{P}\big(a - z(\cdot,t_0)\big)\big\|_\infty < \epsilon(t-t_0).$$

Thus, for $t \in [t_0, t_0+\delta_2)$,

$$\log \frac{y(\cdot,t)}{z(\cdot,t)} \geqslant \log \frac{y(\cdot,t_0)}{z(\cdot,t_0)} - (t-t_0)\mathcal{P}\big(y(\cdot,t_0)-z(\cdot,t_0)\big) - 2\epsilon(t-t_0)$$
$$\geqslant -(t-t_0)\mathcal{P}\big(y(\cdot,t_0)-z(\cdot,t_0)\big) - 2\epsilon(t-t_0),$$

where the last inequality results from (8). Choosing $\epsilon = C/8$ and using (10) we have that for $t \in [t_0, t_0+\delta_2)$,

$$\inf_{\Omega_-} \log \frac{y(\cdot,t)}{z(\cdot,t)} \geqslant \frac{C}{4}(t-t_0) \geqslant 0. \tag{11}$$

Taking $\delta = \min(\delta_1, \delta_2)$ and combining (9) and (11) we obtain that

$$y(\cdot,t) \geqslant z(\cdot,t) \quad \text{for all } t \in [t_0, t_0+\delta).$$

Thus, (7) holds for all $t \in [t_0, t_0+\delta)$ in contradiction with the definition of t_0, which concludes the proof. □

The comparison principle guarantees the boundedness of $y(\cdot,t)$:

Proposition 2.3 *(Boundedness in L^∞). Let $y \in C^1([0,T); L^{\infty,+})$ be a solution of* (1) *with initial data y_0. Then there exists a constant $K > 0$, given by* (13) *and depending on the initial data, such that*

$$\sup_{0 \leqslant t < T} y(\cdot,t) \leqslant a + Kn(x). \tag{12}$$

Proof. Since $\operatorname{ess\,inf}_\Omega n(x) > 0$, then there exists, given y_0, a constant $K > 0$ such that

$$z(x) \equiv a(x) + Kn(x) \geqslant y_0(x) \quad \text{a.e. in } \Omega.$$

Specifically, we can choose

$$K = \operatorname*{ess\,sup}_{x \in \Omega} \frac{y_0(x) - a(x)}{n(x)}. \tag{13}$$

The function z is an equilibrium solution of (1), and by the previous proposition $y(\cdot,t) \leqslant z$ for all $0 \leqslant t < T$. □

Theorem 2.2 *(Global existence). Let $y_0 \in L^{\infty,+}$ be given. Then* (1) *has a unique solution $y \in C^1([0,\infty); L^{\infty,+})$.*

Proof. This is a direct consequence of the short-time existence and uniqueness (Theorem 2.1) and the bound (12) for initial data $y_0 \in L^{\infty,+}$. By the continuation theorem for autonomous ODEs, if $T < \infty$ and $[0,T)$ is the maximal time of existence of the solution y, then

$$\limsup_{t \nearrow T^-} \big\| y(\cdot,t) \big\|_\infty = \infty.$$

Since the norm $\|y(\cdot,t)\|_\infty$ is continuous in time, this violates the bound (12), hence the maximal existence time is infinite. □

3. Asymptotic convergence of $y(\cdot, t)$ to $\mathcal{M}$

Having established the global existence and boundedness of solutions to (1), we proceed to study the long-term behavior of these dynamics. As in the previous section, it is always assumed that system (1) satisfies Assumption 1. The first proposition establishes the existence of an integral of motion:

Proposition 3.1. *The functional $\Gamma : L^{\infty,+} \to \mathbb{R}$ defined by*

$$\Gamma(z) := \int_\Omega n(x) \log z(x)\, d\mu(x),$$

is an integral of motion, that is, if $y \in C^1(\mathbb{R}^+; L^{\infty,+})$ is a solution of (1)*, then*

$$\Gamma\big(y(\cdot, t)\big) = \Gamma(y_0)$$

for all $t \geqslant 0$.

Proof. Differentiating we get

$$\frac{d}{dt}\Gamma\big(y(\cdot, t)\big) = \int_\Omega n(x) \frac{\frac{d}{dt} y(x, t)}{y(x, t)}\, d\mu(x) = \big(n, \mathcal{P}\big(a - y(\cdot, t)\big)\big) = 0,$$

where the last equality follows from the symmetry of $\mathcal{P}$ and the fact that $n \in \mathcal{N}(\mathcal{P})$. $\square$

The next two propositions reveal the "dissipative" nature of (1) through the construction of two Lyapunov functionals. Note that by considering the equilibrium, $\tilde{y}(x) = a(x) + \gamma n(x)$ for sufficiently large γ, we have

$$\operatorname*{ess\,inf}_\Omega \tilde{y}(\cdot) > 0 \quad \text{and} \quad \mathcal{P}(\tilde{y}) = a.$$

Proposition 3.2. *Let $y \in C^1(\mathbb{R}^+; L^{\infty,+})$ be a solution of* (1) *with $\tilde{y}(x)$ defined as above. Then the "entropy" functional*

$$V_a\big[y(\cdot, t)\big] := \int_\Omega \tilde{y}(x)\left[\frac{y(x, t)}{\tilde{y}(x)} - \log \frac{y(x, t)}{\tilde{y}(x)}\right] d\mu(x)$$

is positive and non-increasing in time.

Proof. The positivity of V_a follows from the fact that $z - \log(z) \geqslant 1$ for $z > 0$, and the positivity of $y(x, t)$ and $\tilde{y}(x)$. Differentiating along trajectories we get

$$\begin{aligned}\frac{d}{dt} V_a\big[y(\cdot, t)\big] &= \int_\Omega \frac{y_t(x, t)}{y(x, t)}\big[y(x, t) - \tilde{y}(x)\big]\, d\mu(x) = \big(y(\cdot, t) - \tilde{y}, \mathcal{P}\big(a - y(\cdot, t)\big)\big) \\ &= \big(\mathcal{P}\big(y(\cdot, t) - \tilde{y}\big), \mathcal{P}\big(a - y(\cdot, t)\big)\big) = -\big\|\mathcal{P}\big(a - y(\cdot, t)\big)\big\|_2^2 \leqslant 0,\end{aligned}$$

where we have used the fact that $\mathcal{P}$ is an orthogonal projection and $\mathcal{P}(\tilde{y}) = a$. $\square$

Proposition 3.3. *Let $y \in C^1(\mathbb{R}^+; L^{\infty,+})$ be a solution of* (1). *Then the "energy" functional*

$$V_b\big[y(\cdot,t)\big] := \big\|\mathcal{P}\big(y(\cdot,t)-a\big)\big\|_2^2$$

is non-increasing in time.

Proof. By explicit differentiation along trajectories we get

$$\begin{aligned}\frac{d}{dt}V_b\big[y(\cdot,t)\big] &= 2\big(\mathcal{P}\big(y(\cdot,t)-a\big), \mathcal{P}\big(y(\cdot,t)\mathcal{P}\big(a-y(\cdot,t)\big)\big)\big)\\ &= -2\big(\mathcal{P}\big(y(\cdot,t)-a\big), y(\cdot,t)\mathcal{P}\big(y(\cdot,t)-a\big)\big)\\ &= -2\big\|y^{1/2}(\cdot,t)\mathcal{P}\big(y(\cdot,t)-a\big)\big\|_2^2 \leqslant 0,\end{aligned}$$

where we have used the properties of $\mathcal{P}$ and the positivity of y. $\square$

The identification of the two Lyapunov functionals yields immediately the asymptotic convergence of $y(\cdot,t)$ to the equilibrium manifold $\mathcal{M}$.

Theorem 3.1. *Let $y \in C^1(\mathbb{R}^+; L^{\infty,+})$ be a solution of* (1). *Then*

$$\lim_{t\to\infty}\mathcal{P}\big(y(\cdot,t)\big) = a \quad \text{in } L^2.$$

Proof. We need to prove that

$$\big(\mathrm{dist}_{L^2}\big(y(\cdot,t),\mathcal{M}\big)\big)^2 = \big\|\mathcal{P}\big(y(\cdot,t)-a\big)\big\|_2^2 = V_b\big(y(\cdot,t)\big)$$

tends to zero as $t\to\infty$. Since the functionals V_a, V_b are both non-negative, bounded from above (Proposition 2.3) and non-increasing in time, both must converge to limits as $t\to\infty$. Since, furthermore,

$$\frac{d}{dt}V_a\big[y(\cdot,t)\big] = -V_b\big[y(\cdot,t)\big],$$

the limit of V_b must be zero. $\square$

Example. Assume $\mu(\Omega)=1$ and let $\mathcal{P}$ be the orthogonal projection in L^2 to the space of constants, i.e.,

$$(\mathcal{P}f)(x) = f(x) - \int_\Omega f(x')\,d\mu(x'),$$

and $a \in L^\infty$ satisfies

$$\int_\Omega a(x)\,d\mu(x) = 0.$$

The system (1) takes the form

$$\frac{\partial}{\partial t} y(x,t) = y(x,t)\left(a(x) + \int_\Omega y(x',t)\,d\mu(x') - y(x,t)\right), \tag{14}$$

with initial condition $y(\cdot,0) = y_0 \in L^{\infty,+}$. Theorem 2.2 asserts the existence of a global solution $y \in C^1(\mathbb{R}^+; L^{\infty,+})$. By Proposition 2.3 there exists a constant $K > 0$ such that

$$\sup_{t \geqslant 0} y(\cdot,t) \leqslant a + K.$$

Finally, by Theorem 3.1,

$$\lim_{t\to\infty}\left(y(\cdot,t) - \int_\Omega y(x',t)\,d\mu(x')\right) = a \quad \text{in } L^2.$$

4. Asymptotic convergence of $y(\cdot,t)$

We now inquire under what conditions does $y(\cdot,t)$ converge, as $t \to \infty$, to a specific equilibrium in $\mathcal{M}$. Note that the L^2-convergence of $y(\cdot,t)$ can be decomposed into

$$\lim_{t\to\infty} y(\cdot,t) = \lim_{t\to\infty} \mathcal{P}\big(y(\cdot,t)\big) + \lim_{t\to\infty} \mathcal{Q}\big(y(\cdot,t)\big),$$

where

$$\mathcal{Q}\big(y(\cdot,t)\big) = \big(n, y(\cdot,t)\big)n.$$

We have just proved that the first term on the right-hand side converges to a. It remains to verify under what conditions

$$\beta(t) := \big(y(\cdot,t), n\big) \tag{15}$$

converges as $t \to \infty$.

Since, on the one hand, $\mathcal{M}$ consists of functions of the form $a(x) + \alpha n(x)$, for some $\alpha \in \mathbb{R}$, and on the other hand, by Proposition 3.1 the functional $\Gamma(y(\cdot,t))$ is conserved, the existence of a limiting solution in $\mathcal{M}$ requires the following assumption:

Assumption 2. There exists some $y^* \in \mathcal{M}$ such that

$$\int_\Omega n(x)\log y_0(x)\,d\mu(x) = \int_\Omega n(x)\log y^*(x)\,d\mu(x). \tag{16}$$

Assumption 2 is a restriction on the initial conditions y_0. It assumes the existence of a constant α which solves the equation

$$\int_\Omega n(x)\log\big[a(x) + \alpha n(x)\big]\,d\mu(x) = \int_\Omega n(x)\log y_0(x)\,d\mu(x), \tag{17}$$

under the constraint that $\operatorname{ess\,inf}_\Omega [a(x) + \alpha n(x)] > 0$.

If we define the set $\mathcal{C} \subset \mathbb{R}$ by

$$\mathcal{C} = \Big\{ \xi \in \mathbb{R}\colon \operatorname*{ess\,inf}_{x \in \Omega} \big[a(x) + \xi n(x) \big] > 0 \Big\} \tag{18}$$

and $\Phi : \mathcal{C} \to \mathbb{R}$ by

$$\Phi(\xi) = \int_{\Omega} n(x) \log\big[a(x) + \xi n(x)\big]\, d\mu(x), \tag{19}$$

then Assumption 2 is equivalent to the statement

$$\int_{\Omega} n(x) \log y_0(x)\, d\mu(x) \in \Phi(\mathcal{C}).$$

Note that $\mathcal{C}$ is in fact an unbounded interval, for $\xi \in \mathcal{C}$ implies that $\xi_1 \in \mathcal{C}$ for all $\xi_1 > \xi$.

The next proposition shows that such an α, if it exists, is unique.

Proposition 4.1. *Given an initial data $y_0 \in L^{\infty,+}$, the function y^* satisfying Assumption 2, if it exists, is unique.*

Proof. Uniqueness follows at once from the fact that

$$\frac{d}{d\alpha} \Phi(\alpha) = \int_{\Omega} \frac{n^2(x)}{a(x) + \alpha n(x)}\, d\mu(x) > 0$$

for all $\alpha \in \mathcal{C}$. $\square$

Example. Consider again the example from the previous section. For concreteness set $\Omega = [0, 1]$, with μ the Lebesgue measure and $a(x) = \sin 2\pi x$. Then, since $n \equiv 1$, the equilibria in $\mathcal{M}$ consist of functions of the form

$$\sin 2\pi x + \alpha,$$

where $\alpha > 1$, i.e., $\mathcal{C} = (1, \infty)$. For $\alpha \in \mathcal{C}$,

$$\Phi(\alpha) = \int_0^1 \log\big[\sin 2\pi x + \alpha\big]\, dx > -\log 2.$$

It follows that Assumption 2 is satisfied if and only if

$$\int_0^1 \log y_0(x)\, dx > -\log 2.$$

The following proposition asserts that the convergence of $y(\cdot,t)$ is guaranteed if the solution remains bounded away from the boundaries of the cone of positive solutions $L^{\infty,+}$.

Proposition 4.2. *If*

$$\liminf_{t\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} y(x,t) > 0 \tag{20}$$

then Assumption 2 *is satisfied. Moreover,*

$$\lim_{t\to\infty} \beta(t) = \alpha,$$

where $\beta(t)$ *is given by* (15) *and* α *is the* (*unique*) *solution to* (17)*. Thus,* $y(\cdot,t) \to y^*$ *in* L^2*, where* $y^* = a + \alpha n$*.*

Proof. Take any sequence of times t_m that is increasing to infinity. Since $y(\cdot,t)$ is uniformly bounded in L^∞ (Proposition 2.3), then $\beta(t)$ is bounded, and there exists a subsequence t_{m_k} such that $\beta(t_{m_k})$ converges to a limit γ, hence

$$\lim_{k\to\infty} \mathcal{Q}\big(y(\cdot,t_{m_k})\big) = \gamma n \quad \text{ in } L^\infty.$$

Theorem 3.1 implies that

$$\lim_{k\to\infty} \big[y(\cdot,t_{m_k}) - \mathcal{Q}\big(y(\cdot,t_{m_k})\big) - a\big] = 0 \quad \text{ in } L^2,$$

from which follows that

$$\lim_{k\to\infty} y(x,t_{m_k}) = a(x) + \gamma n(x)$$

in L^2, and so it has a sub-subsequence $y(\cdot,t_{m_{k_j}})$ which converges a.e. in Ω. Note that (20) implies that a.e. $a(x)+\gamma n(x) > 0$. This implies that

$$\lim_{j\to\infty} n(x)\log y(x,t_{m_{k_j}}) = n(x)\log\big[a(x)+\gamma n(x)\big]$$

a.e. Moreover, from (20) and the fact that $y(\cdot,t)$ is uniformly bounded we also have

$$\sup_{t\geqslant 0}\big\|\log y(\cdot,t)\big\|_\infty < \infty.$$

Using Lebesgue's dominated convergence theorem we conclude that

$$\lim_{j\to\infty}\int_\Omega n(x)\log y(x,t_{m_{k_j}})\,d\mu(x) = \int_\Omega n(x)\log\big[a(x)+\gamma n(x)\big]\,d\mu(x).$$

By Proposition 3.1 we have for all m,

$$\int_{\Omega} n(x) \log y(x, t_m)\, d\mu(x) = \int_{\Omega} n(x) \log y_0(x)\, d\mu(x),$$

therefore

$$\int_{\Omega} n(x) \log\bigl[a(x) + \gamma n(x)\bigr]\, d\mu(x) = \int_{\Omega} n(x) \log y_0(x)\, d\mu(x).$$

Thus, Assumption 2 is satisfied and it follows, by the uniqueness of y^*, hence the uniqueness of α in (17), that $\gamma = \alpha$. We have shown that every sequence $\beta(t_m)$ has a subsequence $\beta(t_{m_{k_j}})$ which converges to α. It follows from an elementary theorem of calculus that $\beta(t)$ tends to α as $t \to \infty$. This completes the proof. □

Note the immediate corollary:

Corollary 4.1. *If Assumption* 2 *does not hold then*

$$\liminf_{t\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} y(x,t) = 0.$$

Condition (20) is a sufficient condition for $y(\cdot, t)$ to asymptotically converge to an element of $\mathcal{M}$. The problem is that it is a property of the solution, and it is not clear *a priori* when does it hold. In the remaining part of this section we establish two situations for which (20) holds. In the first case y_0 has to be sufficiently large in the following sense:

Proposition 4.3. *If there exists a constant K such that*

$$y_0(x) > a(x) + Kn(x) > 0 \quad \text{a.e. in } \Omega,$$

then condition (20) *holds.*

Proof. This is an immediate consequence of the fact that $a + Kn$ is a stationary solution of (1), and the comparison principle (Proposition 2.2). □

The second situation that can be analyzed is when a and n are simple functions, i.e., they have the form

$$a(x) = \sum_{i=1}^{m} a_i \chi_{\Omega_i}(x), \qquad n(x) = \sum_{i=1}^{m} n_i \chi_{\Omega_i}(x),$$

where $\Omega_1, \ldots, \Omega_m$ is a measurable disjoint partition of Ω.

Proposition 4.4. *If a and n are simple functions then* (20) *holds.*

Comment. The implication of this proposition is that (20) holds for any finite-dimensional approximation of (1). In particular, the solutions to discrete approximations of (1) with positive initial data always tend to equilibrium solutions as $t \to \infty$.

Proof of Proposition 4.4. We first prove the proposition for the particular case in which $y_0(x) = c > 0$ (a constant function). Note that if y_0 and n are simple functions with respect to the partition (Ω_i), then the right-hand side of (1) is also a simple function, in which case $y(x,t)$ is a simple function, constant on each of the sets Ω_i, for all $t > 0$. We denote by $y_i(t)$ the restriction of $y(x,t)$ to the set Ω_i.

Let M be a bound on $|y(x,t)|$ (such a bound is guaranteed to exist by Proposition 2.3). Then for all $t \geqslant 0$,

$$\int_\Omega n(x)\log y(x,t)\,d\mu(x) = \sum_{i=1}^m \mu(\Omega_i) n_i \log y_i(t)$$

$$\leqslant \log M \int_\Omega n(x)\,d\mu(x) + \Bigl(\min_{1\leqslant i\leqslant m} n_i \mu(\Omega_i)\Bigr) \log\Bigl(\inf_{x\in\Omega} y(x,t)\Bigr).$$

On the other hand, by Proposition 3.1

$$\int_\Omega n(x)\log y(x,t)\,d\mu(x) = \int_\Omega n(x)\log y_0(x)\,d\mu(x) = \log c \int_\Omega n(x)\,d\mu(x),$$

hence

$$\inf_\Omega y(\cdot,t) \geqslant \exp\biggl[\frac{(\log c - \log M)\int_\Omega n(x)\,d\mu(x)}{\min_{1\leqslant i\leqslant m}[n_i \mu(\Omega_i)]}\biggr] > 0.$$

This completes the proof in the case of constant initial conditions. The general case follows at once from the comparison principle, as any solution with initial data $y_0 \in L^{\infty,+}$ can be bounded from below by the solution for constant initial data $c = \operatorname{ess\,inf}_\Omega y_0(x)$. □

5. Discussion

We studied a class of quadratic evolution equations, inspired by models of viscoelastic fluids. Motivated by the physical model, we considered initial data in the cone of positive functions. We showed that the cone of positive L^∞ functions is an invariant set, and that solutions in this set exist for all times. As $t \to \infty$ the solutions tend, in the L^2-norm, to the equilibrium manifold $\mathcal{M}$. The convergence of solutions to specific equilibria in $\mathcal{M}$ could, however, only be proved under additional assumptions.

The following points remain open: (i) Do solutions always tend to a specific equilibrium if Assumption 2 is satisfied? We were unable to prove it, nor to find a counter example. (ii) Do solutions converge, as $t \to \infty$, in situations where Assumption 2 does not hold? While, in such case, the solution cannot converge to an equilibrium in $\mathcal{M}$ (Corollary 4.1), it can, in principle, converge to an equilibrium on the boundary of the cone,

$$\overline{L^{\infty,+}} = \bigl\{ y \in L^\infty:\ y(x) \geqslant 0 \bigr\}.$$

(iii) Does the solution converge to $\mathcal{M}$ in any L^p-norm, for $p > 2$, and in particular, for $p = \infty$?

Another question is whether our results remain valid when the kernel of the projection $\mathcal{P}$ has dimension greater than one. The comparison principle (Proposition 2.2) no longer holds in this case, and as a result, we no longer have a bound on the L^∞ norm, nor do we have a global existence theorem. Assuming, however, that a solution does exist for all times, it is easy to see that Proposition 3.3 still holds, i.e., the "energy" functional V_b is a Lyapunov functional. To prove that the "entropy" functional V_a is also a Lyapunov functional, we need to have a positive function $\tilde{y}$ such that $\mathcal{P}(\tilde{y}) = a$. If such a function exists then Proposition 3.2 remains valid, and $\mathcal{P}(y)$ tends to a in the L^2-norm (Theorem 3.1).

System (1) can be generalized in many different ways, for example, with y being a matrix-valued function and products reinterpreted as matrix products; this is indeed the appropriate setting in the viscoelastic context [3]. Another generalization of (1) is when $\mathcal{P}$ is a general non-negative operator (not necessarily a projection), i.e., $(y, \mathcal{P}(y)) \geqslant 0$ for all $y \in L^2$. We believe that such a system still exhibits global-in-time existence for positive initial data, as well as asymptotic convergence.

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