The Rationale for Measurability*

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Abstract

When modelling large economies by nonatomic measure spaces of agents, one defines "coalitions" as *measurable*—not arbitrary—sets of agents. Here we suggest a rationale for this restriction: "Real" economies have finitely many agents. In them, coalitions are associated with various measures, like total endowment, which play a vital role in the analysis. So in the model, too, one should be able to associate similar measures with coalitions; this means that they must be "measurable." Thus, though in the finite case a coalition is simply an arbitrary set of players, the appropriate generalization to the infinite case is not an arbitrary but a measurable set.

Measure spaces — specifically non-atomic ones — have been used to model large economies for nearly four decades. With his many research and expository contributions to the subject, and the many students whom he inspired with his enthusiasm and intellectual drive, Werner Hildenbrand deserves much of the credit for this development. In this brief note we explore the conceptual underpinnings of the measure-space model.

An economy is modelled as a set A — the set of *agents* — together with a family $\mathcal{A}$ of subsets of A, called *measurable* sets, on which the various measures that characterize the economy are defined. The family $\mathcal{A}$ is taken to be a σ-field, i.e., closed under complementation and countable unions. For definiteness, take A to be the closed unit interval $[0,1]$. If $\mathcal{A}$ is too large — if there are too many

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measurable sets — then it becomes difficult to define measures. For example, if *all* subsets of A are measurable, then under the continuum hypothesis, the only measures are the purely atomic ones — those that assign positive measure to some denumerable subset of A, and 0 to its complement[1]. If we want to allow non-atomic measures like Lebesgue measure, which assigns to each interval the length of that interval, we must restrict the field $\mathcal{A}$ of measurable sets. This can be done, e.g., by taking $\mathcal{A}$ to be the *Borel* field, defined as the smallest σ-field that contains all intervals.

This motivation for measurability is mathematical — it explains why taking all sets as measurable will not allow us to construct the models that we wish to construct. But the conceptual meaning of measurability remains somewhat of a puzzle. In the continuum (non-atomic) model of an economy, individual traders presumably correspond to single points. In thinking of sets of traders (or coalitions), there seems to be no good reason to forbid *any* set from forming, including a set that is not Borel measurable or indeed not Lebesgue measurable. Yet in defining, say, the core of a nonatomic economy, we restrict the "coalitions" to be Borel (or Lebesgue) measurable sets. To be sure, we would have *mathematical* difficulties if we would not do so. But what is the *conceptual* justification of this restriction?

To answer this question, we must rethink the notion of continuum economy. A continuum economy is a model; obviously no real economy actually has a continuum of agents. Individual agents in the real economy may be modelled by individual points[2] in the continuum. But that does *not* necessarily mean that coalitions in the real economy — which may indeed consist of arbitrary sets of real agents — are necessarily modellable by arbitrary subsets of the continuum.

For example, any coalition in the real economy has a total endowment; at given prices p, it has a total demand. These quantities are essential in specifying and analyzing the economy. Therefore coalitions should be modelled as sets for which these quantities can be defined. These quantities are *measures*. Thus coalitions must be modelled as *measurable* sets.

Differently put: The notion of an infinite set depends on the context and the application. In the economic context, the finite sets that arise are endowed with economically significant measures. Therefore, their analogues in the non-atomic

[1] See, e.g., W. Sierpinski, *Hypothèse du Continu,* Chelsea, 1956, Proposition C_{53}.

[2] An alternative is to think of individuals as "infinitesimal sets;" see R. J. Aumann and L. S. Shapley, *Values of Non-Atomic Games,* Princeton University Press, 1974, Section 29, pp. 176-178.

model must also be endowed with measures. And so, they *must* be measurable.

We must not let ourselves get carried away by words, and in particular by the word "set." Unlike finite sets, infinite sets are abstractions. During the century and a half since Cantor, mathematicians have become used to a particular intuitive notion of "set;" even so, this notion has had to be modified and refined several times. At this time a particular notion, defined by a particular axiomatization (ZFC), has become widely accepted. But still it is by no means the only one; there are many variations, even within set theory. For example, one may or may not add the continuum hypothesis to the axioms; one may or may not remove the axiom of choice; and if one does remove it, one may or may not replace it by the axiom of determinateness. These variations are only a few out of many, and all of them profoundly affect the intuitive notion of "set."

Moreover, all these variations deal with only one aspect of sets: their cardinality. Finite sets have other aspects as well. For example, when discussing subsets S of a finite set A, the proportion $|S|/|A|$ — how much of A is in S — is often of interest. Though this sounds closely related to cardinality, Cantorian set theory has no way of generalizing it to infinite sets. In applications such as ours, there may well be various parameters — "weights" — associated with the points in a finite set (like the endowment of an agent); the "total weight" of a finite set then becomes relevant. Again, Cantorian set theory is unable to handle this in the infinite case. Other applications may involve other attributes of finite sets, and then the appropriate infinite generalization should be able to deal with *them*.

In brief, the Cantorian notion of set is just one way of thinking of (infinite) sets; there are other ways. There is no unique "right" way, and the most appropriate way in any given context depends on that context — on the application being discussed.

In the economic application being discussed here, large finite sets S of agents are modelled by infinite sets. Such an S constitutes a certain proportion of the set A of all agents, and it has a certain total endowment vector, which is important when defining, say, the core (as well as the Shapley value and other economic concepts). So in the non-atomic model, S must be modelled in a way that allows one to associate quantities like total endowment with it. Such quantities are *measures*. That is, S must be *measurable.*

What we're suggesting is that for our purposes — in our application — *all* "sets" should be thought of as measurable. The generalization of the concept of a finite set that is appropriate for our purposes is not just any set (in the Cantorian sense), but a measurable set. Large finite coalitions do have a total endowment;

so if we wish to model such coalitions when we go to the non-atomic model, we must restrict ourselves to measurable sets. If it isn't measurable, we shouldn't think of it as a set at all.

Actually, the idea that "all sets are Borel" is not all that revolutionary. For a set to be Borel means that it is constructible in a specified way from intervals. Some kind of constructibility is implicit also in ZFC, otherwise one runs into the classic set-theoretic paradoxes.

But that is not the viewpoint we espouse here. Rather, it is that the notion of "set" (finite or infinite) that is appropriate to the economic application is a *measurable* set, not just a set in the sense of Cantor's set theory.