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UTILITY THEORY WITHOUT THE COMPLETENESS AXIOM

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A utility theory is developed that parallels the von Neumann–Morgenstern utility theory, but makes no use of the assumption that preferences are complete (i.e., that any two alternatives are comparable).

1. INTRODUCTION

BEFORE STARTING out on an exposition of utility theory *without* the completeness axiom, let us briefly review the contents of utility theory *with* the completeness axiom—the by now classical utility theory of von Neumann and Morgenstern [**8**]. This begins with an individual, a set A of "basic alternatives" (or "pure prospects" or "pure outcomes") and the set of all lotteries[1] whose prizes are basic alternatives from A. On this set of lotteries there is defined a preference order, representing the preferences of the individual in question; this preference order is assumed to obey certain axioms.[2] The basic theorem of utility theory asserts that there then exists a real-valued function u on the set of all lotteries,[3] called a *utility* function, which enjoys the following properties:

(a) u represents the preference order, in the sense that a lottery x is preferred to a lottery y if and only if $u(x) > u(y)$; and

(b) u obeys the "expected utility hypothesis",[4] according to which the utility of a lottery is equal to the expected utility of its prizes (for a precise statement see (2.1)).

Furthermore, the u satisfying (a) and (b) is uniquely determined up to an additive and a positive multiplicative constant.

Several of the axioms that govern the preference order in the von Neumann-Morgenstern theory have been questioned, mainly from the viewpoint of their validity as descriptions of real-life behavior.[5] Some authors have

[1] By a "lottery" we mean a situation in which a basic alternative is chosen by means of a random device with known probabilities; technically, it is a random variable with values in A. An inventory policy is an example of a lottery. In a two-person game, a pair of mixed strategies determines a "mixed outcome," i.e., a lottery on the outcomes that can actually occur when the game is played.

[2] For a list of the axioms see [**8**, p. 26]. Different but equivalent systems of axioms have been given by Herstein and Milnor [**4**], Marschak [**7**], and Luce and Raiffa [**5**, pp. 25–28].

[3] Which, of course, includes the basic alternatives, since all but one of the basic alternatives can be assigned probability 0.

[4] This is a misnomer; the expected utility hypothesis is not a hypothesis at all, but a theorem. "Expected utility property" would be more accurate.

[5] For an excellent discussion see [**5**, pp. 25–29].

examined the consequences of dropping or modifying one or another of these axioms. For example, Hausner's multi-dimensional utilities [**3**] result from dropping the so-called continuity axiom. We are concerned here with another one of the axioms, the *completeness axiom.* This axiom says that *given any pair of lotteries, the individual either prefers one to the other or is indifferent between them.* It specifically excludes the possibility that an individual may be willing and able to arrive at preference decisions only for certain pairs of lotteries, while for others he may be unwilling or unable to arrive at a decision;[6] in mathematical phraseology, the preference order is assumed to be *complete.*[7] It is the purpose of this paper to present a variation of the von Neumann-Morgenstern theory which *makes no use of the completeness axiom.* This is the only essential difference between our axioms and those of von Neumann and Morgenstern.

Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable.[8] Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint. Does "rationality" demand that an individual make definite preference comparisons between *all* possible lotteries (even on a limited set of basic alternatives)? For example, certain decisions that our individual is asked to make might involve highly hypothetical situations, which he will never face in real life; he might feel that he cannot reach an "honest" decision in such cases. Other decision problems might be extremely complex, too complex for intuitive "insight," and our individual might prefer to make no decision at all in these problems.[9] Or he might be willing to make rough preference statements such as, "I prefer a cup of cocoa to a 75–25 lottery of coffee and tea, but reverse my preference if the ratio is 25–75"; but he might be unwilling to fix the break-even point between coffee-tea lotteries and cocoa any more precisely.[10] Is it "rational" to force decisions in such cases?

[6] *Indifference* between two alternatives should not be confused with *incomparability*; the former involves a positive decision that it is immaterial whether the one or the other alternative is chosen, whereas the latter means that no decision is reached.

[7] "Total," "connected," and "linear" are sometimes used synonymously.

[8] It should be noted that it is also assumed in the non-numerical indifference curve approach to utility.

[9] Cf. Section 5.

[10] Utility theory is sometimes compared to physics, and it is asserted that the phenomenon described above is nothing but a "limitation of discriminatory capacity" which "cannot be any more serious as an objection to the . . . theory than it would be in the theory of physical measurement" [**10**, p. 182]. We feel that there is, after all, quite a difference in the magnitude of the effects, and that economic theory might be better served if the quite considerable "limitation of discriminatory capacity" would be explicitly recognized.

Other authors have also expressed reservations about the completeness axiom. In discussing observed intransitivities in experimental work on utility theory, Luce and Raiffa mention the possibility of intransitivities occurring "when a subject forces choices between inherently incomparable alternatives" [**5**, p. 25]. Thrall has observed that "from the practical point of view, if the number of judgments needed is finite but large, there is still the time difficulty. By the time the judge has reached the 1,000,000-th choice, his standards of comparison are almost certainly not the same as initially. The theory calls for instantaneous and simultaneous judgments between all pairs . . ." [**10**, p. 183]. Shapley has remarked that "the payoff of a game sometimes most naturally takes the form of a vector having numerical components (such as men, ships, money, etc.) whose relative values cannot be ascertained. The utility spaces of the players can therefore be given only a partial ordering . . ." [**9**, p. 58].[11] In a different context, Shapley has pointed out that partial preference orderings are "useful for describing the preferences of groups, since they enable one to distinguish clearly between indecision and indifference."[12] Finally, von Neumann and Morgenstern themselves say of the completeness axiom that "it is very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient." [**8**, p. 630].

Fortunately, it turns out that much of utility theory stays intact even when the completeness axiom is dropped. However, there *is* a price to pay. We still get a utility function u that satisfies the expected utility hypothesis (item (b) above); and u still "represents" the preference order (item (a) above), but now in a weaker sense: as before, if x is preferred to y then $u(x) > u(y)$, but the opposite implication is no longer true. Indeed, since the real numbers are completely ordered and our lottery space is only partially

[11] Shapley's work in [**9**] was originally motivated by an attempt "to analyze a combat situation in which movement of forces and the inhibition of such movement were critical. It turned out to be feasible to represent essential aspects of this situation by a game-like model with reasonably well defined courses of action corresponding to pure strategies for each side. However, each pair of these strategies, one for each player, generated both a time delay in the movement and losses to the moving forces.... Efforts to obtain estimates of an exchange ratio between attrition and delay failed completely...." (The quotation is from an introductory note to [**9**] by F. D. Rigby, and has been slightly rearranged in the process of condensation). Similar situations abound in economics; examples are problems involving goodwill and immediate profit, or those faced by a division manager who must consider both the income of his division and that of the whole corporation. [**9**] is closely related to this paper, and we shall refer to it repeatedly in the sequel.

[12] From an unpublished abstract of a talk given at the Stanford Symposium on mathematical methods in the Social Sciences, June 1959.

ordered, the opposite implication could not possibly be true. Furthermore, we no longer have uniqueness of the utility.[13]

In spite of these differences, our utility retains many of the useful properties of the von Neumann-Morgenstern utility. For example, we can solve maximization problems with it: Maximization of our utility over a given constraint set will always lead to a maximal[14] element of the constraint set; conversely, for every maximal element x there is a utility whose maximization leads to x. Following up the idea used by Shapley in [**9**], we can even set up a theory of games in which the mixed outcomes to an individual player are only partially ordered. Just as ordinary utilities are used to give a numerical treatment of games in which the mixed outcomes are totally ordered (for each individual player), we will be able to use our "one-way" utilities to give a numerical treatment of such partially ordered games. In fact we will be able to "solve" these games in a manner analogous to the completely ordered case—obtaining an analogue of saddle points for zero-sum games, and of Nash equilibrium points [**6**] for general games.

There is another significant point of similarity with the von Neumann-Morgenstern theory. The representation property of the utility function —item (a) above—can be restated as follows: Given the utility function, we can find the preference order. As we remarked above, this is no longer true in our partially ordered situation, as long as we restrict ourselves to a single utility function. However, under fairly wide circumstances (but not always) we *can* say the following: Given the set of *all* utility functions,[15] we can find the preference order. For example, this is true if we know a priori that the preference order is "finitely generated," i.e., consists of a finite number of "basic" preference statements plus all those statements that follow from these "basic" ones by application of the axioms.

We remark that the present theory is a genuine generalization of the von Neumann-Morgenstern theory, in the sense that in case the space of lotteries does happen to be completely ordered, our utilities are the same as the von Neumann-Morgenstern utilities.

Historically, the first mention of the possibility of a utility theory without the completeness axiom was by von Neumann and Morgenstern: "If the general comparability assumption is not made, a mathematical theory . . . is still possible. It leads to what may be described as a many-dimensional

[13] This too is clear: if for example our partial order makes any two lotteries incomparable, then an *arbitrary* function on the basic alternatives will yield a utility (if the utility of a lottery is defined via the expected utility hypothesis).

[14] I.e., an element to which no other element in the constraint set is preferred. We cannot expect to get a maxim*um* element—i.e., an element preferred or indifferent to all others in the constraint set—because such an element may not exist.

[15] Recall that the utility function is in no sense unique.

vector concept of utility. This is a more complicated and less satisfactory set-up, but we do not propose to treat it systematically at this time" [**8**, p. 29]. Details were never published. What they probably had in mind was some kind of mapping from the space of lotteries to a canonical partially ordered euclidean space, rather than the real-valued mappings we use here; but it is not clear to me how this approach can be worked out. The multidimensional utility of Hausner [**3**], which is a mapping into a completely ordered euclidean space, has nothing to do with this.

More recently Shapley has discussed the same problem in an as yet unpublished manuscript, using an approach basically similar to the present one.[16] Shapley's work and ours are entirely independent.

We have made an attempt to concentrate the less technical part of the paper in the first five sections; the remaining sections become progressively more technical. All proofs are left to the last section.

1. THE AXIOMS

The space on which the utility will be defined is called a *mixture space*: intuitively, this may be thought of as the space of lotteries that we discussed above. Formally, it is a space X with a convex structure; that is, if $\{\gamma_1, \ldots, \gamma_k\}$ is a set of probabilities (i.e., $\gamma_i \geq 0$, $\sum \gamma_i = 1$), and if $x^1, \ldots, x^k \varepsilon X$, then there is defined in X the *convex combination* $\sum_{i=1}^{k} \gamma_i x^i$. One operates with these combinations in all ways as if they were ordinary vector-space sums, keeping in mind only that the coefficients must always be nonnegative and sum to unity.[17] A set $\{x^1, \ldots, x^k\}$ of members of X is said to be *independent* if no two distinct combinations of the x^i are equal, and a maximal independent subset of X is said to *span* X. We shall assume in the sequel that X has a finite spanning subset, or in other words that it is *finite dimensional*. In particular, this condition will always be satisfied when there are only a finite number of basic alternatives.

We assume that on our mixture space X there is defined a transitive and reflexive relation called *preference-or-indifference* and denoted by $\succsim$. If $x \succsim y$ and $y \succsim x$ we shall say that x is *indifferent* to y and write $x \sim y$; if $x \succsim y$ but not $x \sim y$, we shall say that x is *preferred* to y and write $x \succ y$. We assume that the following conditions hold:

(1.1) if $0 < \gamma < 1$ and z is arbitrary, then $x \succsim y$ if and only if $\gamma x + (1-\gamma)z \succsim \gamma y + (1-\gamma)z$;

(1.2) if $\gamma x + (1-\gamma)y \succ z$ for all $\gamma > 0$, then *not* $z \succ y$.

Axiom (1.2) is the "archimidean" or "continuity" axiom.

[16] See footnote 12.

[17] For a set of formal axioms for a mixture space, see [**3**, p. 169]. The treatment here is similar to that of [**5**].

The relation $\succsim$ will be called a *partial order*; the space together with the partial order $\succsim$ will be called a *partially ordered mixture space*. The symbol X (and occasionally Y) will denote a partially ordered mixture space, but may sometimes also be used to denote the underlying (unordered) mixture space; no confusion will result.

2. THE UTILITY

A *utility* on a partially ordered mixture space X is a function from X to the reals for which

$$u(\gamma x + (1 - \gamma)y) = \gamma u(x) + (1 - \gamma)u(y)\,, \tag{2.1}$$

$$x \succ y \quad \text{implies} \quad u(x) > u(y)\,, \tag{2.2}$$

$$x \sim y \quad \text{implies} \quad u(x) = u(y)\,. \tag{2.3}$$

Condition (2.1) is the familiar "expected utility hypothesis," whereas (2.2) and (2.3) state that u represents the preference order.

Our basic result is:

THEOREM A: *There is at least one utility on X.*

3. TWO EXAMPLES

An example of a mixture space is the Euclidean n-space R^n, considered as a vector space over the real numbers. Two of the partial orders most frequently encountered in the literature are the *weak* and the *strong* partial orders on R^n, which we denote by $\succ_W$ and $\succ_S$ respectively. Using subscripts to denote coordinates, we write $x \succ_W y$ if $x_i > y_i$ for all i; we write $x \succ_S y$ if $x_i \geq y_i$ for all i, but $x \neq y$.[18] Both orders satisfy all assumptions of the previous section; they are also both *pure*, i.e., indifference holds only in the case of equality. If we normalize the utilities by setting $u(0) = 0$, then the utilities for $\succ_S$ are of the form $u(x) = \sum_{i=1}^{n} u_i x_i$, where $(u_1, \ldots, u_n) \succ_W 0$; the utilities for $\succ_W$ are of the same form, except that now we need only have $(u_1, \ldots, u_n) \succ_S 0$.

4. DISCUSSION OF THE AXIOMS

(a) We first clear up a point which we glossed over in the introduction, in the interests of brevity and simplicity. The lotteries that are the objects of study in utility theory may have other lotteries as well as basic alternatives

[18] The terminology may sound reversed to the reader, but it has some justification. One partial order is stronger than another if it has more relations; we consider a total order stronger than a partial one.

for their prizes. Lotteries that have other lotteries for their prizes are called *compound*; those that have only basic alternatives as prizes are called *simple*. Every compound lottery is "equivalent" to a simple one; for if one plays out the component lotteries one must eventually get to basic alternatives. This is the equivalence that is asserted by the familiar "algebra-of-combining" axiom in the von Neumann-Morgenstern theory.

In our formulation, the algebra-of-combining axiom is part of the definition of mixture space as given in Section 1. The mixture space X can be thought of as consisting of all formal convex combinations of the basic alternatives, where each convex combination represents not only the appropriate simple lottery, but also all compound lotteries equivalent to it.

(b) If we interpret X as in the previous paragraph, then the finite dimensionality condition demands that there be only finitely many basic alternatives. There are, however, situations in which it is convenient to work with a model in which certain parameters (such as money) take a continuum of values.[19] In such situations we may still be able to obtain a finite dimensional mixture space, by dividing out the indifference relation. The resulting space, though finite dimensional, may not be finitely generated (in the sense that it is the convex hull[20] of finitely many points); for example, the real line (or any open interval on it) is not finitely generated.

The finite dimensionality assumption is very broad; it would be hard to imagine an economic application of utility theory which would require an infinite dimensional mixture space even after the indifference relation is divided out. Formally, however, this assumption cannot be dropped. For an example of an infinite-dimensional partially ordered mixture space without a utility, see Section 8.

(c) Axiom 1.1 is taken from Hausner's set of axioms [**3**, p. 174, W2 and W3]. It asserts that a preference is not changed by "dilution," and conversely that if we have a diluted preference, then the corresponding undiluted preference also holds.

(d) Axiom 1.2 is an extremely weak version of the "archimidean" or "continuity" principle; it is weaker than any variant I have seen. It serves only to exclude the case in which the direction of strict preference between a point z and a closed line segment $[xy]$ goes in one direction for one of the end-points y and in precisely the opposite direction for the entire remainder of the segment. Two cases which are *not* excluded are illustrated by the weak and the strong orders respectively (Figure 1).
Here y and z are on the same horizontal line. For both orders all points in

[19] Usually only finitely many values are actually possible, but the continuous model is often a useful approximation (for example, if one wants to differentiate with respect to price). A similar situation holds in many branches of Physics.

[20] The convex hull of a set D is the set of all convex combinations of members of D.

the half-open[21] segment $[xy)$ are preferred to z; the difference between the two orders is expressed in the relation between z and the end-point y. For the weak order y and z are incomparable, and for the strong order y is preferred to z; in neither case, though, is z actually preferred to y. It can also happen that all points in $[xy)$ are preferred to z, while y and z are indifferent (not pictured).

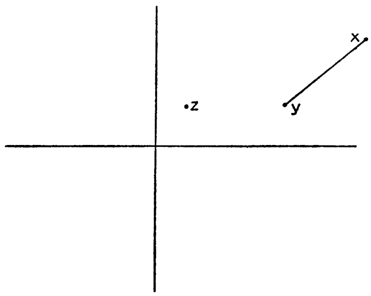

FIGURE 1.

It has been suggested that (1.2) could be slightly weakened, say by making it read

(4.1) if $\gamma x + (1 - \gamma)z \succ y$ for all $\gamma > 0$, then $z \succsim y$.

(4.1) is equivalent to the assertion that for all x, y, and z, the set $\{\gamma : \gamma x + (1 - \gamma)z \succsim y\}$ is closed; it is the form of the archimidean axiom used by Herstein and Milnor [**4**]. The adoption of (4.1) would considerably simplify the proof of the existence of a utility (Theorem A); it is claimed that this simplification would more than make up for the slight loss in intuitive plausibility. But the fact is that there is not only a loss in intuitive plausibility, there is also a quite significant loss in generality; for example, (4.1) excludes the weak order, which has considerable intuitive appeal in some situations and occurs often in the literature. If we had the completeness axiom, there would be no loss of generality at all, and one would then be tempted to use the axiom which makes for the simplest proof; but here, where many cases of interest would be excluded, it is worth a little extra trouble to prove the most general theorem we can. Another point is that (4.1) is no more plausible than the assertion that $\{\gamma : \gamma x + (1 - \gamma)z \succ \gamma\}$ is open,[22] or equivalently that

(4.2) $\gamma_0 x + (1 - \gamma_0)z \succ y$ implies that for all γ sufficiently close to γ_0, $\gamma x + (1 - \gamma)z \succ y$.

[21] I.e., the segment $[xy]$ without the point y.
[22] In the topology of the unit interval.

(4.1) and (4.2) are equally plausible versions of "continuity of preference," and there is no reason to prefer one over the other. For example, whereas (4.1) excludes the weak order, (4.2) excludes the strong order; and when there is completeness, the two are equivalent. But if we were to assume both (4.1) *and* (4.2), then we would be excluding not only some but almost all cases of interest. In fact all that would be left would be orders that are "degenerate," by which we mean orders in which comparability is an equivalence relation.[23] Under these circumstances we prefer the weaker and much more general (1.2).

In practice the effect of (1.2) is to exclude the lexicographic order[24] and other orders inspired by the lexicographic order. We remark that if we drop (1.2) we can still build a utility theory, but the values of the utility functions will be points in a lexicographically ordered euclidean space rather than real numbers; this generalization of the present theory is analogous to Hausner's generalization [**3**] of the von Neumann-Morgenstern theory. It will still be possible to solve maximization problems and games under exactly the same conditions as before (compare [**10**]).[25]

5. MAXIMIZATION PROBLEMS, GAMES, AND SHAPLEY'S THEOREM

The convex hull of a finite set of points in X is called a *convex polyhedron.*

THEOREM B: *Let E be a convex polyhedron in X, and let $x \in E$. Then x is maximal in E under the partial order $\succsim$, if and only if there is a utility u on X such that x maximizes u over E.*

[23] When the underlying mixture space of a degenerate space X is R^n, then X is the direct sum of two linear subspaces Y and Z, such that two elements of X are comparable if and only if they have the same z-component. Thus X is "decomposable" into Y, which is completely ordered, and Z, which is not ordered at all. For yet another characterization of degenerate spaces see Section 6.

[24] The lexicographic order on R^2 is the pure order for which $x \succ y$ if and only if: either $x_1 > y_1$, or $x_1 = y_1$ and $x_2 > y_2$. The definition may be generalized to R^n.

[25] I personally believe the archimidean principle to be very compelling, not withstanding some of the counter-intuitive examples that have been offered in the literature. For example, it is sometimes argued that a trivial prize such as two pins may not be worth any probability of death, no matter how small. But many people drive their cars every day for, say, $50, although they know that this involves a positive probability of death; and by using postulates of utility theory other than the archimidean principle, one can convince oneself that $50 is "comparable" to two pins (by going up a pin at a time, say). The counter-intuitive flavor of the example may be traceable to aspects of the preference axioms other than the archimidean principle; for example, the idealization that asserts the ability to differentiate between probability combinations that are very close to each other may be involved. In spite of all this, there may certainly be situations in which the lexicographic order or something similar constitutes the most convenient model, so it is desirable to have a theory that covers it. (I am indebted to A. Brand for this argument.)

As a typical practical example of the application of Theorem B to an optimization problem, suppose a commanding officer is given a budget, and is asked to decide on an inventory policy for a certain set of military spare parts; say there are hundreds of different kinds of parts involved. The officer is willing to express preferences as between policies each involving at most two or three spare parts (though such policies are obviously not optimal); but "realistic" policies—those involving all or most of the parts—are usually by far too complex to be meaningfully compared. Nevertheless *some* comparisons between complex plans can be made, namely those "generated" by the simpler comparisons via transitivity and condition 1.1; the problem becomes one of choosing an inventory policy which is best possible in the sense that no other policy (within the budget) is preferred to it in the preference order P that is generated by the "simpler" comparisons, which the officer *is* willing to make. Theorem B says that such a "best possible" policy can be obtained by maximizing a utility for the preference order P over the constraint set, and in fact that *every* such policy can be obtained in this way.[26] This approach is not limited to military problems, but applies to any inventory problem in which direct measurements of utility are difficult or impossible to make.

What Theorem B does for maximization problems defined on the partially ordered space X can also be done for two-person zero-sum games played over X. These games are similar to ordinary matrix games in all respects, except that the payoffs are in X rather than being real numbers. As usual, the two players each have a finite set of pure strategies, denoted by $(p_1, \ldots, p_k)$, $(q_1, \ldots, q_l)$; there is a payoff function which associates with each pair of pure strategies p_i and q_j a member a_{ij} of X. If the players use mixed strategies $c = (\gamma_1, \ldots, \gamma_k)$ and $d = (\delta_1, \ldots, \delta_l)$ respectivly, then the outcome is the point $\sum_{i,j} \gamma_i \delta_j a_{ij}$ (abbreviated cAd) in X. The preference order $\gtrsim$ is associated with the first player; the second player has the opposite order, i.e., he prefers x to y or is indifferent between them, if and only if $y \gtrsim x$. Corresponding to a saddle-point in ordinary matrix games, we here have *equilibrium points*; these are pairs of mixed strategies (c°, d°) which are "good against each other" in the sense that $c^\circ A d^\circ$ is *maximal* in the set F of all points in X of the form cAd°, and *minimal* in the set G of all points in X of the form $c^\circ Ad$. These equilibrium points have just about all the nice properties

[26] The approach sketched here has not yet been applied to any actual real-life inventory problem. But an approach that is similar in spirit *has* been applied to a real-life allocation problem; see R. J. Aumann and J. B. Kruskal, "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," *Naval Research Logistics Quarterly*, 6 (1959), pp. 1–16, and the other papers cited there. We mention also that R. G. Davis in his thesis (Princeton, 1960) considered a related approach to aspects of an inventory problem.

of saddle-points in ordinary matrix games. For example, the interchangeability property holds: each player has a set of "good" strategies, such that the equilibrium points are precisely the pairs of good strategies. Furthermore, a player not only achieves "best possible" for himself by playing a good strategy, but he also protects himself against loss; if the other player changes his strategy, the result will either be another equilibrium point, or a point that (from our player's point of view) is actually preferred to an equilibrium point. However, there is nothing in this kind of game that corresponds to the unique value of ordinary matrix games.

Do equilibrium points always exist? If so, how can they be calculated? These questions are answered by the following theorem.

THEOREM C: (c°,d°) *is an equilibrium point in the matrix game* A *if and only if there is a pair* (u, v) *of utilities on* X *such that* (c°, d°) *is an equilibrium point (in the sense of Nash* [**4**]*) in the bimatrix game* $(u(A), -v(A))$.[27]

This theorem generalizes a result of Shapley [**9**]. Shapley considered the case in which the underlying mixture space is R^n and the order is either the weak or the strong order. He defined "weak" and "strong" equilibrium points accordingly, and by exhibiting the utilities explicitly, proved what amounts to Theorem C for each of these two special cases separately. Our proof (see Section 8) is essentially the same as one of Shapley's two proofs. We quote Theorem C here chiefly as an application of our utilities; it serves to unify Shapley's two results, includes a far larger class of preference orders, and, we believe, exhibits his results in their proper context.

Theorem C can be extended to n-person games. Each player i has a finite set P^i of pure strategies, and an outcome space X^i satisfying the assumptions of Section 1. With each n-tuple $(p^1, \dots ,p^n)$ of pure strategies, there is associated an n-tuple $(x^1, \dots, x^n)$ of payoffs, where $x^i \in X^i$. We now define an n-tuple $(c^1, \dots, c^n)$ of mixed strategies to be an equilibrium point if each c^i is "good" against the combination of the $n - 1$ others. The result is that $(c^1, \dots, c^n)$ is an equilibrium point if and only if it is a Nash equilibrium point for some n-tuple of utilities $u^1, \dots, u^n$ on $X^1, \dots, X^n$.

6. THE STRUCTURE OF PARTIALLY ORDERED MIXTURE SPACES

We first note that as in the von Neumann-Morgenstern theory, if u is a utility, then so is $\alpha u + \beta$, where $\alpha > 0$ and β is an arbitrary real number. Two utilities connected in this way are called *equivalent*.

[27] I.e., the two-person nonzero sum game whose strategy spaces are the same as in the original game, but in which the payoff to (s_i,t_j) is $u(a_{ij})$ to player 1 and $-v(a_{ij})$ to player 2.

In this section we will give a constructive characterization of spaces X satisfying the assumptions of Section 1. Let us first consider the case in which the mixture space involved is R^n. Assumptions (1.1) and (1.2) may then be restated as follows:

(6.1.1) $$x \gtrsim y \quad \text{implies} \quad x + z \gtrsim y + z\,;$$

(6.1.2) $$x \gtrsim y \quad \text{and} \quad \alpha > 0 \quad \text{implies} \quad \alpha x \gtrsim \alpha y\,;$$

(6.2) $$x \succ kz \quad \text{for all positive integers } k \text{ implies } \textit{not} \quad z \succ 0\,.$$

A utility in this context is merely a real function on X which represents the order in the sense of (2.2) and (2.3), and which is linear in the ordinary (vector-space) sense; that is, there is a vector $(u_1, \ldots, u_n)$ and a scalar c, such that $u(x) = c + \sum_{i=1}^{n} u_i x_i$. Different c's yield equivalent utilities; we shall usually normalize[28] by setting $c = 0$. We shall denote the vector $(u_1, \ldots, u_n)$ by u, and call it a *utility* as well. Thus $u(x)$ is the same as the inner product ux; no confusion will result.

For a geometric characterization, we turn to the set S ($= S_X$) of points in R^n that are $\gtrsim 0$. It is not difficult to see that the order is completely determined by S. Of more significance than S in the analysis, however, is the set T ($= T_X$) of points in R^n that are $\succ 0$; this may be defined in terms of S by $T = S \setminus (-S)$, where $\setminus$ denotes set-theoretic subtraction. From (1.1) it follows that

(6.3) $$S \text{ is a convex cone,}^{29}$$

and from (1.2) that

(6.4) $$\bar{T} \cap (-T) = \phi\,,$$

where the bar denotes closure; conversely, if these conditions are satisfied, then S defines a partial order. Note that T is also a convex cone, but does not contain the origin. A utility is geometrically characterized by an *open support* of T, i.e., an open half-space containing T, and whose bounding hyperplane contains the origin; the inner normal to the bounding hyperplane provides the utility.

For the examples of Section 3, T is the open positive orthant for the weak order, and for the strong order it is the closed positive orthant minus the origin. Orders on R^n "between" the weak and the strong order are obtained by choosing T to be between these two; for example, for $n = 2$ we could

[28] This normalization sets $u(0) = 0$; there is also a multiplicative parameter that could be normalized, but there seems to be no unique natural way in which to do this. Note that the "natural" way in which we have fixed the additive parameter depended on the existence of an origin; this is a feature of R^n when considered as a vector space, but it is not inherent in the mixture space structure of R^n.

[29] A *cone* is a subset C of R^n such that $x \in C$ and $\alpha > 0$ imply $\alpha x \in C$.

stipulate $T = \{x : x_1 > 0, x_2 \geq 0\}$. Other possibilities for T are open half spaces, open half spaces of linear subspaces of R^n, and circular cones. Excluded are closed half spaces, or half spaces that are partly open and partly closed (such as the open half plane $x_1 > 0$ to which has been adjoined the positive x_2-axis, which would yield the lexicographic order on R^2). Candidates for S can sometimes be obtained from candidates for T by judiciously adding to T points from $\bar{T}$; details are omitted. We mention that the order is "degenerate" (in the sense of Section 4, paragraph (d)) if and only if S is a closed half space of a linear subspace of R^n.

Up to now we have assumed that the underlying mixture space of X is R^n. Hausner [**2**] has proved that any mixture space may be imbedded in a real vector space, and from our finite-dimensionality assumption it follows that the vector space will be an R^n. It is not difficult to extend the partial order as well.[30] Thus any partially ordered mixture space X can be described as a convex subset of a partially ordered copy of R^n, such that the order on X is the restriction to X of the order on R^n. Furthermore, the utilities on X will be precisely the restrictions to X of the utilities on this copy of R^n.

7. DUALITY

In this section we wish to answer the question: To what extent does the set of utilities on X determine the order on X?

To this end, we introduce the duality notion. The dual of a cone C in R^n is defined to be the cone C^* consisting of all $u \in R^n$ such that $ux > 0$ for all $x \in C$. For example, the open positive orthant and the closed positive orthant without the origin are mutually dual, as are R^n and ϕ, an open half space and the ray normal to its bounding hyperplane, and concentric open and closed right circular cones (the latter without the origin) whose half-angles add to 90°. The cone $\{x \in R^2 : x_1 > 0, x_2 \geq 0\}$ is self-dual.

In all the above examples $C^{**} = C$. It is of interest to ask under what general conditions this holds. If we calculate C^{**}, we find that it is precisely the intersection of the open supports of C. Thus we have

THEOREM D: *A necessary and sufficient condition that $C^{**} = C$ is that C be the intersection of its open supports.*

The importance of Theorem D lies in the fact that the condition given is of wide applicability. Let us call a cone satisfying the condition *regular*. A regular cone must be convex, and unless it is all of R^n, it may not contain the origin; but aside from these restrictions, almost any cone "likely to come

[30] Let $X \subset R^n$. The cone S is defined to be the smallest convex cone in R^n containing all vectors of the form $x - y$, where x and y are in X and $x \succsim y$.

up in practice'' is regular. Of course the examples above all involve regular cones. More generally: Any open cone is regular. If C is a convex cone obtained from a closed cone by removing the origin, then C is regular. The set of all x satisfying a given set of homogeneous linear inequalities, *which may contain both weak and strong inequalities,* is regular if it contains at least one strong inequality (which may, for example, serve only to remove the origin). If C is an open circular cone, then any cone between C and $\bar{C}$ that does not contain 0 is regular. On the other hand, if we add the positive half of one of the axes to the open positive octant in R^3, the result is a cone which is not regular, though it is convex and does not contain the origin.

A concept closely related to regularity is Fenchel's "even convexity" [**2**]; a set is said to be *evenly convex* if it is the intersection of open half spaces. Clearly a regular cone is the same thing as an evenly convex cone without the origin.[31]

If X is a partially ordered copy of R^n, then the set of all (normalized) utilities on X is precisely T_X^* (where T_X is the set of all points preferred to 0). Hence if we know that T_X is regular, we can recover the order from the set of all utilities. Thus the set of all utilities on a given X "almost" determines the order, and determines it completely if the set of orders under consideration is suitably restricted.

Our definition of duality is somewhat different from the ordinary definition, in which C^* is defined to be the set of all u such that $ux \leq 0$ for all $x \in C$. Under that definition the necessary and sufficient condition that $C^{**} = C$ is that C be the intersection of its closed supports, or equivalently that it be convex and closed.

8. PROOFS AND EXAMPLES

Characterization of Partially Ordered Euclidean Spaces. We wish to prove that when $X = R^n$, Assumptions (1.1) and (1.2) are equivalent to (6.1.1), (6.1.2), and (6.2). The proof is a straightforward computation; some readers may prefer to skip straight to the proof of Theorem A.

First assume (1.1) and (1.2). Then $x \succsim y$ implies $\frac{1}{2}(x + z) \succsim \frac{1}{2}(y + z)$, and hence $\frac{1}{2}(x + z) + \frac{1}{2}0 \succsim \frac{1}{2}(y + z) + \frac{1}{2}0$; using the converse form of (1.1), we then get $x + z \succsim y + z$, which completes the proof of (6.1.1). For (6.1.2), assume first that $\alpha < 1$. Then by (1.1), $x \succsim y$ implies $\alpha x = \alpha x + (1 - \alpha)0 \succsim \alpha y + (1 - \alpha)0 = \alpha y$. If $\alpha = 1$ there is nothing to prove, and if $\alpha > 1$ then $x \succsim y$ is equivalent to $(1/\alpha)(\alpha x) + (1 - (1/\alpha))0 \succsim (1/\alpha)(\alpha y) + (1 - (1/\alpha))0$, and the converse form of (1.1) then yields $\alpha x \succsim \alpha y$. To prove (6.2), assume $x \succ kz$ for all positive integers k. Then from (1.1) it follows that $x \succ \alpha z$ for all real numbers $\alpha \geq 1$. Hence from (6.1.2)—which we have already

[31] I am indebted to V. L. Klee for pointing this out.

proved—it follows by setting $\gamma = 1/\alpha$ that $\gamma x \succ z$ for all $0 < \gamma \leq 1$. (6.2) now follows from (1.2) by setting $y = 0$.

Conversely, assume (6.1.1), (6.1.2), and (6.2). If $x \succsim y$ and $0 < \gamma < 1$, then by (6.1.1) we get

$$x + \frac{(1-\gamma)}{\gamma} z \succsim y + \frac{(1-\gamma)}{\gamma} z \,;$$

applying (6.1.2), we deduce

$$\gamma x + (1-\gamma)z = \gamma\Big(x + \frac{(1-\gamma)}{\gamma} z\Big) \succsim \gamma\Big(y + \frac{(1-\gamma)}{\gamma} z\Big) = \gamma y + (1-\gamma)z\,.$$

If $\gamma x + (1-\gamma)z \succsim \gamma y + (1-\gamma)z$ and $0 < \gamma < 1$, then by (6.1.1) we get

$$\gamma x = \gamma x + (1-\gamma)z + (-(1-\gamma)z) \succsim \gamma y + (1-\gamma)z + (-(1-\gamma)z) = \gamma y \,;$$

applying (6.1.2) with $\alpha = 1/\gamma$, we deduce $x \succsim y$; this completes the proof of (1.1). To prove (1.2), assume that $\gamma x + (1-\gamma)y \succ z$ for all $1 \geq \gamma > 0$. Applying (6.1.1), we deduce that $\gamma(x-y) \succ z - y$ for all $1 \geq \gamma > 0$. Now set $\gamma = 1/k$ and apply (6.1.2) with $\alpha = k$; it follows that $x - y \succ k(z-y)$ for all positive integers k. Hence by (6.2), $z - y$ not $\succ 0$; hence by (6.1.1), z not $\succ y$.

Proof of Theorem A. We assume that the underlying mixture space of X is R^n; this involves no loss of generality because any finite-dimensional mixture space can be imbedded in such a mixture space. The proof is by induction on n. If $n = 1$ the order must either be complete or all elements are incomparable; in either case the theorem is trivial. Suppose the theorem has been proved for all dimensions up to but not including n. If there is an element of X other than 0 that is indifferent to 0, then we may "divide out" the indifference relation, i.e., consider equivalence classes under indifference; this yields a space of lower dimension, to which the induction hypothesis applies. We may therefore assume without loss of generality that the order on X is pure, so that $S = T \cup \{0\}$ (cf. Section 6). Suppose first that the closure of $T \cup (-T)$ is not all of X, and let w be a point not in that closure. Let Y be a subspace of X such that every $x \in X$ is uniquely of the form $\beta w + y$, where β is real and $y \in Y$; for example, take Y to be the orthogonal complement of the line L_w spanned by w. Define an order on Y by $y \succsim 0$ if and only if there is a β such that $y + \beta w \succsim 0$ in X; geometrically, S_Y is the projection of S_X on Y in the direction of L_w. Hence S_Y is a convex cone, and to prove that the order on Y satisfies our assumptions, it remains only to establish that $\bar{T}_Y$ and $-T_Y$ do not meet. Indeed, suppose y is in their intersection. Noting that T_Y is the projection of T_X on Y in the direction of L_w, we deduce that there is a β and sequences $\{\beta_1, \beta_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ such that $y_i \to y$, $y_i + \beta_i w \succ 0$ in X, and $0 \succ y + \beta w$ in X. Let β_∞ be a limit point (possibly infinite) of the β_i; without loss of generality we may assume that it is actually the limit. If $\beta_\infty = \beta$, then $y + \beta w \varepsilon \bar{T}_X \cap (-T_X)$, contrary to (6.4). If $\beta_\infty \neq \beta$ but is finite, then

$$w = \frac{(y_i + \beta_i w) - (y + \beta w)}{\beta_\infty - \beta} + \frac{y - y_i}{\beta_\infty - \beta} + \frac{\beta_\infty - \beta_i}{\beta_\infty - \beta} w ,$$

and hence w is the sum of a term that is either $\succ 0$ or $\prec 0$ (according as $\beta_\infty > \beta$ or $\beta_\infty < \beta$) and terms that tend to 0, contrary to the assumption that w is not in the closure of $T_X \cup (-T_X)$. If $\beta_\infty = \pm\infty$, then

$$w = \frac{y_i + \beta_i w}{\beta_i} + \frac{y - y_i}{\beta_i} - \frac{y}{\beta_i} ,$$

and again w is the sum of a term that is $\succ 0$ or $\prec 0$ and terms that tend to 0, yielding a contradiction. This proves that the order on Y satisfies our assumptions, and since Y is of lower dimension than X, we can apply the induction hypothesis to construct a utility on Y. This utility can now be extended to X by setting $u(y + \beta w) = u(y)$.

Finally, suppose that the closure of $T \cup (-T)$ exhausts X. If we could show that T is open, then since $0 \notin T$, it would follow that there must be a hyperplane through 0 that does not intersect T (cf. for example [**1**, p. 29, Theorem 7]); the normal to this hyperplane in the direction of the half space occupied by T would then provide a utility. It remains therefore to show that T is open. Contrariwise, suppose $x \varepsilon T$ is on the boundary of T. Let H be a support hyperplane for T through x. Any neighborhood of x will contain points on both sides of H, and therefore in particular it will contain a point that is not in $\bar{T}$; this point must therefore be in the closure of $-T$. Therefore x itself will be in the closure of the closure of $-T$, which is the same as $-\bar{T}$. Therefore $-x \varepsilon \bar{T}$. But since $x \varepsilon T$, it follows that $-x \varepsilon -T$; so $-x \varepsilon \bar{T} \cap (-T)$, contradicting $\bar{T} \cap (-T) = \phi$.

Proof of Theorem B. The "if" statement follows from the definition of utility (2.2). To prove the "only if" half, we assume again that the underlying space is R^n and that the order is pure. The remainder of the proof follows Shapley's proof precisely; it is included only for the sake of completeness. Let D be the set of points dominated by members of E, i.e.,

$$D = \{y : \exists z \varepsilon E \quad \text{such that} \quad z \succsim y\} .$$

D is a polyhedral set; let D^r be the unique r-dimensional face of D whose (relative) interior contains x. Let H be a supporting hyperplane for D that meets D precisely in D^r. Then if u is the normal to H, we have $u(x - y) \geq 0$

for all $y \varepsilon D$, with equality only if $y \varepsilon D^r$. We claim that $u(x - y) > 0$ whenever $x \succ y$. If not, there would be a $y \varepsilon D^r$ such that $x \succ y$. But, since x is in the relative interior of D^r, there would also be a $z \varepsilon D^r$ (of the form $z = x + \delta(x - y)$, $\delta > 0$) such that $z \succ x$. This contradicts the maximality of x. Thus our claim is substantiated, and it follows that u is a utility.

Theorem C follows at once from Theorem B.

An Infinite-Dimensional Partially Ordered Mixture Space Without a Utility. Let X be the set of all infinite sequences $x = \{x_1, x_2, \ldots\}$ of real numbers. Define a pure order on X by stipulating that $x \succ y$ if and only if $x_i \geq y_i$ for all i, but $x \neq y$ (this is the strong order). It is easily verified that all the axioms except finite dimensionality are satisfied. Suppose there were a utility; set

$$u_1 = u\{1, 0, \ldots\}$$
$$u_2 = u\{0, 1, 0, \ldots\}$$

From (2.2) it follows that all the u_i are positive. Set $x_i = 1/u_i$ for all i. Then for all positive integers k we have

$$u(x) = u\{x_1, x_2, \ldots\}$$
$$= x_1 u_1 + \ldots + x_k u_k + u\{0, \ldots, 0, x_{k+1}, \ldots\} > k ,$$

again by (2.2). Hence $u(x)$ is larger than any positive integer, an absurdity.

In this example, the dimensionality of X is the cardinality c of the continuum. To see this, let B be a maximal independent subset of X consisting of dyadic sequences only; B exists by Zorn's lemma. Denote the cardinality of B by b. Every dyadic sequence must be a linear combination of one and only one finite subset of B; because of independence the coefficients are unique, and therefore rational. Since there are c dyadic sequences altogether it follows that $c \leq \sum_{n=1}^{\infty} \aleph_0 b^n$. Hence $c \leq b$. But $b \leq c$ by definition, and hence $b = c$. It would be interesting to know whether or not there is a counter-example in which X has denumerable dimensionality.

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