53 Power and Taxes in a Multi-Commodity Economy

with M. Kurz

Taxation and redistribution in a democratic majority-rule society are analyzed, using the Harsanyi–Shapley non-transferable utility value. The context is that of a multi-commodity pure exchange economy. Two approaches are treated: one in which taxes are in kind and exchange takes the form of barter; and one in which taxes are in money, exchange takes the form of sale and purchase, and prices are determined by a process of supply and demand. It is shown that in the presence of a non-atomic continuum of agents, the two approaches are equivalent, but that this is not so when there are only finitely many agents. It is also shown that the value exists under both approaches, and a characterization is found in the non-atomic case.

Most of modern economic theory treats the public sector as a "benevolent" social agent who behaves so as to maximize some social welfare function. In a different paper [2] we propose an alternative view in which the public sector with its fiscal structure emerges as an endogenous consequence of the power structure of society. We thus propose to regard taxation and the redistribution of wealth as determined simultaneously with the process of exchange, resulting in an outcome that is in equilibrium in both the economic and political spheres.

In the earlier paper we considered a relatively simple, one-commodity (i.e. "money") economy, where the entire struggle focused on income redistribution. Here we extend the characterization to a general l-commodity exchange economy. The classical competitive equilibrium in this economy is replaced by a game theoretic equilibrium, based on the Shapley value, that takes into account not only the economic function of exchange but also the political functions of voting and majority rule.

This work was supported by U.S. National Science Foundation Grant SOC74–11446 at the Institute for Mathematical Studies in the Social Sciences, Stanford University, and by a grant from the Israel National Council for Research and Development at the Hebrew University of Jerusalem.

Received June 24, 1976

This chapter originally appeared in *Israel Journal of Mathematics* 27 (1977): 185–234. Reprinted with permission.

Within this framework, two different approaches will be investigated. In the first, called the "Commodity Redistribution" approach, the entire economic-political system is considered as a single game, and the equilibrium we propose is simply the non-transferable-utility Shapley value of this one game (see [14]). In the second, called the "Income Redistribution" approach, the economic side of the model — consumption and exchange — is assumed to take place in a normal competitive environment, whereas the political side — the redistribution of income — is assumed governed by game theoretic considerations; specifically, by the Shapley value.

One of our major results (Theorem B) is that when the space of agents is a non-atomic continuum — representing the idea of many agents, each individually insignificant — then these two approaches lead to the same result. We will find that this is emphatically *not* true when the number of agents is finite. Our other major results are a characterization of the resulting allocations in the non-atomic case, and a general existence theorem that covers both approaches, for both non-atomic and finite populations.

Sections 2, 3, and 4 are devoted to game theoretic preliminaries and definitions. In Section 5 we present the basic exchange model. Sections 6 and 7 describe in detail the two approaches we have just outlined. The major results are stated in Section 8 and discussed in Section 9. Sections 10 and 19 are devoted to examples, and the remainder of the paper to proofs.

2. Values of finite coalitional games

Let T be a finite set; call the members of T *players* and the subsets of T *coalitions*. A *coalitional game* on T (or simply *game*) is a function v that associates with each coalition S a real number $v(S)$ (the *worth* of S), such that $v(\emptyset) = 0$. A *payoff vector* on T is a measure on the subsets of T. Intuitively, a payoff vector is simply a function that assigns a real number (the payoff) to each player; such functions are in an obvious 1–1 correspondence with the measures on T. The number of members of T is denoted $|T|$.

A *null player* in a game v is a player i such that $v(S \cup \{i\}) = v(S)$ for all coalitions S. Players i and j are called *substitutes* if $v(S \cup \{i\}) = v(S \cup \{j\})$ whenever S contains neither i nor j. For a fixed player set T, a *value* is a function ϕ that associates with each game v a payoff vector ϕv satisfying the following conditions:

(2.1) *Additivity*: $\phi(v + w) = \phi v + \phi w$.

(2.2) *Symmetry*: $(\phi v)(\{i\}) = (\phi v)(\{j\})$ *whenever* i *and* j *are substitutes.*

(2.3) *Efficiency*: $(\phi v)(T) = v(T)$.

(2.4) *Null player condition*: $(\phi v)(\{i\}) = 0$ *whenever* i *is a null player.*

The definition of value is due to Shapley [13], who also proved the following basic characterization of values:

PROPOSITION 2.5. *For each finite player set* T *there is one and only one value* ϕ; *it is given by the formula*

$$(\phi v)(\{i\}) = E(v(S_i \cup \{i\}) - v(S_i)), \tag{2.6}$$

where S_i *is the set of all players preceding* i *in a random order on* T, *and* E *is the expectation operator when all* $|T|!$ *such orders are assigned equal probability.*

For a proof of Proposition 2.5 that is even simpler than the original proof (Shapley [13]), see appendix A of [4].

Define the *dual*[1] v^* of a finite game v by $v^*(S) = v(T) - v(T \backslash S)$. By reversing orders in Proposition 2.5, it is easy to see that

$$\phi v^* = \phi v. \tag{2.7}$$

3. Values of non-atomic games

In much of this paper we shall be working with a non-atomic continuum of players, and for this reason it is important to examine the extension of the above model to the non-atomic case.

Let $(T, \mathscr{C}, \mu)$ be a non-atomic measure space; i.e. a set T, together with a σ-field $\mathscr{C}$ of subsets of T, and a non-atomic, non-negative measure μ on $\mathscr{C}$ with $\mu(T) = 1$. One should not think of the points t of T as individual players; rather, one should think of an individual player as an infinitesimal subset dt of T. The measure μ is the population measure, i.e. $\mu(S)$ represents the proportion of the total population in S.

A *coalitional game* (or simply *game*) on the measurable space $(T, \mathscr{C})$ is a function v from $\mathscr{C}$ to the real numbers such that $v(\emptyset) = 0$. There are various ways of extending the definition of value from the finite games above to the situation we have here; see [4]. Here we will adopt a variant of a definition due to Kannai [9]. In this variant a special role is played by the family $\mathscr{C}'$ of S in $\mathscr{C}$ with $\mu(S)$ rational; these S will be called *coalitions.*

[1] Cf. [10], or [4, p. 140].

Let v be a game and S a coalition; we wish to define $(\phi v)(S)$. Intuitively, one proceeds by dividing both S and $T\backslash S$ into a large but finite number of "small" sets W_i, all of equal μ-measure; let there be n in all (see Fig. 1).

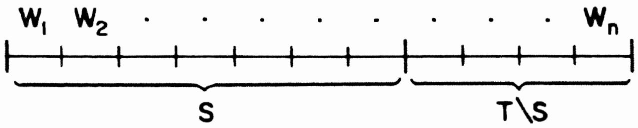

Fig. 1.

If one considers each of these small sets as an individual player, then one gets a finite game, by restricting v to unions of the W_i. In this finite game the coalition S — or more precisely the coalition of those W_i whose union is S — has a value, which we denote $\phi_n(S)$. Now if we let $n \to \infty$ and the W_i shrink, $\phi_n(S)$ may or may not tend to a limit. If it does, and if this limit is independent of the various choices that must be made, then we denote the limit $(\phi v)(S)$, and call it the "μ-value of S."

Formally, let $\Pi_1, \Pi_2, \cdots$ be a sequence of partitions of T into measurable sets (i.e. members of $\mathscr{C}$) of equal μ-measure, such that S is a union of members of Π_1, and each Π_m refines the previous partition Π_{m-1}. Assume moreover that the sequence is *separating*, i.e. that if s and t are distinct points in T, then for m sufficiently large, s and t are in different members of Π_m. For example, if Π_m is the partition of the unit interval $[0,1]$ into the subintervals $[0, 1/2^m]$, $(1/2^m, 2/2^m], \cdots, (1-1/2^m, 1]$, then the sequence $\{\Pi_m\}$ is separating. For each m, let v_m be the finite game on Π_m defined by $v_m(\Xi) = v(\bigcup_{W \in \Xi} W)$, and let ϕv_m be its value. Further, let $S_m = \{W \in \Pi_m : W \subset S\}$; S_m is the coalition in the finite game v_m corresponding to S in the original game v. Now let $m \to \infty$; if $(\phi v_m)(S_m)$ has a limit, and if this limit is independent of the choice of the sequence $(\Pi_1, \Pi_2 \cdots)$ (when chosen in accordance with the above conditions), then the limit is denoted $(\phi v)(S)$. If this is the case for all coalitions S, then the function ϕv is called the ***μ-value*** of v.

The game model of this section differs from the standard model of non-atomic games [4] in that in addition to T, $\mathscr{C}$ and v, we are here given an underlying measure μ. The μ-value is a variant of the asymptotic value [4, pp. 126–127]; it is obtained from the latter by considering only partitions of T into sets of equal μ-measure. Obviously if v has an asymptotic value, then it also has a μ-value, and it equals the asymptotic value; but the converse is false.[2]

[2] Example 19.2 of [4] has a λ-value but no asymptotic value.

In the applications below we deal with voting games in which we stress the democratic "one man—one vote" principle. In this context it is natural to use finite approximations in which the elements of the partitions are assigned equal amounts of the population measure, and this leads directly to the μ-value defined above.

As in the case of finite games, we define the *dual* v^* of a game v by $v^*(S) = v(T) - v(T \backslash S)$. Then by using a limiting argument and (2.7), one can easily show that v^* has a μ-value if and only if v does, and in that case

$$\phi v^* = \phi v. \tag{3.1}$$

The μ-value satisfies conditions analogous to the axioms (2.1)–(2.4) defining the finite value. For future reference we quote here only the efficiency axiom

$$(\phi v)(T) = v(T), \tag{3.2}$$

which follows easily from the efficiency axiom (2.3) for finite games.

It is sometimes convenient to treat finite and non-atomic games in the same context, and in particular to refer to the "μ-value" of a game that may either be non-atomic or finite. In that case, the μ-value of a finite game will be taken to be simply the value.

4. Threats

A *strategic game* Γ consists of

i) A measure space $(T, \mathscr{C}, \mu)$ (T is the *player space*, and μ the *population measure*; S in $\mathscr{C}$ with $\mu(S)$ rational are *coalitions*).

ii) For each coalition S, a set X^S (the *strategies* of S).

iii) For each coalition S, each strategy σ of S, each strategy τ of $T \backslash S$, and each t in T, a number $\boldsymbol{h}^S_{\sigma\tau}(t)$ (the *payoff* to t), such that $\boldsymbol{h}^S_{\sigma\tau}(t) = \boldsymbol{h}^{T\backslash S}_{\tau\sigma}(t)$.

When T is finite, this is something very similar to the traditional definition of "games in normal form" [17]. However, it is not entirely the same; the point is that here strategies are assigned to coalitions, and not just to individual players. Though formally it is easy to derive coalitional strategies from individual strategies and vice versa, there may be certain strategies that are more naturally described in terms of coalitions than individuals, e.g. those involved in the imposition of taxes. Moreover, in the non-atomic case, coalitional strategies enable us to bypass the technical complexities that would arise from the need to define payoffs to "infinity-tuples" of individual strategies. For these two reasons we prefer the definition as given.

Strategic games will be assumed to satisfy the following four conditions, which are not substantive but merely a matter of convenience:

(4.1) $\boldsymbol{h}^S_{\sigma\tau}(t) \geqq 0.$

(4.2) $\mu(T) = 1.$

(4.3) $\boldsymbol{h}^S_{\sigma\tau}$ *is measurable in t for all S, σ, and τ.*

(4.4) *The empty coalition $\varnothing$ has exactly one strategy.*

In view of (4.4), we shall write $\boldsymbol{h}^T_\sigma$ instead of $\boldsymbol{h}^T_{\sigma\tau}$. The meaning of (4.4) is that $\varnothing$ has no real choice of strategies and cannot affect the payoff.

All strategic games treated here will either have finite T, or non-atomic μ. If T is finite, we assume that $\mathscr{C}$ consists of all subsets of T, and $\mu(\{t\}) = 1/|T|$ for all t in T.

Before we go further let us agree on some notational conventions. The family of coalitions (i.e. S in $\mathscr{C}$ with $\mu(S)$ rational) is denoted $\mathscr{C}'$. If $\boldsymbol{f}$ is a (vector or scalar) function on T, we shall write $\int_S \boldsymbol{f}$ for $\int_S \boldsymbol{f}(t)\mu(dt)$, and $\int \boldsymbol{f}$ for $\int_T \boldsymbol{f}$. The set of non-negative real numbers will be denoted R^+. W.r.t. means "with respect to"; w.l.o.g. means "without loss of generality"; integrable means "μ-integrable."

For $S \in \mathscr{C}'$ and $(\sigma, \tau) \in X^S \times X^{T\setminus S}$, write

$$H^S(\sigma, \tau) = \int_S \boldsymbol{h}^S_{\sigma\tau} - \int_{T\setminus S} \boldsymbol{h}^S_{\sigma\tau}. \tag{4.5}$$

We may view H^S as a 2-person 0-sum game, the players being the coalitions S and $T\setminus S$. If this game has a saddle point (σ_0, τ_0) — i.e., if[3]

$$H^S(\sigma, \tau_0) \leqq H^S(\sigma_0, \tau_0) \leqq H^S(\sigma_0, \tau) \tag{4.6}$$

for all σ in X^S and τ in $X^{T\setminus S}$ — then we shall denote the minmax value $H^S(\sigma_0, \tau_0)$ of this game by $w(S)$. Note that

$$w(T) = \max\left\{\int \boldsymbol{h}^T_\sigma : \sigma \in X^T\right\}. \tag{4.7}$$

Finally, write

$$v(S) = \frac{1}{2} w(T) + \frac{1}{2} w(S) \tag{4.8}$$

[3] Implicit in formula (4.6) is the assumption that the three expressions appearing therein are defined as extended real numbers, i.e., that none of them is of the form $\infty - \infty$.

for all S in $\mathscr{C}'$. We call v the *Harsanyi coalitional form* of the strategic game Γ, and say that it is *defined* whenever all the games H^S have saddle points.

Intuitively, the games H^S, w and v are meaningful only when utility is "transferable." In that case one can meaningfully speak of $\int_S \boldsymbol{h}^S_{\sigma\tau}$ as the total payoff to S, since S can divide that sum in an arbitrary way among its members.

If a coalition S acts in concert in such a situation, what total payoff can it expect? The answer given by von Neumann and Morgenstern [17] is

$$\min_{\tau \in X^{T\setminus S}} \max_{\sigma \in X^S} \int_S \boldsymbol{h}^S_{\sigma\tau}. \tag{4.9}$$

This has been criticized as "too pessimistic" because it assumes the worst possible, as if the only object of the complementary coalition $T\setminus S$ is to minimize the payoff to S. A more sophisticated answer, based on a subtle interplay of threats, counterthreats, and compromises, was suggested by Harsanyi [6], who followed up the pioneering work of Nash [12] on the subject. Suppose the members of S have decided to act in concert, as have the members of $T\setminus S$. Presumably S and $T\setminus S$ will eventually wish to cooperate so that they will jointly receive the maximum payoff $w(T)$ (see (4.7)); the only question is how this amount is to be divided between them. Before deciding on this, each side will wish to put itself in as good a bargaining position as possible vis-à-vis its opponent. To this end, each side makes a threat — i.e., announces a strategy to be carried out in case of disagreement. If the threats are σ and τ, the payoffs in case of disagreement will be $\int_S \boldsymbol{h}^S_{\sigma\tau}$ to S, and $\int_{T\setminus S} \boldsymbol{h}^S_{\sigma\tau}$ to $T\setminus S$. Thus the total payoff is $\int \boldsymbol{h}^S_{\sigma\tau}$, which is in general less than the maximum amount $w(T)$ that T can achieve. It therefore seems reasonable for the sides to compromise by splitting the difference between $w(T)$ and $\int \boldsymbol{h}^S_{\sigma\tau}$, and adding this amount to the disagreement payoff of each side. The final payoff to S will then be $(w(T)+H^S(\sigma,\tau))/2$, and to $T\setminus S$, it will be $(w(T)-H^S(\sigma,\tau))/2$. Thus if in choosing their threats, the sides take into account their effect on the final outcome of bargaining, then S will try to maximize, and $T\setminus S$ to minimize, the quantity $H^S(\sigma,\tau)$. Hence the final amount that S can expect to obtain is precisely $v(S)$.

Suppose now that v is defined and has a μ-value ϕv. Then ϕv is called the *Harsanyi–Shapley transferable utility* (*TU*) *value*, or simply *TU value*, of the strategic game Γ.

Next, let us refer to a positive real-valued measurable function on $(T, \mathscr{C})$ by the term "comparison function." If $\boldsymbol{\lambda}$ is a comparison function, denote by $\boldsymbol{\lambda}\Gamma$ the strategic game obtained from Γ by multiplying the payoffs $\boldsymbol{h}^S_{\sigma\tau}(t)$ by $\boldsymbol{\lambda}(t)$, and by v_λ the Harsanyi coalitional form of $\boldsymbol{\lambda}\Gamma$. If v_λ is defined and has a μ-value ϕv_λ, and if there is a σ in X^T such that

$$\int_S \boldsymbol{\lambda}\boldsymbol{h}_\sigma^T = (\phi v_\lambda)(S) \tag{4.10}$$

for all S in $\mathscr{C}'$, then $\boldsymbol{h}_\sigma^T$ is called a *Harsanyi–Shapley non-transferable utility* (*NTU*) *value*, or simply a *value*, of the strategic game Γ.

The reasoning behind this definition of value may be briefly described as follows: if side payments were permitted with the "exchange rates" $\boldsymbol{\lambda}$, then the value for the resulting coalitional game would dictate paying each player dt an amount $(\phi v_\lambda)(dt)$, where v_λ is the Harsanyi coalitional form of $\boldsymbol{\lambda}\Gamma$. Formula (4.10) may be rewritten

$$\boldsymbol{\lambda}(t)\boldsymbol{h}_\sigma^T(t)\mu(dt) = (\phi v_\lambda)(dt); \tag{4.11}$$

that means that at the exchange rates $\boldsymbol{\lambda}$, the value ϕv_λ is achievable without any "transfers of utility." This may therefore be taken as a value even in the absence of transferability, since no transfers are called for.

A more thorough discussion may be found in Shapley[4] [14], and in [1, section 6]. It is worthwhile to note that for $|T| = 2$, the Harsanyi–Shapley NTU value coincides with the bargaining solution of Nash [12] for two-person cooperative games.

The value notion differs in a fundamental way from solution notions such as the core, the von Neumann–Morgenstern solution, and the bargaining set, which are based on the concept of domination. Recall that an outcome $\boldsymbol{x}$ *dominates* an outcome $\boldsymbol{y}$ if there is a coalition S that prefers $\boldsymbol{x}$ to $\boldsymbol{y}$ and can achieve *for itself* an outcome at least as good as $\boldsymbol{x}$. Thus domination expresses dissatisfaction on the part of a coalition because *it can do better by itself*. But the value, though it is based partly on this kind of consideration, also takes into account the opportunities of a coalition to cause harm to players outside it — i.e. to threaten. Thus the core, von Neumann–Morgenstern solution and bargaining set take into account arguments of the form "I should get more because I do not need you to do better"; whereas the value takes into account not only this kind of argument, but also the kind that says "I should get more because you need me to get what you're getting."

[4] Shapley permits some (but not all) of the exchange rates $\boldsymbol{\lambda}(t)$ to vanish. Vanishing exchange rates are awkward to interpret; it seems best to avoid them when possible. A value in our sense (with non-vanishing $\boldsymbol{\lambda}(t)$) is of course also a value in Shapley's sense.

5. The market model

A *market*[5] M consists of

i) A measurable space $(T, \mathscr{C})$ (the space of *agents*[6]) together with a σ-additive, non-negative measure μ on $\mathscr{C}$ with $\mu(T)=1$ (the *population measure*[7]).

ii) The non-negative orthant Ω — called the *consumption set* — of a Euclidean space E^l (l represents the number of different commodities in the market).

iii) An integrable function $\boldsymbol{e}$ from T to Ω (the *endowment function* or *initial allocation*).

iv) For each t in T, a function u_t on Ω (the *utility function* of t).

A market is called *finite* if T is finite, and *non-atomic* if μ is non-atomic. In this paper we will assume that every market is either finite or non-atomic.[8]

We will assume that the measurable space $(T, \mathscr{C})$ is finite or isomorphic[9] to the unit interval $[0,1]$ with the Borel sets. This assumption is less restrictive than it sounds; any non-denumerable Borel subset of any Euclidean space (or indeed, of any complete separable metric space) is isomorphic to $[0,1]$.

If x and y are in E^l, we write $x \geqq y$ if $x^i \geqq y^i$ for all i, $x > y$ if $x^i > y^i$ for all i, and $x \geq y$ if $x \geqq y$ and $x \neq y$. The origin of E^l, as well as the number zero, are denoted 0. A function u on Ω is *increasing* if $x \geq y$ implies $u(x) > u(y)$. The partial derivative $\partial u/\partial x^i$ of a function u on Ω is denoted u^i, and the gradient $(u^1, \cdots, u^l)$ is denoted u'. The following assumptions will be made throughout:

(5.1) *For each t, u_t is increasing, concave, and continuous on Ω.*

(5.2) $u_t(0)=0$.

(5.3) *$u_t(x)$ is simultaneously measurable*[10] *in t and x.*

(5.4) $\int \boldsymbol{e} > 0$.

[5] The terminology differs somewhat from that of [1]. There a "market" is defined by preference relations rather than utility functions.

[6] "Agents" are the same as "players"; we prefer the former in the current politico-economic context, the latter in purely game theoretic contexts.

[7] In [1], μ was not interpreted as a population measure.

[8] This excludes the case in which μ has a denumerable infinity of atoms, as well as the "mixed" case, in which μ has some atoms as well as a non-atomic part.

[9] Two measurable spaces are *isomorphic* if there is a one-one transformation from one onto the other that preserves measurability in both directions.

[10] I.e. measurable in the product field $\mathscr{B} \times \mathscr{C}$, where $\mathscr{B}$ is the Borel field on Ω.

(5.5) *For each t and i, the partial derivative $u_t^i(x)$ exists and is continuous at each x in Ω with $x^i > 0$.*

A market is called *bounded* if

(5.6) *u_t is uniformly bounded,*[11]

and

(5.7) *$u_t(1, \cdots, 1)$ is uniformly positive.*[12]

We close this section by describing some notation and terminology that will be used throughout. For S in $\mathscr{C}$, we will sometimes write $e(S)$ for $\int_S \boldsymbol{e}$. An *S-allocation* is a measurable function $\boldsymbol{x}$ from S to Ω with $\int_S \boldsymbol{x} = e(S)$. An *allocation* is a T-allocation. If $\boldsymbol{x}$ is a function from T to Ω then we will sometimes write $u(\boldsymbol{x})$ for the function on T whose value at t is $u_t(\boldsymbol{x}(t))$. A *price vector* is a member p of Ω with $p > 0$; it is called *normalized* if $\Sigma_{i=1}^l p^i = 1$. If p is a price vector and $x \in E^l$, then $\Sigma_{i=1}^l p^i x^i$ is denoted px. The expressions "almost everywhere" (a.e.), "almost all" (a.a.), and so on will refer to the measure μ. If $\boldsymbol{\lambda}$ is a comparison function, denote by $\boldsymbol{\lambda} M$ the market obtained from M by multiplying each utility function u_t by $\boldsymbol{\lambda}(t)$.

6. Commodity redistribution

We wish to define a game that embodies the redistribution process. First we describe the game verbally. Taxation[13] decisions are reached by majority vote. This means that any coalition S with $\mu(S) > 1/2$ can impose taxes in any way it pleases. In particular, it may divide among its own members the taxes collected from the minority. The taxes discussed in this section are imposed on commodities and paid in commodities.

It might be argued that in a democracy, the tax laws must be uniform, so that two people cannot be taxed differently just because one is a member of a ruling coalition while the other is not. But it must be recalled that what we are discussing here is not merely taxation but rather net income redistribution due to government activity. Though taxation is required to be uniform, there is no such requirement on government spending, nor would such a requirement be feasible. The net effect of this is that the majority can tax the minority as heavily as it wishes, and distribute the proceeds among its own members.

[11] $\operatorname{Sup}\{u_t(x): t \in T, x \in \Omega\} < \infty$.
[12] $\operatorname{Inf}\{u_t(1, \cdots, 1): t \in T\} > 0$.
[13] Actually, we are studying the net effect of both taxation and redistribution.

This being the case, it would seem at first sight that the entire income $e(T)$ of society becomes available to whoever is in the majority, since the majority can, in principle, tax the minority at 100% and return nothing to it. If one calculates the resulting value, one gets an allocation in which no account is taken of the fact that different individuals may have different endowments. Though the calculation is not without interest, the result appears too extreme to be considered relevant to the problem of the distribution of wealth in a democracy.

We now appear to be caught in a dilemma. If the majority rules, one gets an entirely egalitarian outcome. If, on the other hand, one ignores the political structure and does not allow the majority to redistribute, one is simply left with the initial allocation. Neither case seems realistic. How, then, can one account for the type of taxation scheme that one observes?

The answer lies in what we said at the end of Section 4. *The power of the minority lies in its threat possibilities.* We are going to assume that there is no "forced labor" — i.e. that the minority can, if it wishes, destroy part or all of its endowment. This is a powerful threat, which can force the majority to compromise.

The formal treatment starts out with a market M. Given M, define a strategic game $\Gamma = \Gamma(M)$, called the *redistribution game*, as follows: T, $\mathscr{C}$, and μ are as in the market M. As for the strategy spaces and payoff functions, we will not describe these fully, because that would lead to irrelevant complications; but we will make three assumptions about them, which suffice to characterize completely the games v_λ and their values.

The first of the three assumptions is:

(6.1) *If $\mu(S) > 1/2$, then for each S-allocation $\boldsymbol{x}$ there is a strategy σ of S such that for each strategy τ of $T \setminus S$,*

$$\boldsymbol{h}^S_{\sigma\tau}(t) \begin{cases} \geqq u_t(\boldsymbol{x}(t)), & t \in S \\ = 0, & t \notin S. \end{cases}$$

This means that a coalition in the majority can force every member outside of it down to the zero level, while reallocating to itself its initial bundle in any way it pleases.

Next, we assume

(6.2) *If $\mu(S) \geqq 1/2$, then there is a strategy τ of $T \setminus S$ such that for each strategy σ of S, there is an S-allocation $\boldsymbol{x}$ such that*

$$\boldsymbol{h}^S_{\sigma\tau}(t) \leqq u_t(\boldsymbol{x}(t)), \qquad t \in S.$$

This means that a coalition in the minority can prevent the majority from making use of any endowment other than its own (the majority's). Finally, we assume

(6.3) *If* $\mu(S) = 1/2$, *then for each* S*-allocation* $\boldsymbol{x}$ *there is a strategy* σ *of* S *such that for each strategy* τ *of* $T\backslash S$ *there is a* $T\backslash S$*-allocation* y *such that*

$$h^S_{\sigma\tau}(t)\begin{cases} \geqq u_t(\boldsymbol{x}(t)), & t \in S \\ \leqq u_t(y(t)), & t \in T\backslash S. \end{cases}$$

This simply means that if neither S nor its complement are in the majority, then each side can divide its endowment in any way it pleases, while at the same time not giving anything to the other side.

This completes the definition of the redistribution game. Values for all strategic games have been defined in Section 4, and the definition applies in particular to this game. More interesting than the values themselves, though, are the allocations $\boldsymbol{x}$ such that $u(\boldsymbol{x})$ is a value of $\Gamma(M)$. These are the commodity redistributions to which we are most directly led by value considerations; we call them *commodity tax allocations* for M.

7. Income redistribution

In the Redistribution Game of Section 6, it was assumed that in taxing an individual, Society could take specific cognizance of the *vector* of his endowments. It is possible to take a different approach, in which Society would only be allowed to tax income — i.e. the monetary worth of the endowment vector at prevailing prices. That is what we will do in this section.

Given a price vector p, define the *indirect utility function* u^p_t of trader dt to be the function from R^+ to itself given by

$$u^p_t(y) = \max\{u_t(x) \colon x \in \Omega \quad \text{and} \quad px \leqq y\}. \tag{7.1}$$

Intuitively, $u^p_t(y)$ is the highest utility dt can attain by buying goods at prices p with a maximum expenditure of y.

If all traders are assured that they can always trade at the fixed prices p, then the given l-good economy becomes an economy with only one good, namely money. In this economy the initial endowments are $p\boldsymbol{e}(t)$, and the utility functions are u^p_t; this may be analyzed as a redistribution game with only one commodity, and this analysis yields a certain taxation-redistribution system.

Taxation and redistribution in this one-good "money" economy, as well as the ordinary incentives for trading, will in general create a situation in which at prices p, the supply and demand in the original l-good economy are out of balance. One therefore would like to know whether there exists a price vector p such that if the above procedure is carried out, supply and demand for each of the l goods will match *after* the taxation and redistribution are carried out, where the same price vector p is used both in assessing the endowment for purposes of taxation and in trading after taxes have been collected. Such a p, together with the resulting tax scheme, is called a "competitive tax equilibrium."

Formally, given a market M and a price vector p, define a market M^p, called the *market derived from M at prices p* (or simply the *derived market*) as follows: $(T, \mathscr{C}, \mu)$ is as in M; the number of commodities is 1; the initial allocation is $p\boldsymbol{e}$; and the utility functions are the indirect utilities u_t^p defined in (7.1). We will see below (Lemma 16.1) that M^p does indeed satisfy all the conditions required of markets as defined in Section 5 ((5.1) through (5.5)). A *competitive tax equilibrium* in M is a pair consisting of an allocation $\boldsymbol{x}$ and a price vector p such that

(7.2) $\boldsymbol{x}(t)$ *a.e. maximizes* u_t *over* $\{x \in \Omega : px \leqq p\boldsymbol{x}(t)\}$,

and

(7.3) $p\boldsymbol{x}$ *is a commodity tax allocation in* M^p.

If $(\boldsymbol{x}, p)$ is a competitive tax equilibrium, then $\boldsymbol{x}$ will be called an *income tax allocation*. Note that in M^p there is just one "commodity," namely money, and that the quantities $p\boldsymbol{x}(t)$ appearing in (7.3) are in units of that "commodity." Note also that in any market in which $l = 1$ — i.e. in which there is just one commodity — the commodity and income tax allocations are the same. Hence when $l = 1$, we will sometimes refer simply to *tax allocations*.

8. Statement of major results

Three basic results are proved in this paper. Theorem A is a general existence theorem that covers both finite and non-atomic markets, and both commodity and income tax allocations. Theorem B is an equivalence theorem, which asserts that in non-atomic markets, the commodity and income redistribution approaches lead to the same result; this result does *not* hold for finite markets. Theorem C provides a characterization of tax allocations in one-commodity non-atomic markets, and asserts that in such markets, there is a unique tax allocation. Theorems B and C together yield a characterization of commodity tax allocations in many-commodity non-atomic markets: By Theorem B, every such allocation is associated with a tax allocation in a derived market M^p; and these, in turn, are characterized by Theorem C (see Proposition 9.14).

A market is called *trivial* if there are just two agents ($|T| = 2$) and one of them has an endowment vector equal[14] to 0.

THEOREM A (Existence). *Every non-trivial bounded*[15] *market has a commodity tax allocation and an income tax allocation.*

THEOREM B (Equivalence). *In a non-atomic bounded market, the commodity tax allocations coincide with the income tax allocations.*

THEOREM C (Characterization). *A non-atomic bounded market with a single commodity ($l = 1$) has a unique tax allocation $\boldsymbol{x}$. This allocation is characterized*[16] *by a.e. $\boldsymbol{x}(t) > 0$ and*

(8.1) $$\boldsymbol{x}(t) + \frac{u_t(\boldsymbol{x}(t))}{u_t'(\boldsymbol{x}(t))} = \boldsymbol{e}(t) + \int \frac{u(\boldsymbol{x})}{u'(\boldsymbol{x})}.$$

Theorem A is proved in Section 18, Theorem B in Section 16, and Theorem C in Section 15. A counter-example to Theorem B when T is finite is given in Section 19.

9. Interpretation, discussion, further results

We start out by recalling some basic facts about "efficient" allocations. An allocation $\boldsymbol{x}$ in a market M is called *efficient* if there is no allocation $\boldsymbol{y}$ such that a.e. $u_t(\boldsymbol{y}(t)) > u_t(\boldsymbol{x}(t))$. With each efficient allocation there is associated an essentially[17] unique pair $(\boldsymbol{\lambda}, p)$, consisting of a measurable function $\boldsymbol{\lambda}$ from $(T, \mathscr{C})$ to R^+, and a price vector p, such that

(9.1) *the maximum of $\boldsymbol{\lambda}(t)u_t(x) - px$ over Ω is a.e. achieved when $x = \boldsymbol{x}(t)$.*

This $(\boldsymbol{\lambda}, p)$ is called an *efficiency pair* for $\boldsymbol{x}$; we will call it *normalized* if p is normalized. From (9.1) it follows that p is an *efficiency price vector*, i.e. that

(9.2) *the maximum of $u_t(x)$ over $\{x \in \Omega: px \leqq p\boldsymbol{x}(t)\}$ is a.e. achieved when* $x = \boldsymbol{x}(t)$,

and that

[14] When $|T| > 2$, a vanishing endowment does not make an agent powerless, because he still has his vote. But when $|T| = 2$, the vote plays no role because neither player alone has a majority, and so lacks the power to tax the other agent.

[15] I.e. obeying (5.6) and (5.7).

[16] I.e., an allocation $\boldsymbol{x}$ is a tax allocation if and only if a.e. $\boldsymbol{x}(t) > 0$ and (8.1).

[17] Unique up to multiplication by a positive constant.

(9.3) *the maximum of* $\int \boldsymbol{\lambda} u(\boldsymbol{y})$ *over all allocations* $\boldsymbol{y}$ *is achieved when* $\boldsymbol{y} = \boldsymbol{x}$.

Conversely, any one of the three statements (9.1), (9.2), or (9.3) implies that $\boldsymbol{x}$ is efficient; moreover, given an efficient $\boldsymbol{x}$, the p satisfying (9.2) is essentially unique, as is the $\boldsymbol{\lambda}$ satisfying (9.3). From (9.1) it also follows that a.e.

$$\boldsymbol{\lambda}(t)u_t^i(\boldsymbol{x}(t)) \leqq p^i \text{ for all } i, \tag{9.4}$$

and

$$\boldsymbol{\lambda}(t)u_t^i(\boldsymbol{x}(t)) = p^i \text{ when } \boldsymbol{x}^i(t) > 0. \tag{9.5}$$

Note, incidentally, that (9.2) and (7.2) are the same.

From (9.5) it follows that a.e.

$$\boldsymbol{x}(t) \neq 0 \Rightarrow \boldsymbol{\lambda}(t) > 0.$$

In particular, if $\boldsymbol{x}(t) \neq 0$ a.e., then $\boldsymbol{\lambda}$ is a comparison function (i.e. $\boldsymbol{\lambda}(t) > 0$ a.e.). The converse, however, is false: if $\boldsymbol{\lambda}$ is a comparison function, $\boldsymbol{x}$ may still vanish at a set of positive measure.

The existence of an efficiency price vector p is well known[18]; its essential uniqueness follows from the differentiability of the u_t. One can then use (9.5) to define $\boldsymbol{\lambda}$, and deduce (9.1) (cf. the proof of lemma 14.1 of [1]). Our other assertions follow without difficulty from these considerations.

Given an efficient allocation $\boldsymbol{x}$, the efficiency comparison function $\boldsymbol{\lambda}$ can be thought of as providing "coefficients of importance" for the players[19]: the redistribution $\boldsymbol{x}$ would result if one would want to maximize total utility when the individual utilities u_t are weighted by $\boldsymbol{\lambda}(t)$.

In case $l = 1$, we may w.l.o.g. take $p = 1$. If $\boldsymbol{x}$ is a tax allocation, then by Theorem C, $\boldsymbol{x}(t) > 0$ a.e., so by (9.5), $\boldsymbol{\lambda}(t) = 1/u_t'(\boldsymbol{x}(t))$. Thus (8.1) becomes

$$\boldsymbol{\lambda}(t)u_t(\boldsymbol{x}(t)) - \int \boldsymbol{\lambda} u(\boldsymbol{x}) = \boldsymbol{e}(t) - \boldsymbol{x}(t). \tag{9.6}$$

The right side of (9.6) represents the net[20] taxes of dt. The left side is the excess of dt's final (i.e. after-tax) utility over the average final utility of all agents, when the utilities are compared using $\boldsymbol{\lambda}$. Thus (9.6) says that one's taxes are

[18] See e.g. Hildenbrand [7].

[19] Or rather for the players' utilities. If $\boldsymbol{\lambda}(t) = 2\boldsymbol{\lambda}(s)$, a unit of dt's utility is considered equivalent to two of ds's. Of course any rescaling of the u_t's involves a corresponding rescaling of the $\boldsymbol{\lambda}(t)$'s.

[20] I.e., the net decrement in dt's worth after both taxation and redistribution are taken into account. In fact, both sides of (9.6) are densities; to get actual amounts of tax, one should multiply by $\mu(dt)$.

proportional to how much one is better off than the average person, when utilities are weighted by one's "importance."

An alternative interpretation of (8.1) can be given in terms of marginal utilities. Since $u_t'(x(t))$ is the marginal utility of income, its inverse $1/u_t'(x(t))$ is the money worth of a unit of utility. Hence $u_t(x(t))/u_t'(x(t))$ is the monetary worth of dt's final utility when evaluated at the marginal rate. Thus (8.1) says that dt's taxes are equal, in dollars, to the amount by which the monetary worth of his utility exceeds the average monetary worth of everybody's utility.

Yet another interpretation of (8.1), in terms of the "fear of ruin," was discussed in [2]; we will not repeat this discussion here.

We note that

(9.7) *if* x *is an allocation satisfying* (8.1) *a.e., then* $x(t)>0$ *a.e.*;

this follows from the fact that $\int x = \int e$, hence x cannot a.e. vanish, and hence $\int u(x)/u'(x)>0$. Thus (8.1) alone is sufficient for an allocation x to be a tax allocation. Note also that if c is any constant such that for some allocation x, we have

(9.8) $$u_t(x(t))/u_t'(x(t)) = e(t) - x(t) + c$$

a.e., then x must be a tax allocation, since (9.8) implies $c = \int u(x)/u'(x)$.

Next, we show how Theorems B and C can be combined to yield a characterization of commodity tax allocations when there are many commodities ($l \geqq 1$). We require two lemmas.

LEMMA 9.9. *If a market* M *is bounded, so are all its derived markets* M^p.

This will be proved below (Lemma 16.1).

LEMMA 9.10. *Let* x *be an efficient allocation in a market* M, *with efficiency pair* $(\boldsymbol{\lambda}, p)$. *Then*

(9.11) px *is an efficient allocation in* M^p, *with efficiency pair* $(\boldsymbol{\lambda}, 1)$;

(9.12) $u(x) = u^p(px)$; *and*

(9.13) $\boldsymbol{\lambda}(t) = 1/(u^p)'(px(t))$ *whenever* $x(t) \neq 0$.

PROOF. (9.12) follows from (9.2) and (7.1). To prove (9.11), let $y \in R^+$, and let the maximum in the definition of $u^p(y)$ be achieved at x. Then $y = px$ and $u^p(y) = u(x)$; since $(\boldsymbol{\lambda}, p)$ is an efficiency pair for x, we deduce, using (9.12), that

$$\lambda(t)u_t^p(y) - y = \lambda(t)u_t(x) - px \leqq \lambda(t)u_t(x(t)) - px(t)$$
$$= \lambda(t)u_t^p(px(t)) - px(t).$$

But this means precisely that $(\lambda, 1)$ is an efficiency pair for px. (9.13) follows immediately from (9.11) and (9.5) applied to M^p.

The characterization of commodity tax allocations for $l \geqq 1$ is as follows:

PROPOSITION 9.14. *In a non-atomic bounded market M, an allocation* x *is a commodity tax allocation if and only if it is efficient and a.e.*

$$\lambda(t)u_t(x(t)) - \int \lambda u(x) = p(e(t) - x(t)), \tag{9.15}$$

where (λ, p) *is an efficiency pair for* x.

Note that (9.15) is subject to exactly the same interpretations as (9.6), except that the taxes are expressed in "dollar" (rather than commodity) terms, the prices being simply the efficiency prices for x.

To derive Proposition 9.14 from Theorems B and C, first let x be a commodity tax allocation. Then x is efficient. By Theorem B, x is an income tax allocation, so there is a normalized p satisfying (7.2) and (7.3). By (7.3) and Theorem C, $px(t) > 0$ a.e., and hence $x(t) \neq 0$ a.e. By (7.2), p is the normalized efficiency price vector for x, so there is a comparison function λ such that (λ, p) is the normalized efficiency pair for x. By (7.3), Lemma 9.9, and Theorem C, we have a.e.

$$\frac{u_t^p(px(t))}{u_t^{p\prime}(px(t))} + px(t) = pe(t) + \int \frac{u^p(px)}{u^{p\prime}(px)}. \tag{9.16}$$

Together, (9.12), (9.13) and (9.16) yield (9.15).

Conversely, let x be an efficient allocation, with normalized efficiency pair (λ, p) satisfying (9.15). Since $\int x = \int e$, x cannot vanish a.e., hence $\int \lambda u(x) > 0$, hence by (9.15) a.e. $x(t) \neq 0$, and hence a.e. $px(t) > 0$. From (9.12), (9.13) and (9.15) we then deduce (9.16), and so from Lemma 9.9 and Theorem C it follows that (7.3) holds. (7.2) follows from the fact that p is an efficiency price vector. Hence x is an income tax allocation, and so by Theorem B a commodity tax allocation. This completes the derivation of Proposition 9.14 from Theorems B and C.

In *presenting* our results, we stated Theorems B and C first and derived Proposition 9.14 from them. Our strategy of *proof* will be the reverse. First we will prove Proposition 9.14 (characterization in the many commodity case), in

Sections 11 through 14. This is the longest and deepest part of the paper; a reader seeking an intuitive overview of this part is referred to [2], Section 9, Steps 1 through 5.[21] From this we derive Theorem C (existence, uniqueness, and characterization in the non-atomic case) in Section 15, and Theorem B (equivalence in the non-atomic case) in Section 16. Theorem B and the second part of Theorem C — i.e. the characterization (8.1) — are derived rather easily from Proposition 9.14, in much the same way that we just did the reverse derivation. The first part of Theorem C (existence and uniqueness in the non-atomic one-commodity case), however, requires a separate, though not particularly difficult argument. Theorem A (existence in the many-commodity case) is then proved in two parts. The non-atomic case is done in Section 17: One finds an income tax allocation by applying Theorem C to calculate demand in the derived markets M^p, allowing p to vary, and using Debreu's lemma [5, p. 82]; by Theorem B, the income tax allocation thus found is also a commodity tax allocation. In the finite-player case there is no equivalence; we apply in two different ways a basic lemma of Shapley [14] about the existence of NTU values, to find both kinds of tax allocations (Section 18).

Additional discussion of these results, and especially of Theorem C, will be found in [2].

10. Examples

Several one-dimensional examples were discussed in [2]. In this section we wish to examine a class of multidimensional examples. We will confine ourselves to the non-atomic case.

We start with a lemma that will simplify the calculations.

LEMMA 10.1. *Let $\boldsymbol{x}$ be a commodity (or equivalently, income) tax allocation, with efficiency pair $(\boldsymbol{\lambda}, p)$. Set $c = \int \boldsymbol{\lambda} u(\boldsymbol{x})$. Assume $\boldsymbol{x}(t) > 0$ a.e., and set*[22]

$$\varepsilon^i(t) = u_t^i(\boldsymbol{x}(t))\boldsymbol{x}^i(t)/u_t(\boldsymbol{x}(t)),$$

$$\varepsilon(t) = \sum_{i=1}^{l} \varepsilon^i(t).$$

Then a.e.

$$p^i \boldsymbol{x}^i(t)/\varepsilon^i(t) = c + p(\boldsymbol{e}(t) - \boldsymbol{x}(t)) \tag{10.2}$$

[21] [2] treats only $l = 1$; but the intuitive considerations for general l are in many respects similar, though somewhat more complex.

[22] $\varepsilon^i(t)$ is the elasticity of t's utility w.r.t. x^i at $\boldsymbol{x}^i(t)$.

and a.e.

$$px(t) = \frac{\varepsilon(t)}{1+\varepsilon(t)}(c + pe(t)). \tag{10.3}$$

If ε is a constant — say $\varepsilon(t) = \varepsilon$ — then

$$c = \int pe/\varepsilon. \tag{10.4}$$

PROOF. (10.2) follows from (9.15) and (9.5). Multiplying both sides of (10.2) by $\varepsilon^i(t)$, summing over i, and dividing by $1+\varepsilon(t)$, we obtain (10.3). To obtain (10.4), we integrate (10.3) and remember that $\int x = \int e$. This completes the proof of the lemma.

Let u be a utility function that is homogeneous of degree α, where $0 < \alpha \leqq 1$. For our first example, we would like to choose $u_t = u$ for all t. Because u is homogeneous, the induced preferences are homothetic; hence all efficient allocations consist of bundles lying on the ray from the origin through the aggregate initial endowment $\int e$, and we may write

$$p = u'(\textstyle\int e). \tag{10.5}$$

Let x be a commodity tax allocation; then x is efficient, and hence $x(t)$ is a scalar multiple of $\int e$ for each t. From this and the homogeneity we obtain $\varepsilon^i(t) = u^i(\int e)\int e^i/u(\int e)$. By Euler's formula, it follows that

$$\varepsilon(t) = \sum_{i=1}^{l} \varepsilon^i(t) = \sum_{i=1}^{l} (\textstyle\int e^i)u^i(\int e)/u(\int e) = \alpha.$$

From (10.3) and (10.4) we obtain

$$px(t) = \frac{1}{1+\alpha}\int pe + \frac{\alpha}{1+\alpha}pe(t), \tag{10.6}$$

i.e. the net income of each agent is a mix of his gross income and the average gross income of all agents, in the ratio α:1. For the tax we get

$$pe(t) - px(t) = \frac{1}{1+\alpha}(pe(t) - \textstyle\int pe), \tag{10.7}$$

and this means that we have a linear tax with a rate of at least 50% (since $\alpha \leqq 1$). The bundle $x(t)$ itself (as distinguished from its worth $px(t)$) may be calculated by recalling that it must be a scalar multiple of $\int e$; hence by (10.6),

$$(10.8)\qquad x(t)=\frac{px(t)}{p\int e}\int e=\left(\frac{1}{1+\alpha}+\frac{\alpha}{1+\alpha}\frac{pe(t)}{\int pe}\right)\int e,$$

where $p=u'(\int e)$.

There is, however, a difficulty; we cannot take $u_t=u$, because homogeneous utility functions are not bounded. To get around this, assume e is bounded, say $e(t)\leqq b\in\Omega$ for all t. Define u_t by $u_t(x)=f(u(x))$ for all x, where f is bounded and $f(y)=y$ for $y\leqq u((pb/\int pe)\int e)$, where $p=u'(\int e)$. Though the u_t are no longer homogeneous, all agents still have the same homothetic preferences; hence all efficient allocations are on the ray through $\int e$ and the efficiency prices are $p=u'(\int e)$. Since by Theorem C the derived market M^p has only one tax allocation, it follows that there is exactly one income (and hence commodity) tax allocation in M. But by (10.8), $x(t)\leqq(pb/\int pe)\int e$, hence u_t and its derivatives are the same as u and its derivatives at $x(t)$, and so from Proposition 9.14 it follows that the x of (10.8) is the unique commodity tax allocation in the given market.

For a more specific example, we may let $a_1,\cdots,a_k$ be l-dimensional vectors, and define

$$(10.9)\qquad u(x)=(a_1x)^{\alpha_1}(a_2x)^{\alpha_2}\cdots(a_kx)^{\alpha_k},$$

where $\alpha_j>0$ and $\Sigma_{j=1}^k\alpha_j\leqq 1$. If $k=l$ and a_j is the j-th unit vector, this is a Cobb–Douglas utility; such utilities are however excluded, since they are not monotonic on the boundary of Ω. If $a_j>0$ (i.e. all components are positive) for all j, monotonicity is restored. In this case the degree α of homogeneity is simply $\Sigma_{j=1}^k\alpha_j$.

Another example is given by

$$(10.10)\qquad u(x)=\sum_{j=1}^{k}(a_jx)^{\alpha};$$

here it is only required that $\Sigma_{j=1}^k a_j>0$ and $a_j\geqq 0$ for all j.

To some extent the above method may be used even when the u_t are different, as long as they are homogeneous, all with the same degree of homogeneity α, in the "relevant" part of Ω (i.e. up to an appropriately chosen bound). This would be the case, for example, if the right side of (10.9) were replaced by $(a_1(t)x)^{\alpha_1(t)}\cdots(a_k(t)x)^{\alpha_k(t)}$, subject to the restriction $\Sigma_{j=1}^k\alpha_j(t)=\alpha$; or if the right hand side of (10.10) were replaced by $\Sigma_{j=1}^k(a_j(t)x)^{\alpha}$. In this kind of situation the elasticities $\varepsilon^i(t)$ may be different for different t, but their sum $\varepsilon(t)$ will always equal α. From (10.3) and (10.4) we can then deduce (10.6) and (10.7), i.e. we can calculate the net income and the tax in terms of gross income, average gross income, and the degree α of homogeneity. But calculating the prices p and the actual allocation $\boldsymbol{x}$ is quite another matter, since we can no longer say that $\boldsymbol{x}(t)$ must be on the ray through $\int \boldsymbol{e}$. We will not go further into this matter here.

As before, we must modify the utilities to make them bounded; the appropriate way to do this is in this case a little complex, and it may be necessary to assume that the normalized gradient $u'_t(x)/\Sigma^l_{i=1}u^i_t(x)$ is bounded away from 0 (as will be the case for utility functions of type (10.9), but not for those of type (10.10) if the a_i are permitted to have vanishing components). Unlike before, we have no way of knowing what prices in the modified economy will look like. Therefore, though we can be sure that there is a commodity tax allocation satisfying (10.6) and (10.7), we cannot be sure that there are no others.

11. The optimal threat

Let M be a market, $\Gamma = \Gamma(M)$ the corresponding Redistribution Game, H^S, v, and w as in Section 4; in particular, v is the Harsanyi coalitional form of Γ. In this section we shall show that the minority's optimal threat — i.e., optimal strategy in H^S — is actually to destroy its entire endowment.[23] We shall then show that the value of v is the same as that of a coalitional game q in which $q(S)$ is the total utility of S when this threat is carried out.

Define coalitional games $r = r_M$ and $q = q_M$ by

$$r(S) = \sup\left\{\int_S u(\boldsymbol{x}) : \int_S \boldsymbol{x} = e(S)\right\} \tag{11.1}$$

and

$$q(S) = \begin{cases} r(S) & \textit{if} \quad \mu(S) \geqq \frac{1}{2} \\ 0 & \textit{if} \quad \mu(S) < \frac{1}{2}. \end{cases} \tag{11.2}$$

Note that if $r(T)$ is finite then $r(S)$ is finite for all S. We shall say that $r(S)$ is *attained* if it is finite and there is an S-allocation $\boldsymbol{x}$ with $r(S) = \int_S u(\boldsymbol{x})$.

PROPOSITION 11.3. *Assume that v is defined. Then $r(S)$ is attained whenever $\mu(S) > 1/2$, and for all S we have*

$$w(S) = q(S) - q(T \setminus S).$$

PROOF. That v is defined means that all the H^S have saddle points. Let (σ_0, τ_0) be a saddle point for H^S. Suppose first that $\mu(S) > 1/2$. Using the τ of (6.2), we

[23] Of course this threat is not carried out in the final outcome.

find that there is an S-allocation $\boldsymbol{x}$ such that $\boldsymbol{h}^S_{\sigma_0\tau}(t) \leqq u_t(\boldsymbol{x}(t))$ for all $t \in S$. Hence from (4.6) and (4.5) we get

$$H^S(\sigma_0, \tau_0) \leqq H^S(\sigma_0, \tau) \leqq \int_S \boldsymbol{h}^S_{\sigma_0\tau} \leqq \int_S u(\boldsymbol{x}).$$

Applying (6.1) to this same S-allocation $\boldsymbol{x}$, we get a strategy σ of S such that

$$\boldsymbol{h}^S_{\sigma\tau_0}(t) \begin{cases} \geqq u_t(\boldsymbol{x}(t)) & \text{if} \quad t \in S \\ = 0 & \text{if} \quad t \notin S. \end{cases}$$

Again using (4.6) and (4.5) we get

$$H^S(\sigma_0, \tau_0) \geqq H^S(\sigma, \tau_0) = \int_S \boldsymbol{h}^S_{\sigma\tau_0} - \int_{T\setminus S} \boldsymbol{h}^S_{\sigma\tau_0} \geqq \int_S u(\boldsymbol{x}).$$

Hence

(11.4) $$w(S) = H^S(\sigma_0, \tau_0) = \int_S u(\boldsymbol{x}).$$

Suppose now that $r(S)$ is not attained at $\boldsymbol{x}$, i.e. that there is an S-allocation $\boldsymbol{x}'$ with $\int_S u(\boldsymbol{x}') > \int_S u(\boldsymbol{x})$. If σ' corresponds to this $\boldsymbol{x}'$ in accordance with (6.1), then again using (4.6) and (4.5) we get

$$H^S(\sigma_0, \tau_0) \geqq H^S(\sigma', \tau_0) = \int_S \boldsymbol{h}^S_{\sigma'\tau_0} - \int_{T\setminus S} \boldsymbol{h}^S_{\sigma'\tau_0} \geqq \int_S u(\boldsymbol{x}') > \int_S u(\boldsymbol{x}),$$

contradicting (11.4). Hence when $\mu(S) > 1/2$, $r(S)$ is attained at $\boldsymbol{x}$, and

(11.5) $$q(S) - q(T\setminus S) = r(S) = \int_S u(\boldsymbol{x}) = w(S).$$

Next, note that $H^S(\sigma, \tau) = -H^{T\setminus S}(\tau, \sigma)$. Hence (σ_0, τ_0) is a saddle point of $H^{T\setminus S}$, and therefore

(11.6) $$w(S) = -w(T\setminus S).$$

This shows that when $\mu(S) < 1/2$,

$$q(S) - q(T\setminus S) = -r(T\setminus S) = -w(T\setminus S) = w(S).$$

Consider finally the case $\mu(S) = 1/2$. We have already shown that $r(T)$ is attained and in particular is finite, and hence $r(S)$ is also finite. Given $\varepsilon > 0$, let $\boldsymbol{x}$ be an S-allocation with $\int_S u(\boldsymbol{x}) > r(S) - \varepsilon$. By (6.3), there is a strategy σ of S and a $T\setminus S$-allocation $\boldsymbol{y}$ such that

$$h^S_{\sigma\tau_0}(t)\begin{cases} \geqq u_t(x(t)), & t \in S \\ \leqq u_t(y(t)), & t \in T\backslash S\,. \end{cases}$$

Hence

$$w(S) = H^S(\sigma_0, \tau_0) \geqq H^S(\sigma, \tau_0) \geqq \int_S u(x) - \int_{T\backslash S} u(y) \tag{11.7}$$

$$> r(S) - \varepsilon - r(T\backslash S) = q(S) - q(T\backslash S) - \varepsilon.$$

Since also $\mu(T\backslash S) = 1/2$, we deduce

$$w(T\backslash S) > q(T\backslash S) - q(S) - \varepsilon\,;$$

hence by (11.6),

$$w(S) < q(S) - q(T\backslash S) + \varepsilon. \tag{11.8}$$

Letting $\varepsilon \to 0$ in (11.7) and (11.8), we obtain the desired result.

COROLLARY 11.9. *Assume that v is defined. Then v has a μ-value if and only if q does, and in that case the μ-values are equal.*

PROOF. From Proposition 11.3 and (4.8) we obtain

$$v(S) = \tfrac{1}{2}[q^*(S) + q(S)], \tag{11.10}$$

and the result then follows from (3.1).

PROPOSITION 11.11. *Assume that all the $r(S)$ are attained. Then all the games H^S have saddle points, i.e. v is defined.*

PROOF. First let $\mu(S) > 1/2$. Let $r(S)$ be attained at x. By (6.1), there is a strategy σ_0 of S such that for all strategies τ of $T\backslash S$, $\int_S h^S_{\sigma_0\tau} \geqq \int_S u(x)$ and $\int_{T\backslash S} h^S_{\sigma_0\tau} = 0$; hence

$$H^S(\sigma_0, \tau) \geqq \int_S u(x).$$

On the other hand, by (6.2) there is a strategy τ_0 of $T\backslash S$ such that for all strategies σ of S,

$$\int_S h^S_{\sigma\tau_0} \leqq \max\left\{\int_S u(y) \colon \int_S y = e(S)\right\} = r(S) = \int_S u(x)$$

and hence

$$H^S(\sigma, \tau_0) \leqq \int_S u(x).$$

Hence (σ_0, τ_0) is a saddle point of H^S. Since $H^{T\setminus S}(\tau, \sigma) = -H^S(\sigma, \tau)$, we have completed the proof when $\mu(S) > 1/2$ or $\mu(S) < 1/2$.

Assume now that $\mu(S) = 1/2$. Then also $\mu(T\setminus S) = 1/2$. Then by (6.3), there is a strategy σ_0 of S such that for all strategies τ of $T\setminus S$,

$$H^S(\sigma_0, \tau) \geqq \int_S u(\boldsymbol{x}) - r(T\setminus S) = r(S) - r(T\setminus S).$$

Similarly, there is a strategy τ_0 of $T\setminus S$ such that for each strategy σ of S,

$$H^{T\setminus S}(\tau_0, \sigma) \geqq r(T\setminus S) - r(S)$$

and hence

$$H^S(\sigma, \tau_0) \leqq r(S) - r(T\setminus S).$$

Hence (σ_0, τ_0) is a saddle point. This completes the proof of Proposition 11.11.

PROPOSITION 11.12. *A necessary and sufficient condition that an allocation* $\boldsymbol{x}$ *in* M *be a commodity tax allocation is that there exists a comparison function* $\boldsymbol{\lambda}$ *such that* $v_{\boldsymbol{\lambda}}$ *is defined and has a* μ*-value given by*

$$(\phi v_{\boldsymbol{\lambda}})(S) = \int_S \boldsymbol{\lambda} u(\boldsymbol{x}). \tag{11.13}$$

PROOF. The necessity follows immediately from the definitions. To prove the sufficiency, note that by (6.1), there is a strategy σ of T such that

$$\boldsymbol{h}_\sigma^T(t) \geqq u_t(\boldsymbol{x}(t)) \quad \textit{for all } t. \tag{11.14}$$

Multiplying (11.14) by $\boldsymbol{\lambda}(t)$, integrating over T, and using (3.2), we obtain

$$\int \boldsymbol{\lambda} \boldsymbol{h}_\sigma^T \geqq \int \boldsymbol{\lambda} u(\boldsymbol{x}) = (\phi v_{\boldsymbol{\lambda}})(T) = v_{\boldsymbol{\lambda}}(T). \tag{11.15}$$

But by applying (4.8) to the game $\boldsymbol{\lambda}\Gamma$, we may deduce $v_{\boldsymbol{\lambda}}(T) = w_{\boldsymbol{\lambda}}(T)$; and then applying (4.7) to $\boldsymbol{\lambda}\Gamma$, we deduce $v_{\boldsymbol{\lambda}}(T) \geqq \int \boldsymbol{\lambda} \boldsymbol{h}_\sigma^T$. Combining this with (11.15), we deduce $\int \boldsymbol{\lambda} \boldsymbol{h}_\sigma^T = \int \boldsymbol{\lambda} u(\boldsymbol{x})$, and hence equality holds a.e. in (11.14). Hence

$$\int_S \boldsymbol{\lambda} \boldsymbol{h}_\sigma^T = \int_S \boldsymbol{\lambda} u(\boldsymbol{x}) = (\phi v_{\boldsymbol{\lambda}})(S)$$

for all S in $\mathscr{C}$, and hence $\boldsymbol{h}_\sigma^T$ is a value; since equality holds a.e. in (11.14), it follows that $u(\boldsymbol{x})$ is a value and so $\boldsymbol{x}$ a commodity tax allocation. This completes the proof of Proposition 11.12.

The major conclusion of this section is Corollary 11.9, which enables us to replace the rather complex strategic game Γ, for purposes of calculating the value, by the relatively transparent coalitional game q. What enables us to do this is the explicit calculation, in Proposition 11.3, of the optimal threat strategies σ_0 and τ_0 of the majority and the minority: namely, for the majority to tax at 100%, and for the minority to destroy its entire endowment.

12. Preliminaries on non-atomic games

In proving our results, we shall make extensive use of the theory of non-atomic games developed in [4]. This section is devoted to reviewing some of the relevant results of that theory.

The *asymptotic value* of a game ([9], [4, section 17]) is defined in exactly the same way as the μ-value (Section 3), except that the W_i need not have equal μ-measure, and the $\mu(S)$ need not be rational. If under these conditions, $\phi_n(S)$ still tends to a limit $(\phi v)(S)$ and the limit is independent of the choice of the sequence of partitions, and if this is true for all S in $\mathscr{C}$, then ϕv is called the *asymptotic value* of v. Obviously we have

REMARK 12.1. *If the asymptotic value of v exists, so does the μ-value, and they are equal.*

Throughout this section, r will refer to a general (coalitional) game, not necessarily of the form (11.1). A game r is said to be *monotonic* if $S \supset U$ implies $r(S) \geqq r(U)$. It is of *bounded variation* if it is the difference of monotonic games; the linear space of games of bounded variation on $(T, \mathscr{C})$ is denoted BV. The *variation norm* (or simply *norm*) on BV is defined by

$$\| r \| = \sup \sum_{i=1}^{n} | r(S_i) - r(S_{i-1}) |,$$

where the supremum is over all chains of coalitions $\emptyset = S_0 \subset \cdots \subset S_n = T$.

The bounded games on $(T, \mathscr{C})$ form a linear space called BS; clearly $BS \supset BV$. The *supremum* (or *sup*) norm on BS is defined by

$$\| r \|' = \sup | r(S) |,$$

where the supremum is over all S in $\mathscr{C}$.

The non-atomic σ-additive measures form a subspace of BV called NA; the subset of NA consisting of non-negative measures ν with $\nu(T) = 1$ is called NA^1. The set of all linear combinations of positive integer powers of NA^1

measures is called P; games in P are called *measure polynomials.* The variation closure of P in BV is called pNA, and the sup closure of P in BS is called pNA'. Clearly $pNA \subset pNA'$.

The set of measurable functions from $(T, \mathscr{C})$ to $[0,1]$ is denoted $\mathscr{I}$. It is useful to think of a member of $\mathscr{I}$ as an "ideal" subset of T; the number $f(t)$ is the "degree" to which the point t belongs to the ideal set f. Ordinary sets correspond to functions whose value is either 1 or 0, which is interpreted as meaning that the point either "completely belongs" or "completely fails to belong" to the set. If $S \in \mathscr{C}$, we denote by χ_S the characteristic function, defined by $\chi_S(t) = 1$ if $t \in S$, and $\chi_S(t) = 0$ if $t \notin S$. An *ideal game* is a function from $\mathscr{I}$ to the real numbers that vanishes at 0. The linear subspace of all bounded ideal games is denoted IBS; on it we define the *supremum* (or *sup*) norm by

$$\| r \|' = \sup\{| r(f)| : f \in \mathscr{I}\}.$$

PROPOSITION 12.2. *There is a unique linear, sup-continuous mapping that associates with each game r in pNA' an ideal game r^*, so that*

$$(\nu^k)^* = (\nu^*)^k, \quad \textit{and} \tag{12.3}$$

$$\nu^*(f) = \int_T f(t)\nu(dt) \tag{12.4}$$

for all measures ν in NA^1, positive integers k, and ideal sets f.

This is proposition 22.16 of [4]. The operator $r \to r^*$ is called the *extension operator*; this name is justified by the following proposition:

PROPOSITION 12.5. *If $r \in pNA'$, then $r^*(\chi_S) = r(S)$.*

PROOF. If $r \in NA^1$, the proposition follows from (12.4); hence by (12.3), it follows for all powers of NA^1 measures; by linearity of the extension operator, for all measure polynomials; and finally, by sup-continuity, for all of pNA'.

If $r \in pNA'$, $t \in (0,1)$, and $S \in \mathscr{C}$, denote

$$\partial r^*(t, S) = \lim_{\tau \to 0} \frac{r^*(t\chi_T + \tau\chi_S) - r^*(t\chi_T)}{\tau}.$$

At this point we are of course making no claim about the existence of this limit, we are merely introducing a notation. If $\nu \in NA^1$ and $r = \nu^k$ for some positive integer k, then

$$\partial r^*(t, S) = kt^{k-1}\nu(S). \tag{12.6}$$

Define $DIAG$ as the linear space of all r in BV for which

(12.7) *there is a positive integer* k, *a* k*-dimensional vector* ζ *of measures in* NA^1, *and a neighborhood* U *in* E^k *of the diagonal* $[(0,\ldots,0),(1,\ldots,1)]$ *such that if* $\zeta(S)\in U$ *then* $r(S)=0$.

Motivation for this definition may be found in [4], on p. 252. Define $pNAD$ to be the variation closure of $pNA + DIAG$ (or equivalently, of $P + DIAG$).

PROPOSITION 12.8. *Let* $r \in pNAD \cap pNA'$; *then* r *has an asymptotic value* ϕr. *Furthermore, for each coalition* S, *the derivative* $\partial r^*(t,S)$ *exists for almost all* t *in* $[0,1]$ *and is integrable over* $[0,1]$ *as a function of* t; *and*

$$(\phi r)(S)=\int_0^1 \partial r^*(t,S)dt. \tag{12.9}$$

Finally,

$$\|r\| \geqq \int_0^1 |\partial r^*(t,S)|dt. \tag{12.10}$$

PROOF. The three sentences are, respectively, corollary 43.12, proposition 44.22 and formula (44.23) of [4].

PROPOSITION 12.11. *Suppose* r *in* $BV\cap pNA'$ *is of the form* r_1+r_2, *where* $r_1\in DIAG$. *Then* $\int_0^1|\partial r^*(t,S)|dt \leqq \|r_2\|$, *and* $|r^*(\alpha\chi_T)|\leqq\|r_2\|$ *for all* α *between* 0 *and* 1.

PROOF. The first assertion follows from formula[24] (44.23) and lemma 44.14 of [4]; the second from lemma 44.14 of [4].

A game r in pNA' is called *homogeneous of degree* 1 if for all α in $[0,1]$ and all S in $\mathscr{C}$, we have $r^*(\alpha\chi_S)=\alpha r(S)$.

PROPOSITION 12.12. *Let* r *in* $pNAD\cap pNA'$ *be homogeneous of degree* 1. *Then for all* S *in* $\mathscr{C}$ *and all* t *in* $(0,1)$,

$$\partial r^*(t,S)=(\phi r)(S).$$

PROOF. Existence of $\partial r^*(t,S)$ for almost all t follows from Proposition 12.8. Then reasoning precisely as in the proof of lemma 27.2 of [4] we deduce that $\partial r^*(t,S)$ exists and is the same for all t in $(0,1)$. The fact that it must be ϕr then follows from (12.9).

[24] The δ-norm $\|\ \ \|_\delta$ appearing in this formula is defined on p. 262 of [4].

13. The value of truncated games

Corollary 11.9 indicates the importance of the game q defined in (11.2), which we might say is derived from r by "truncating" below $\mu(S) = 1/2$. In this section we prove a proposition that relates the value of a "truncated" game q to that of the corresponding untruncated game r in a more general context, one in which r is not necessarily derived from a market.

PROPOSITION 13.1. *Let $\mu \in NA^1$, let $r \in pNAD \cap pNA'$, and let $0 < \alpha < 1$. Define*

$$q(S) = \begin{cases} r(S) & \text{if } \mu(S) \geqq \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{13.2}$$

Then q has a μ-value, given by

$$(\phi q)(S) = r^*(\alpha\chi_T)\mu(S) + \int_\alpha^1 \partial r^*(t, S)dt. \tag{13.3}$$

The proof is along lines similar to those used in section 18 of [4]. Let $S \in \mathscr{C}'$, and let $\mathscr{P} = \{\Pi_1, \Pi_2, \cdots\}$ be a separating sequence of partitions of T into coalitions of equal μ-measure, such that S is a union of members of Π_1. Let $\mathfrak{R}_m$ be a *random order* on Π_m, i.e. a random variable whose values are orders on Π_m, in which all of the $|\Pi_m|!$ possible orders have the same probability. Let $\boldsymbol{B}_h^m$ be the h-th member of Π_m in the order $\mathfrak{R}_m$, and set

$$\boldsymbol{Q}_h^m = \boldsymbol{B}_1^m \cup \cdots \cup \boldsymbol{B}_h^m.$$

The following lemma is proved in [4, p. 132, corol. 18.10]:

LEMMA 13.4. *Let ζ be a vector of NA^1 measures. Let U be a neighborhood of the "diagonal" in the range of ζ, i.e. the line segment with end points $(0, \cdots, 0)$ and $(1, \cdots, 1)$. Then for every ε, there is an m_0, such that for $m \geqq m_0$,*

$$\operatorname{Prob}\{\zeta(\boldsymbol{Q}_h^m) \in U \quad \text{for all} \quad h \quad \text{with} \quad 1 \leqq h \leqq |\Pi_m|\} > 1 - \varepsilon.$$

In words, this says that for sufficiently fine partitions, the sequence $\zeta(\boldsymbol{Q}_h^m)$ will with high probability remain in an arbitrarily small neighborhood of the diagonal.

We now proceed with the

PROOF OF PROPOSITION 13.1. Fix m, set $\Pi = \Pi_m$, $\mathfrak{R} = \mathfrak{R}_m$, $\boldsymbol{B}_h = \boldsymbol{B}_h^m$, $\boldsymbol{Q}_h = \boldsymbol{Q}_h^m$, and $n = |\Pi|$. For each h between 1 and n, let

$$z_h = \begin{cases} 1 & \text{if } B_h \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

Let h^* be the smallest integer not smaller than αn. Set

$$\begin{aligned} \Delta = \Delta^m(r) &= \sum_{h=1}^{n} [q(Q_h) - q(Q_{h-1})] z_h \\ &= r(Q_{h^*}) z_{h^*} + \sum_{h > h^*} [r(Q_h) - r(Q_{h-1})] z_h, \end{aligned} \tag{13.5}$$

where $Q_0 = \emptyset$. The expression Δ is the total contribution of the players in S_m (i.e. the players of the finite game q_m who are included in S) to q_m, when Π is ordered according to $\mathcal{R}$; thus

$$(\phi q_m)(S_m) = E\Delta^m(r). \tag{13.6}$$

Set

$$\theta r = r^*(\alpha \chi_T)\mu(S) + \int_\alpha^1 \partial r^*(t, S) dt;$$

we wish to prove that $E\Delta^m(r) \to \theta r$ as $m \to \infty$.

Consider first the special case in which $r = \nu^k$, where $\nu \in NA^1$ and k is a positive integer. Setting $f(x) = x^k$, we obtain

$$r(Q_h) - r(Q_{h-1}) = f(\nu(Q_{h-1}) + \nu(B_h)) - f(\nu(Q_{h-1})) = \nu(B_h) f'(x_h),$$

where

$$\nu(Q_{h-1}) \leqq x_h \leqq \nu(Q_h); \tag{13.7}$$

thus (13.5) becomes

$$\Delta = f(\nu(Q_{h^*})) z_{h^*} + \sum_{h > h^*} \nu(B_h) z_h f'(x_h). \tag{13.8}$$

Let $\eta(\varepsilon)$ be the minimum of the moduli of uniform continuity of f and f' on $[0, 1]$. Let $\zeta = (\mu, \nu)$, and for each $\varepsilon > 0$ define a neighborhood U of the diagonal in the range of ζ by $U = \{(x, y): |x - y| < \eta(\varepsilon)/2\}$. From Lemma 13.4 it then follows that for m sufficiently large, we have with probability $> 1 - \varepsilon$ that for all h,

$$|\nu(Q_h) - h/n| = |\nu(Q_h) - \mu(Q_h)| < \eta(\varepsilon)/2.$$

From this and (13.7) it follows that for m sufficiently large,

$$|x_h - h/n| < \eta(\varepsilon)/2 + 1/n < \eta(\varepsilon).$$

Hence if throughout (13.8), we replace $\nu(\boldsymbol{Q}_h)$ and $\boldsymbol{x}_h$ by h/n, then we are with probability $>1-\varepsilon$ making an error $<\varepsilon$. Thus we may write

$$\Delta = f(h^*/n)z_{h^*} + \sum_{h>h^*} \nu(\boldsymbol{B}_h)z_h f'(h/n) + \boldsymbol{\psi}$$

for m sufficiently large, where $|\boldsymbol{\psi}|<\varepsilon$ with probability $>1-\varepsilon$ and $|\boldsymbol{\psi}|$ is bounded; the bound — let us call it C — is the maximum of $|f'|$ on $[0,1]$. Since the $\boldsymbol{B}_h$ have equal μ-measure, it follows that the number of $\boldsymbol{B}_h$ such that $\boldsymbol{B}_h \subset S$ is precisely $n\mu(S)$; from this we conclude that $Ez_{h^*} = \mu(S)$ and $E(\nu(\boldsymbol{B}_h)z_h) = \nu(S)/n$ for all h. Hence

$$E(\Delta) = f(h^*/n)\mu(S) + \sum_{h>h^*} f'(h/n)\nu(S)/n + \psi,$$

where $|\psi|<2\varepsilon C$ for m sufficiently large. The fact that $|\psi|$ becomes arbitrarily small for m sufficiently large means that $\psi=\psi^m \to 0$ as $m\to\infty$; from the definition of the Riemann integral it then follows that

$$\text{(13.9)} \qquad \lim_{m\to\infty} E\Delta^m(r) = f(\alpha)\mu(S) + \int_\alpha^1 f'(t)\nu(S)dt.$$

But by (12.3), $f(\alpha)=\alpha^k=(\nu^*(\alpha\chi_T))^k = r^*(\alpha\chi_T)$; applying this and (12.6) to (13.9), we deduce that when $r=\nu^k$, then indeed $E\Delta^m(r)\to\theta r$. From this and the additivity of $E\Delta(r)$ as a function of r, it follows that

$$\text{(13.10)} \qquad E\Delta^m(r)\to\theta r \quad \textit{for} \quad r\in P,$$

i.e. when r is a measure polynomial.

Suppose now that r is an arbitrary member of $pNAD \cap pNA'$. Let $\varepsilon>0$ be given. Since $pNAD$ is the variation closure of $P+DIAG$, we have $r = r_0+r_1+r_2$, where $r_0\in P$, $r_1\in DIAG$, and $\|r_2\|<\varepsilon$. Let ζ and U correspond to r_1 in accordance with (12.7). Then by Lemma 13.4, for m sufficiently large we have with probability $>1-\varepsilon$ that $\zeta(\boldsymbol{Q}_h)\in U$ for all h, and hence $\Delta^m(r_1)=0$. From Proposition 12.11 it follows that $|\theta(r_1+r_2)|\leqq 2\|r_2\|$ (note that $r_1+r_2=r-r_0\in pNA'$). Hence by (13.10) and Proposition 12.11, we have that for m sufficiently large,

$$|E\Delta^m(r)-\theta r| \leqq |E\Delta^m(r_0)-\theta r_0| + |E\Delta^m(r_1)| + |E\Delta^m(r_2)| + |\theta(r_1+r_2)|$$

$$< \varepsilon + 0 + \|r_2\| + 2\|r_2\| < 4\varepsilon.$$

Hence $E\Delta^m(r)\to\theta r$ as $m\to\infty$, and so by (13.6), $(\phi q_m)(S_m)\to\theta r$ as $m\to\infty$. This completes the proof of Proposition 13.1.

COROLLARY 13.11. *Let r in $pNAD \cap pNA'$ be homogeneous of degree* 1. *Define q by* (13.2). *Then q has a μ-value, given by*

$$(\phi q)(S) = \alpha r(T)\mu(S) + (1-\alpha)(\phi r)(S). \tag{13.12}$$

PROOF. Propositions 13.1 and 12.12.

14. Characterization of commodity tax allocations in the many-commodity non-atomic case: Proof of Proposition 9.14

Let us be given a non-atomic market M, and let $\Gamma = \Gamma(M)$ be the corresponding commodity redistribution game. Boundedness of M will be assumed only when specified.

The market M will be called *integrably sublinear*[25] if for each $\varepsilon > 0$ there is a μ-integrable function η on T such that $u_t(x) \leqq \varepsilon \|x\|$ whenever $\|x\| > \eta(t)$. A *transferable utility competitive equilibrium* (t.u.c.e.) in M (see Aumann and Shapley [4], section 32) is a pair $(\boldsymbol{x}, p)$ where $\boldsymbol{x}$ is an allocation and p a price vector, such that a.e. $u_t(x) - px$ attains its maximum over x in Ω at $x = \boldsymbol{x}(t)$. If $\boldsymbol{\lambda}$ is a comparison function, then clearly $(\boldsymbol{x}, p)$ is a t.u.c.e. in $\boldsymbol{\lambda}M$ if and only if $(\boldsymbol{\lambda}, p)$ is an efficiency pair for $\boldsymbol{x}$ in M.

Define the coalitional games v, w, r and q by (4.8), (4.7), (11.1), and (11.2).

PROPOSITION 14.1. *Let M be integrably sublinear. Then*

$$r(S) \textit{ is attained for all } S, \textit{ and} \tag{14.2}$$

$$v \textit{ is defined.} \tag{14.3}$$

Furthermore, if $(\boldsymbol{x}, p)$ is a t.u.c.e. for M, then v has a μ-value ϕv, given by

$$(\phi v)(S) = \frac{1}{2}\int_S [r(T) + u(\boldsymbol{x}) - p(\boldsymbol{x} - \boldsymbol{e})]. \tag{14.4}$$

PROOF. (14.2) is a special case of the main theorem of [3] (see also [4, prop. 36.1]). Assertion (14.3) follows from (14.2) and Proposition 11.11. Next, we note that $r \in pNAD \cap pNA'$ and is homogeneous of degree 1; this follows from [4, corol. 45.8 and prop. 45.10]. Hence by Corollary 13.11 with $\alpha = 1/2$, the game q has a μ-value, given by

$$(\phi q)(S) = \tfrac{1}{2} r(T)\mu(S) + \tfrac{1}{2}(\phi r)(S). \tag{14.5}$$

[25] Shapley and Shubik [15] use the term "sublinear" to describe a function that is $o(\|x\|)$ as $\|x\| \to \infty$. The concept of *integrable* sublinearity was introduced by Aumann and Perles [3], though they used somewhat different terminology ($u_t(x) = o(\|x\|)$, *integrably* in t). For a discussion of the concept, see [4, p. 183].

Proposition 32.3 of [4] asserts that r has an asymptotic value, given by

$$(\phi r)(S) = \int_S [u(x) - p(x - e)]. \tag{14.6}$$

By (14.3), v is defined, and so from Corollary 11.9 we deduce that v has a μ-value ϕv equal to ϕq. Formula (14.4) then follows from (14.5) and (14.6). This completes the proof of Proposition 14.1.

Note that $\Gamma(\boldsymbol{\lambda} M) = \boldsymbol{\lambda}\Gamma(M)$ for all comparison functions $\boldsymbol{\lambda}$. Recall that v_λ is the Harsanyi coalitional form of $\boldsymbol{\lambda}\Gamma$. Similarly, define $r_\lambda = r_{\lambda M}$ and $q_\lambda = q_{\lambda M}$; explicitly,

$$r_\lambda(S) = \sup\left\{\int_S \boldsymbol{\lambda} u(x) \colon \int_S x = e(S)\right\}, \tag{14.7}$$

$$q_\lambda(S) = \begin{cases} r_\lambda(S), & \text{when } \mu(S) \geqq \frac{1}{2}, \\ 0 \quad , & \text{otherwise}. \end{cases} \tag{14.8}$$

PROPOSITION 14.9. *Let M be bounded, and assume that $r_\lambda(T) < \infty$. Then $\boldsymbol{\lambda} M$ is integrably sublinear.*

PROOF. Let $\theta = \min\{1, \int e^1, \cdots, \int e^l\}$ and $\delta = \inf_t u_t(1, \cdots, 1)$; by (5.4), $\theta > 0$, and by (5.7), $\delta > 0$. Define an allocation y by $y(t) = \int e$ for all t. Then

$$u_t(y(t)) \geqq u_t(\theta, \cdots, \theta) \geqq \theta u_t(1, \cdots, 1) + (1 - \theta) u_t(0, \cdots, 0) \geqq \theta\delta.$$

Hence $\theta\delta \int \boldsymbol{\lambda} \leqq \int \boldsymbol{\lambda} u(y) \leqq r_\lambda(T)$, and so $\boldsymbol{\lambda}$ is integrable.

Let β be the uniform bound on u_t provided by (5.6). Then for each $\varepsilon > 0$,

$$\| x \| > \boldsymbol{\lambda}(t)\beta/\varepsilon \Rightarrow \boldsymbol{\lambda}(t) u_t(x) < \varepsilon \| x \|.$$

Since $\boldsymbol{\lambda}(t)\beta/\varepsilon$ is integrable, it follows that $\boldsymbol{\lambda} M$ is integrably sublinear.

PROPOSITION 14.10. *Let M be bounded, let x be an allocation in M, and let $\boldsymbol{\lambda}$ be a comparison function. Then a necessary and sufficient condition that v_λ be defined and have a μ-value given by* (11.13) *is that there exists a price vector p satisfying* (9.1) (*the definition of efficiency pair*) *and* (9.15).

PROOF. First assume (9.1) and (9.15) a.e. From (9.1) we obtain

$$r_\lambda(T) = \int \boldsymbol{\lambda} u(x), \tag{14.11}$$

and in particular $r_\lambda(T) < \infty$. Hence by Proposition 14.9, $\boldsymbol{\lambda} M$ is integrably sublinear. Hence by Proposition 14.1, v_λ has a μ-value ϕv_λ, given by

(14.12) $$(\phi v_\lambda)(S) = \frac{1}{2}\int_S [r_\lambda(T) + \boldsymbol{\lambda} u(\boldsymbol{x}) - p(\boldsymbol{x} - \boldsymbol{e})].$$

On the other hand, by integrating (9.15) over S and rearranging, we find

(14.13) $$\int_S \boldsymbol{\lambda} u(\boldsymbol{x}) = \frac{1}{2}\int_S \left[\left(\int \boldsymbol{\lambda} u(\boldsymbol{x})\right) + \boldsymbol{\lambda} u(\boldsymbol{x}) - p(\boldsymbol{x} - \boldsymbol{e})\right].$$

Combining this with (14.11) and (14.12), we deduce (11.13).

Conversely, assume that v_λ is defined and has a μ-value given by (11.13). Then by Proposition 11.3, $r_\lambda(T)$ is attained and in particular $r_\lambda(T) < \infty$. Hence by Proposition 14.9, $\boldsymbol{\lambda}M$ is integrably sublinear. From (11.13) and (3.2) it follows that $\int \boldsymbol{\lambda} u(\boldsymbol{x}) = v_\lambda(T)$. From (4.8) and Proposition 11.3 it follows that $v_\lambda(T) = r_\lambda(T)$, and hence (14.11) holds. Hence there is a p such that $(\boldsymbol{\lambda}, p)$ is an efficiency pair for $\boldsymbol{x}$, i.e. such that $(\boldsymbol{x}, p)$ is a t.u.c.e. in $\boldsymbol{\lambda}M$. From the integrable sublinearity of $\boldsymbol{\lambda}M$ and (14.4) we then obtain (14.12). Combining this with (11.13) and rearranging, we get

$$\int_S \boldsymbol{\lambda} u(\boldsymbol{x}) = \int_S [r_\lambda(T) - p(\boldsymbol{x} - \boldsymbol{e})]$$

for all S. Hence a.e.

$$\boldsymbol{\lambda}(t) u_t(\boldsymbol{x}(t)) = r_\lambda(T) - p(\boldsymbol{x}(t) - \boldsymbol{e}(t)).$$

Combining this with (14.11), we deduce (9.15). This completes the proof of Proposition 14.10.

Proposition 9.14 follows immediately from Propositions 11.12 and 14.10.

15. Non-atomic one-commodity markets: Proof of Theorem C

Since $l = 1$, we may in Proposition 9.14 take $p = 1$. For any allocation $\boldsymbol{x}$ we have $\int \boldsymbol{x} = \int \boldsymbol{e} > 0$, hence $\int \boldsymbol{\lambda} u(\boldsymbol{x}) > 0$; so if $\boldsymbol{x}$ satisfies (9.15), then a.e. $\boldsymbol{x}(t) > 0$. Then applying (9.5), we deduce (8.1). Conversely, if $\boldsymbol{x}(t) > 0$ a.e. and (8.1) holds, then from (9.5) we get (9.15). Hence we have established the second sentence of Theorem C, i.e. that the tax allocations $\boldsymbol{x}$ are precisely those for which $\boldsymbol{x}(t) > 0$ a.e. and (8.1) holds. It remains to prove that there is precisely one such $\boldsymbol{x}$.

Define

$$g_t(x) = \frac{u_t(x)}{u_t'(x)} + x;$$

g_t is increasing and continuous, is defined for all non-negative numbers, vanishes at 0, and tends to infinity as $x \to \infty$. Hence its inverse g_t^{-1} is defined and has the same properties. Moreover, $g_t(x) > x$ when $x > 0$, and so

$$g_t^{-1}(y) < y \quad \textit{when} \quad y > 0. \tag{15.1}$$

For each non-negative number γ, define

$$f(\gamma) = \int_T g_t^{-1}(\gamma + \boldsymbol{e}(t))\mu(dt); \tag{15.2}$$

by (15.1) the integrand is integrable, so there is no difficulty with the finiteness of the integral. Again using (15.1) we get

$$f(0) < \int \boldsymbol{e}.$$

By the monotone convergence theorem we may go to the limit under the integration sign in (15.2); then using the fact that $g_t^{-1}(x) \to \infty$ as $x \to \infty$, we deduce that for sufficiently large γ,

$$f(\gamma) > \int \boldsymbol{e}.$$

Moreover f is strictly increasing (since the g_t^{-1} are); and it is continuous, since by the definition of g_t, $|g_t^{-1}(y_2) - g_t^{-1}(y_1)| < |y_2 - y_1|$. Hence there is one and only one γ with $f(\gamma) = \int \boldsymbol{e}$; denote this γ by c and set

$$\boldsymbol{x}(t) = g_t^{-1}(c + \boldsymbol{e}(t)). \tag{15.3}$$

By construction $\boldsymbol{x}$ is an allocation, and satisfies $\boldsymbol{x}(t) > 0$ for all t. From (15.3) we obtain

$$\boldsymbol{x}(t) + \frac{u_t(\boldsymbol{x}(t))}{u_t'(\boldsymbol{x}(t))} = \boldsymbol{e}(t) + c,$$

and so integrating and using the fact that $\boldsymbol{x}$ is an allocation, we deduce $c = \int u(\boldsymbol{x})/u'(\boldsymbol{x})$. Thus there is an allocation satisfying the conditions of the corollary. Conversely, if $\boldsymbol{x}$ is an allocation satisfying (8.1), then it also satisfies (15.3) with $c = \int u(\boldsymbol{x})/u'(\boldsymbol{x})$. Integrating (15.3) yields

$$\int \boldsymbol{e} = \int_T g_t^{-1}(c + \boldsymbol{e}(t))\mu(dt) = f(c).$$

Thus c is the same as the one previously found, and so $\boldsymbol{x}$ also is. This completes the proof of Theorem C.

16. Equivalence: Proof of Theorem B

Let M be a market. First we settle an old debt: the proof that the derived markets M^p are indeed markets, and in particular satisfy the differentiability condition.

LEMMA 16.1. *For each price vector p, the derived market M^p satisfies conditions* (5.1) *through* (5.5); *and if M is bounded, so is M^p.*

PROOF. The verification of (5.1) through (5.4) is straightforward. To prove the differentiability condition (5.5), write $u_t = u$, and let $y > 0$. Since u^p is concave, it possesses a right derivative $D^+u^p(y)$ and a left derivative $D^-u^p(y)$ at y, and

$$D^+u^p(y) \leqq D^-u^p(y). \tag{16.2}$$

Let the maximum in the definition of $u^p(y)$ be achieved at x, i.e. $u^p(y) = u(x)$ and $px = y$. Let a_i be the i-th unit vector in Ω, i.e. $a_i^i = 1$, $a_i^j = 0$ for $j \neq i$. For $\delta > 0$ we then have $u^p(y + p^i\delta) \geqq u(x + \delta a_i)$, hence

$$[u^p(y + p^i\delta) - u^p(y)]/\delta \geqq [u(x + \delta a_i) - u(x)]/\delta,$$

and hence letting $\delta \to 0$, we deduce

$$p^iD^+u^p(y) \geqq u^i(x), \tag{16.3}$$

where $u^i = \partial u/\partial x^i$. If, moreover, $x^i > 0$, then for $0 < \delta < x^i$ we have $u^p(y - p^i\delta) \geqq u(x - \delta a_i)$, hence

$$[u^p(y - p^i\delta) - u^p(y)]/(-\delta) \leqq [u(x - \delta a_i) - u(x)]/(-\delta),$$

and hence letting $\delta \to 0$, we deduce

$$p^iD^-u^p(y) \leqq u^i(x). \tag{16.4}$$

Since $y > 0$, there must be an i with $x^i > 0$; for this i we may combine (16.2), (16.3) and (16.4) to deduce $D^+u^p(y) = D^-u^p(y)$, i.e. u^p is differentiable at y. The continuity of the derivative is then a consequence of the concavity of u^p.

If M is bounded, then the verification of (5.6) for M^p is straightforward. To prove (5.7), write Σp for $\Sigma_{i=1}^l p^i$. Then

$$u^p(\Sigma p) = \max\{u(x): px = \Sigma p\} \geqq u(1, \cdots, 1).$$

If $\Sigma p \geqq 1$, then by the concavity of u^p it follows that

$$u^p(1) \geqq \frac{1}{\Sigma p} u^p(\Sigma p) + \left(1 - \frac{1}{\Sigma p}\right)u^p(0) \geqq \frac{1}{\Sigma p} u(1, \cdots, 1).$$

If $\Sigma p \leqq 1$, then by the monotonicity of u^p it follows that

$$u^p(1) \geqq u^p(\Sigma p) \geqq u(1, \cdots, 1).$$

Since $u(1, \cdots, 1) = u_t(1, \cdots, 1)$ is uniformly positive by assumption, it follows that so is $u^p(1) = u_t^p(1)$, and the proof of (5.7) — and so of Lemma 16.1 — is complete.

For future reference we note the following corollary.

COROLLARY 16.5. *Let the maximum in the definition of $u_t^p(y)$ be achieved at x. Then $(u_t^p)'(y) \geqq u_t^i(x)/p^i$; and if $x^i > 0$, then $(u_t^p)'(y) = u_t^i(x)/p^i$.*

PROOF. Follows from (16.3), (16.4), and the differentiability of u_t^p.

PROOF OF THEOREM B. First let $\boldsymbol{x}$ be a commodity tax allocation in M. Then $\boldsymbol{x}$ is efficient, say with efficiency pair $(\boldsymbol{\lambda}, p)$, and obeys (9.15) (by Proposition 9.14). Hence $\boldsymbol{x}(t) \neq 0$ a.e., and so from (9.12) and (9.13) we deduce (9.16). By Lemma 16.1, M^p is bounded, and so Theorem C applies to it; thus (9.16) shows that $p\boldsymbol{x}$ is a tax allocation in M^p, i.e. (7.3) is satisfied. The other condition for $(\boldsymbol{x}, p)$ to be a competitive tax equilibrium — Condition (7.2) — follows from the fact that p is an efficiency price vector for $\boldsymbol{x}$. Hence $\boldsymbol{x}$ is an income tax allocation.

Conversely, let $\boldsymbol{x}$ be an income tax allocation. Then there is a p satisfying (7.2) and (7.3), and also a $\boldsymbol{\lambda}$ such that $(\boldsymbol{\lambda}, p)$ is an efficiency pair for $\boldsymbol{x}$. By (7.3) and Theorem C we have (9.16), hence a.e. $\boldsymbol{x}(t) \neq 0$, hence (9.12) and (9.13) yield (9.15), and hence by Proposition 9.14, $\boldsymbol{x}$ is a commodity tax allocation. This completes the proof of Theorem B.

17. Existence in the non-atomic case

The idea of the proof is as follows: For each price vector p, consider the market M^p derived from M at prices p. By Theorem C, M^p has a unique tax allocation, which assigns to each trader a certain income. This income generates a certain demand, and so we get a certain total excess demand. We can then apply Debreu's lemma [5, p. 82], and deduce that there is a p for which the excess demand contains 0. But this yields a competitive tax equilibrium, hence an income tax allocation in M, and so by Theorem B a commodity tax allocation in M.

Let $\Delta = \{p \in \Omega : \Sigma_{i=1}^{l} p^i = 1\}$. Let Int Δ denote the relative interior of Δ; every point in Int Δ is a price vector. Let $\partial\Delta = \Delta \backslash \text{Int } \Delta$. We will make use of the following variant of Debreu's lemma, proved in [8, p. 150, lemma 1]:

LEMMA 17.1. *Let Z be an uppersemicontinuous,*[26] *compact-valued correspondence*[27] *from Int Δ to E^l that is bounded from below,*[28] *such that*

$$pz = 0 \quad \text{for all} \quad z \in Z(p), \tag{17.2}$$

and such that

(17.3) *if the sequence $\{p_n\}$ in* Int Δ *converges to p_0 in $\partial\Delta$ then* $\inf\left\{\sum_{i=1}^{l} z^i : z \in Z(p_n)\right\} > 0$ *for n large enough.*

Then there is a p in Int Δ *such that 0 is in the convex hull of $Z(p)$.*

We proceed in a series of lemmas.

LEMMA 17.4. *For all t in T, $y > 0$ and p in* Int Δ, *$u_t^p(y)$ and $(u_t^p)'(y)$ are continuous in (p, y).*

PROOF. Fix t and write $u_t = u$, $u_t^p = u^p$. The continuity of $u^p(y)$ follows from the definition (7.1). To prove the continuity of $(u^p)'(y)$, let $p_n \to p$, $y_n \to y$, and let the maximum in the definition of $u^{p_n}(y_n)$ be attained at x_n, i.e. $u^{p_n}(y_n) = u(x_n)$ and $p_n x_n = y_n$. Then $\limsup x_n^i \leqq y/p^i$ for all i, so that $\{x_n\}$ is bounded; let x be a limit point of $\{x_n\}$, w.l.o.g. the limit. Then $px = y$ and $u^p(y) = u(x)$ (by the continuity of $u^p(y)$). Since $px = y$, there is an i with $x^i > 0$; then for n sufficiently large, also $x_n^i > 0$. Applying Corollary 16.5 and the continuity of u^i, we deduce

$$(u^{p_n})'(y_n) = \frac{u^i(x_n)}{p_n^i} \to \frac{u^i(x)}{p^i} = (u^p)'(y);$$

this completes the proof of the lemma.

LEMMA 17.5. *Suppose that for each p in* Int Δ, *h^p is an increasing continuous function from R^+ onto R^+, such that $h^p(y)$ is continuous in (p, y). Then the inverse functions $(h^p)^{-1}$ are defined, increasing, continuous, and take R^+ onto R^+; and $(h^p)^{-1}(z)$ is continuous in (p, z).*

PROOF. Except for the continuity of $(h^p)^{-1}(z)$ in (p, z), the lemma is readily verified. To prove the continuity, let $(p_n, z_n) \to (p_0, z_0) \in \text{Int}\,\Delta \times R^+$. Set $y_n = (h^{p_n})^{-1}(z_n)$ for $n = 0, 1, 2, \cdots$. We wish to show $y_n \to y_0$. Since

[26] By this we mean that each p has a neighborhood in which the graph of Z is closed.
[27] A *correspondence* is a set-valued function with non-empty values.
[28] I.e., there is a y in E^l such that $x \geqq y$ for all p and all x in $Z(p)$.

$$h^{p_n}(y_0+1) \to h^{p_0}(y_0+1) > h^{p_0}(y_0) = z_0$$

and $z_n \to z_0$, it follows that $z_n < h^{p_n}(y_0+1)$ for sufficiently large n. Hence

$$y_n = (h^{p_n})^{-1}(z_n) < y_0 + 1,$$

and so $\{y_n\}$ has a limit point y'. Allowing $n \to \infty$ through values for which $y_n \to y'$, we obtain from the continuity of $h^p(y)$ in (p, y) that

$$z_0 = \lim z_n = \lim h^{p_n}(y_n) = h^{p_0}(y').$$

Hence $y' = (h^{p_0})^{-1}(z_0) = y_0$. Thus $\{y_n\}$ has a limit point, and every such limit point is y_0, which means $y_n \to y_0$. This completes the proof of Lemma 17.5.

For $p \in \operatorname{Int} \Delta$, let y^p be the unique tax allocation of M^p provided by Theorem C.

LEMMA 17.6. *$y^p(t)$ is a.e. continuous in p.*

PROOF. We closely follow the proof of Theorem C (Section 15), keeping track of the parameter p. Set

$$g_t^p(y) = y + \frac{u_t^p(y)}{(u_t^p)'(y)}. \tag{17.7}$$

Then g_t^p satisfies the hypotheses of Lemma 17.5, and so its inverse $(g_t^p)^{-1}$ is defined, continuous and increasing, takes R^+ onto R^+, and $(g_t^p)^{-1}(z)$ is continuous in (p, z); moreover from (17.7) we get

$$(g_t^p)^{-1}(z) < z \quad \text{for} \quad z > 0. \tag{17.8}$$

Defining $f^p(\gamma) = \int_T (g_t^p)^{-1}(\gamma + pe(t))\mu(dt)$, we deduce from Lebesgue's dominated convergence theorem (using (17.8)) that $f^p(\gamma)$ is continuous in (p, γ). Moreover f^p is strictly increasing, and from the monotone convergence theorem it follows that $f^p(\gamma) \to \infty$ as $\gamma \to \infty$. If we define $h^p(\gamma) = f^p(\gamma) - f^p(0)$, then h^p obeys all the conditions of Lemma 17.5, and so $(h^p)^{-1}$ is defined, increasing, continuous, takes R^+ onto R^+, and $(h^p)^{-1}(\beta)$ is continuous in (p, β). From (17.8) and (5.4) it follows that $h^p(0) = 0 < \int pe - f^p(0)$. Hence $(h^p)^{-1}(\int pe - f^p(0))$ is defined and continuous in p; call it c^p. Then $f^p(c^p) = h^p(c^p) + f^p(0) = \int pe$. From (15.3) we then deduce

$$y^p(t) = (g_t^p)^{-1}(c^p + pe(t)), \tag{17.9}$$

and so from the continuity properties of $(g_t^p)^{-1}$ and c^p we deduce the assertion of our lemma.

COROLLARY 17.10. *For each compact subset C of* Int Δ, *there is a constant c such that $y^p(t) \leqq c + pe(t)$ for all t and all p in C.*

PROOF. From (17.8) and (17.9) we get $y^p(t) \leqq c^p + pe(t)$. From the continuity of c^p and the compactness of C it follows that c^p is bounded in C, which proves the corollary.

Let

$$B^p(t) = \{x \in \Omega \colon px \leqq y^p(t)\}, \tag{17.11}$$

and let $D^p(t)$ be the set of elements of $B^p(t)$ at which u_t is maximized; $D^p(t)$ is t's (after tax) *demand* at prices p. Let $D(p) = \int D^p$, and let $Z(p) = D(p) - \int e$; $D(p)$ is aggregate demand, and $Z(p)$ aggregate excess demand.

We note for future reference that

$$\frac{u_t^p(y)}{(u_t^p)'(y)} \geqq y \tag{17.12}$$

for all p, y, and t; this follows from the concavity of u_t^p.

LEMMA 17.13. *If the sequence $\{p_n\}$ in* Int Δ *converges to p_0 in $\partial \Delta$, then a.e.*

$$\inf\left\{\sum_{i=1}^{l} x^i \colon x \in D^{p_n}(t)\right\} \to \infty.$$

PROOF. Let $\delta = \min_i \int e^i$; by (5.4), $\delta > 0$. From (17.12) and the fact that y^p is an allocation in M^p it follows that for all p in Int Δ,

$$\int \frac{u^p(y^p)}{(u^p)'(y^p)} \geqq \int y^p = \int pe \geqq \delta.$$

Hence from Theorem C it follows that for all p in Int Δ, a.e.

$$y^p(t) + \frac{u_t^p(y^p(t))}{(u_t^p)'(y^p(t))} = pe(t) + \int \frac{u^p(y^p)}{(u^p)'(y^p)} \geqq \delta. \tag{17.14}$$

If the lemma is false, then there is a set U of positive measure such that for each t in U there is a compact set $C(t)$ such that

$$D^{p_n}(t) \cap C(t) \neq \emptyset \quad \textit{for arbitrarily large} \quad n. \tag{17.15}$$

W.l.o.g. we can choose U so that (17.14) holds for *all* t in U (not only a.e.) whenever p is one of the p_n. Fix a t in U, and set $u = u_t$, $u^p = u_t^p$, $y^p = y^p(t)$, $D^p = D^p(t)$, $C = C(t)$; thus we have

(17.16) $$y^{p_n} + \frac{u^{p_n}(y^{p_n})}{(u^{p_n})'(y^{p_n})} \geqq \delta$$

for all $n \geqq 1$. We next assert that there are *positive* numbers β and δ' such that

(17.17) $$u(x) \leqq \beta$$

for all x in C, and

(17.18) $$u^i(x) \geqq \delta'$$

for all i and all x in C at which $u^i(x)$ exists. Assertion (17.17) follows immediately from (5.6), or alternatively from the continuity of u and the compactness of C. To prove (17.18), define $\xi^i = \max\{x^i : x \in C\} + 1$ for each i, and let

$$C^i = \{z \in \Omega : z^i = \xi^i, z^j \leqq \xi^j \quad \text{for} \quad j \neq i\}.$$

Then u^i is defined and continuous on C^i, which is compact; hence u^i attains its minimum, which we call δ^i, on C^i. But for each x in C and each i there is a z in C^i with $x^i < z^i$ and $x^j = z^j$ for $j \neq i$. From the concavity of u it then follows that $u^i(x) \geqq u^i(z)$ when $u^i(x)$ exists (which is always the case when $x^i > 0$). Since $u^i(z) \geqq \delta^i$, we may set $\delta' = \min_i \delta^i$, and thus (17.18) is proved.

By Corollary 16.5, $(u^p)'(y^p) \geqq u^i(x)/p^i$ for all i and all x in D^p. Hence from (17.15) and (17.18) we get

(17.19) $$(u^p)'(y^p) \geqq \delta'/p^i$$

for all i. From (17.15) and (17.17) we get $u^p(y^p) \leqq \beta$. Combining this with (17.16) and (17.19), we get

(17.20) $$y^p \geqq \delta - p^i\beta/\delta'$$

for all i and all p in Int Δ.

Since $p_0 \in \partial\Delta$, there is a coordinate of p_0 that vanishes; w.l.o.g. let $p_0^1 = 0$. Then $p_n^1 \to 0$, and so from (17.20) we deduce

(17.21) $$y^{p_n} \geqq \delta/2$$

for n sufficiently large.

For each n, let $x_n \in D^{p_n} \cap C$; by (17.15), there is such an x_n. The sequence $\{x_n\}$ has a limit point x_0; w.l.o.g. let it be the limit. By (17.21), $p_n x_n = y^{p_n} \geqq \delta/2$ for all sufficiently large n; hence $p_0 x_0 \geqq \delta/2$. Hence there is an i with $p_0^i > 0$ and $x_0^i > 0$; w.l.o.g. let $i = 2$. Let $C' = \{x \in C : x^2 \geqq x_0^2/2\}$, and let $\beta' = \max\{u^2(x) : x \in C'\}$; this max is achieved, because u^2 is continuous on the compact set C'. For sufficiently large n, we have $x_n^2 > x_0^2/2$, and hence $x_n \in C'$ and $u^2(x_n) \leqq \beta'$. From Corollary 16.5 we get for these n that

$$u^2(x_n) = p_n^2(u^{p_n})'(y^{p_n}) \geqq p_n^2\delta'/p_m^1$$

hence $p_n^2 \leqq p_n^1\beta'/\delta'$. Since $p_n^1 \to 0$ it follows that $p_n^2 \to 0$, and hence $p_0^2 = 0$. This contradiction proves Lemma 17.13.

LEMMA 17.22. *There is a price vector p such that $0 \in Z(p)$.*

PROOF. We will use Lemma 17.1. To show that the hypotheses of Lemma 17.1 hold, note first that from Lemma 17.6 and $p \in \operatorname{Int}\Delta$ it follows that $\boldsymbol{B}^p(t)$ is continuous (in the set sense) in p for a.a. t. Hence from a known theorem [5, p. 19, comment 4], it follows that

(17.23) $\boldsymbol{D}^p(t)$ *is uppersemicontinuous in* p *for a.a.* t.

Next, from the definition of $\boldsymbol{D}^p$ it follows that $px \leqq \boldsymbol{y}^p(t)$ for all $x \in \boldsymbol{D}^p(t)$. Hence from Corollary 17.10 it follows that if C is a compact subset of Int Δ, then there is a constant c such that for all i,

$$0 \leqq x^i \leqq \left(c + \sum_{j=1}^{l} \boldsymbol{e}^j(t)\right) \Big/ \min\{p^i : p \in C,\ 1 \leqq i \leqq l\}$$

whenever $p \in C$ and $x \in \boldsymbol{D}^p(t)$. Thus

(17.24) $\boldsymbol{D}^p$ *is* integrably bounded *throughout* C,

i.e. there is an integrable function $\boldsymbol{h}$ such that $\|x\| \leqq \boldsymbol{h}(t)$ whenever $x \in \boldsymbol{D}^p(t)$ and $p \in C$. Now it is known (see e.g. [8, p. 73, prop. 8]) that the integral of an integrably bounded uppersemicontinuous set-valued function is uppersemicontinuous. Thus (17.23) and (17.24) yield the uppersemicontinuity of $D(p)$ and hence of $Z(p)$ throughout C. Since every point in Int Δ has a compact neighborhood in Int Δ, it follows that

(17.25) Z *is uppersemicontinuous in* $\operatorname{Int}\Delta$.

The values of an uppersemicontinuous set-valued function are necessarily closed, so that Z is closed-valued. From (17.24) it follows that each $Z(p)$ is also bounded; hence

(17.26) Z *is compact-valued.*

Moreover $x \in Z(p) \Rightarrow x \geqq -\int \boldsymbol{e}$, whence

(17.27) Z *is bounded from below.*

To establish the non-emptiness of $Z(p)$, we will use a measurability argument. Let's use the phrase *a.e. Borel-measurable* for a function (point- or set-valued) that differs from one with Borel graph on a set of measure 0 only. Then since $\boldsymbol{y}^p$ is an allocation it is a.e. Borel-measurable. A fairly standard argument then shows that $\boldsymbol{D}^p$ is also a.e. Borel-measurable. Then from the selection theorem of von Neumann [16] it follows that $\boldsymbol{D}^p$ has a measurable selection. Since by (17.24), $\boldsymbol{D}^p$ is integrably bounded, this measurable selection is integrable, and its integral is a member of $D(p)$. Hence $D(p)$, and so also $Z(p)$, are non-empty, i.e.

(17.28) Z *is a correspondence.*

Next, from the monotonicity of u_t it follows that $px = \boldsymbol{y}^p(t)$ for all p and t and all $x \in \boldsymbol{D}^p(t)$. Since $\int \boldsymbol{y}^p = \int p\boldsymbol{e}$, it follows that $px = p\int \boldsymbol{e}$ for all $x \in D(p)$, whence

(17.29) $$pz = 0 \quad \textit{for all} \quad z \in Z(p).$$

It remains only to establish the boundary condition (17.3). Let $p_n \to p_0$ where $p_n \in \operatorname{Int}\Delta$ and $p_0 \in \partial\Delta$. Let $x_n \in D(p_n)$; it is sufficient to prove that $\Sigma_{i=1}^{l} x_n^i \to \infty$. Since $D(p_n) = \int \boldsymbol{D}^{p_n}$, there is a sequence $\{\boldsymbol{x}_n\}$ such that $\boldsymbol{x}_n(t) \in \boldsymbol{D}^{p_n}(t)$ for all t and $\int \boldsymbol{x}_n = x_n$. Then by Fatou's lemma and Lemma 7.13,

$$\liminf \sum_{i=1}^{l} x_n^i = \liminf \int \sum_{i=1}^{l} \boldsymbol{x}_n^i \geqq \int \liminf \sum_{i=1}^{l} \boldsymbol{x}_n^i = \infty.$$

This proves (17.3). Hence by Lemma 17.1, there is a p in Int Δ such that 0 is in the convex hull of $Z(p)$. But the integral of any set-valued function is convex [8, p. 62, theor. 3]. Hence $D(p)$ is convex, and hence so is $Z(p)$. Hence $Z(p)$ is its own convex hull, and so $0 \in Z(p)$. This completes the proof of Lemma 17.22.

Let p be such that $0 \in Z(p)$. Then there is an $\boldsymbol{x}$ such that $\boldsymbol{x}(t) \in \boldsymbol{D}^p(t)$ for all t, and $\int \boldsymbol{x} = \int \boldsymbol{e}$. But this means precisely that $\boldsymbol{x}$ is an allocation satisfying (7.2) and (7.3). Thus $(\boldsymbol{x}, p)$ is a competitive tax equilibrium, hence $\boldsymbol{x}$ is an income tax allocation, and so by Theorem B a commodity tax allocation. This completes the proof of Theorem A in the non-atomic case.

18. Existence for finite T: Completion of the proof of Theorem A

Suppose the agent space T is finite. Then there is no equivalence theorem, and the existence of commodity tax allocations and income tax allocations must be established separately. But though the two proofs are different, they use similar ideas and many of the same tools. We proceed first to develop these tools.

Let E^T be the set of all real valued functions on T; since T is finite, E^T is a Euclidean space of dimension $|T|$. Define a *generalized comparison function* to be a non-negative (rather than positive) valued function on T, that does not vanish identically; the set of all such functions is denoted Λ. One can think of Λ as the non-negative orthant of E^T, excluding the origin. Notations for generalized comparison functions are similar to those for comparison functions. Thus if $\boldsymbol{\lambda} \in \Lambda$ and M is a market with agent space T, then the coalitional games $r_{\boldsymbol{\lambda}}$ and $q_{\boldsymbol{\lambda}}$ are given by (14.7) and (14.8).

At the basis of both proofs lies the following lemma, proved by L. S. Shapley [14, p. 261], and used by him to establish the existence of NTU values in a general context.

PROPOSITION 18.1. *Let C be a convex and compact subset of E^T. Let $\boldsymbol{\lambda} \to \psi_{\boldsymbol{\lambda}}$ be a continuous mapping*[29] *from Λ to E^T such that for all $\boldsymbol{\lambda}$,*

(18.2) $$\sum_{t \in T} \psi_{\boldsymbol{\lambda}}(t) = \max \left\{ \sum_{t \in T} \boldsymbol{\lambda}(t) z(t) \colon z \in C \right\},$$

and for all t and $\boldsymbol{\lambda}$,

(18.3) $$\boldsymbol{\lambda}(t) = 0 \Rightarrow \psi_{\boldsymbol{\lambda}}(t) \geqq 0.$$

Then there is a $\boldsymbol{\lambda}$ in Λ such that

(18.4) $$\psi_{\boldsymbol{\lambda}} \in C.$$

Let M be a fixed market with the finite agent space T. In both of our applications of Proposition 18.1, we will choose[30]

(18.5) $$C = \{z \in E^T \colon \text{there is an allocation } \boldsymbol{x} \text{ with } u(\boldsymbol{x})/|T| = z\}$$

(recall that $u(\boldsymbol{x})$ is the function on T whose value at t is $u_t(\boldsymbol{x}(t))$). The convexity of C follows from the concavity of the u_t, and the compactness of C from the fact that the set of all allocations is compact and the u_t are continuous, so that C is the continuous image of a compact set.

To prove the existence of a commodity tax allocation, set $\psi_{\boldsymbol{\lambda}}(t) = (\phi q_{\boldsymbol{\lambda}})(\{t\})$, where ϕ is the value. To apply Proposition 18.1, we shall have to establish the continuity of $\boldsymbol{\lambda} \to \psi_{\boldsymbol{\lambda}}$, (18.2), and (18.3).

By (14.7), for all $S \subset T$ we have

[29] The proposition remains true if the mapping is defined on the unit simplex only.

[30] We divide $u(\boldsymbol{x})$ by $|T|$ because we wish to think of the payoff to an individual agent t as $u_t(\boldsymbol{x}(t))\mu(\{t\})$, just as in the non-atomic case we think of the payoff to dt as $u_t(\boldsymbol{x}(t))\mu(dt)$.

$$r_{\lambda}(S) = \sup\left\{\int_S \lambda u(x): \int_S x = e(S)\right\}.$$

Note that because of the finiteness of T, the integrals appearing on the right are in fact sums. The sup is over all S-allocations x; this is a compact subset of Euclidean space of dimension $l|S|$, and so the sup is attained and is a continuous function of λ in Λ. Hence by (14.8), $q_{\lambda}(S)$ is also a continuous function of λ. Hence by Proposition 2.5, $\psi_{\lambda}(t) = (\phi q_{\lambda})(\{t\})$ is indeed continuous in λ.

The right side of (18.2) is simply $r_{\lambda}(T)$; thus (18.2) merely is the efficiency axiom (2.3) for the value, since $q_{\lambda}(T) = r_{\lambda}(T)$.

Next, the game q_{λ} is monotonic, i.e. $S \supset U \Rightarrow q_{\lambda}(S) \geqq q_{\lambda}(U)$. Together with Proposition 2.5, this shows that $\psi_{\lambda}(t) = (\phi q_{\lambda})(\{t\}) \geqq 0$ for all λ and t, and this in particular implies (18.3).

From Proposition 18.1 we then deduce (18.4). Taking into account the definition (18.5) of C, we find that (18.4) says that there is a generalized comparison function λ and an allocation x such that

(18.6) $$(\phi q_{\lambda})(\{t\}) = \lambda(t)u_t(x(t))/|T|$$

for all t in T.

We show next that the λ obeying (18.6) is in fact a comparison function, i.e. $\lambda(t) > 0$ for all t. Recall that given an ordering of the agents and an agent t, S_t represents the set of agents preceding t in the ordering. Assume first that $|T| > 2$. Let t_1 be an agent with $e(t_1) \neq 0$; there must be such an agent since $\int e > 0$ (Assumption (5.4)). Since λ does not vanish identically, either $\lambda(t_1) > 0$, or there is an agent t_0 different from t_1 with $\lambda(t_0) > 0$. In the latter case, consider an order on the agents in which t_0 is first, and t_1 is the first t such that $\mu(S_t \cup \{t\}) \geqq 1/2$; i.e., t_1 is number $|T|/2$ in the order if $|T|$ is even, number $(|T|+1)/2$ if $|T|$ is odd. Then $q_{\lambda}(S_{t_1}) = 0$ and

$$q_{\lambda}(S_{t_1} \cup \{t_1\}) = r_{\lambda}(S_{t_1} \cup \{t_1\}) \geqq \lambda(t_0)u_{t_0}(e(t_1))/|T| > 0.$$

Hence $q_{\lambda}(S_{t_1} \cup \{t_1\}) - q_{\lambda}(S_{t_1}) > 0$ for this order, and so $(\phi q_{\lambda})(\{t_1\}) > 0$. Hence by (18.6), $\lambda(t_1) > 0$ in this case as well.

Suppose now t_2 is any agent other than t_1. Consider the order in which t_1 is first and t_2 is the first t such that $\mu(S_t \cup \{t\}) \geqq 1/2$. Then $q_{\lambda}(S_{t_2}) = 0$ and

$$q_{\lambda}(S_{t_2} \cup \{t_2\}) = r_{\lambda}(S_{t_2} \cup \{t_2\}) \geqq \lambda(t_1)u_{t_1}(e(t_1))/|T| > 0.$$

Hence $q_{\lambda}(S_{t_2} \cup \{t_2\}) - q_{\lambda}(S_{t_2}) > 0$ for this order, and so $(\phi q_{\lambda})(\{t_2\}) > 0$. Hence by (18.6), $\lambda(t_2) > 0$. Since t_2 was chosen to be any agent, we have proved

$$(18.7) \qquad \boldsymbol{\lambda}(t) > 0$$

when $\boldsymbol{\lambda}$ satisfies (18.6) and $|T| > 2$.

The case $|T| = 2$ requires special treatment since the first t in the order is also the first t such that $\mu(S_t \cup \{t\}) \geqq 1/2$. Let $T = \{t_1, t_2\}$; w.l.o.g. $\boldsymbol{\lambda}(t_1) > 0$. Since M is non-trivial (see Section 8), we must have $\boldsymbol{e}(t_2) \neq 0$. Hence $q_{\boldsymbol{\lambda}}(\{t_1, t_2\}) - q_{\boldsymbol{\lambda}}(\{t_1\}) > 0$, hence $(\phi q_{\boldsymbol{\lambda}})(\{t_2\}) > 0$ and so by (18.6), $\boldsymbol{\lambda}(t_2) > 0$. When $|T| < 2$, (18.7) is trivial. Thus we have shown (18.7) whenever $\boldsymbol{\lambda}$ satisfies (18.6); in other words, there is a comparison function (not only a generalized comparison function) obeying (18.6).

We have already noted that the sup in the definition of $r_{\boldsymbol{\lambda}}(S)$ is attained for all $\boldsymbol{\lambda}$ in Λ and all $S \subset T$, and so in particular for $S = T$ and for the $\boldsymbol{\lambda}$ obeying (18.6). Since this $\boldsymbol{\lambda}$ is a comparison function, it follows from Proposition 11.11 that $v_{\boldsymbol{\lambda}}$ is defined, and hence from Corollary 11.9 that $\phi q_{\boldsymbol{\lambda}} = \phi v_{\boldsymbol{\lambda}}$. Thus from (18.6) we get $(\phi v_{\boldsymbol{\lambda}})(\{t\}) = \boldsymbol{\lambda}(t) u_t(\boldsymbol{x}(t))/|T|$ for all t in T. Because of the finiteness of T this implies (11.13), and so from Proposition 11.12 it follows that $\boldsymbol{x}$ is a commodity tax allocation.

We turn next to the proof of existence of an income tax allocation. This is in many respects similar to the proof just completed, and we will on several occasions refer the reader to an argument in "the previous case," rather than going through it in detail again.

Given a $\boldsymbol{\lambda}$ in Λ, let $\boldsymbol{x}$ be an allocation at which $r_{\boldsymbol{\lambda}}(T)$ is attained, i.e. such that $\int \boldsymbol{\lambda} u(\boldsymbol{x}) = r_{\boldsymbol{\lambda}}(T)$. Such an allocation is efficient, and so there is associated with it a unique normalized efficiency price vector p. We now show that though there may be more than one $\boldsymbol{x}$ associated with $\boldsymbol{\lambda}$, all these different $\boldsymbol{x}$'s associated with the same $\boldsymbol{\lambda}$ will have the same normalized efficiency price vector p, which we may call $p(\boldsymbol{\lambda})$.

Indeed, suppose $r_{\boldsymbol{\lambda}}(T)$ is attained at two different allocations, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Let $(\boldsymbol{\lambda}, p_1)$ and $(\boldsymbol{\lambda}, p_2)$ be efficiency pairs for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively. For a given i, let t be such that $\boldsymbol{x}_1^i(t) > 0$. Applying (9.1) first with $\boldsymbol{x} = \boldsymbol{x}_1$ and $p = p_1$, and then with $\boldsymbol{x} = \boldsymbol{x}_2$ and $p = p_2$, we find

$$p_1(\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)) \leqq \boldsymbol{\lambda}(t)(u_t(\boldsymbol{x}_1(t)) - u_t(\boldsymbol{x}_2(t))) \leqq p_2(\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)).$$

Integrating (i.e. summing) over t yields 0 both on the extreme right and on the extreme left, since $\int \boldsymbol{x}_1 = \int \boldsymbol{x}_2 = \int \boldsymbol{e}$. Hence we must have equality throughout for each t, and so in particular

$$\boldsymbol{\lambda}(t) u_t(\boldsymbol{x}_1(t)) - p_2 \boldsymbol{x}_1(t) = \boldsymbol{\lambda}(t) u_t(\boldsymbol{x}_2(t)) - p_2 \boldsymbol{x}_2(t).$$

Thus the maximum of $\boldsymbol{\lambda}(t)u_t(x)-p_2x$ is taken on at $\boldsymbol{x}_1(t)$ as well as at $\boldsymbol{x}_2(t)$. From this it follows that $\boldsymbol{\lambda}(t)u_t^i(\boldsymbol{x}_1(t))=p_2^i$. But by (9.5), $\boldsymbol{\lambda}(t)u_t^i(\boldsymbol{x}_1(t))=p_1^i$; hence $p_1^i=p_2^i$. Since this holds for each i we conclude[31] that $p_1=p_2$. From this it follows that the *normalized* efficiency price vectors associated with $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are also the same, so that $p(\boldsymbol{\lambda})$ is indeed well defined. Note that $p(\boldsymbol{\lambda})>0$ even when $\boldsymbol{\lambda}$ has some vanishing components.

Now if p is any price vector, let $r_{\boldsymbol{\lambda}}^p$ and $q_{\boldsymbol{\lambda}}^p$ be the coalitional games defined by

(18.8)
$$r_{\boldsymbol{\lambda}}^p(S)=\sup\left\{\int_S \boldsymbol{\lambda}u^p(p\boldsymbol{x}):\int_S p\boldsymbol{x}=pe(S)\right\}$$
$$=\sup\left\{\int_S \boldsymbol{\lambda}u(\boldsymbol{x}):\int_S p\boldsymbol{x}=pe(S)\right\},$$

$$q_{\boldsymbol{\lambda}}^p(S)=\begin{cases} r_{\boldsymbol{\lambda}}^p(S) & \textit{if} \quad \mu(S)\geqq\frac{1}{2} \\ 0 & \textit{if} \quad \mu(S)<\frac{1}{2}. \end{cases}$$

We may think of $r_{\boldsymbol{\lambda}}^p$ and $q_{\boldsymbol{\lambda}}^p$ as being associated with $\boldsymbol{\lambda}M^p$ in the same way that r and q are associated with M; but it should be remembered that $\boldsymbol{\lambda}M^p$ is not really a market, since $\boldsymbol{\lambda}(t)$ may vanish for some t, and those t will not have increasing utilities in $\boldsymbol{\lambda}M^p$. Set $\psi_{\boldsymbol{\lambda}}(t)=\phi q_{\boldsymbol{\lambda}}^{p(\boldsymbol{\lambda})}(\{t\})$. To apply Proposition 18.1, we must, as in the previous case, prove the continuity of $\boldsymbol{\lambda}\to\psi_{\boldsymbol{\lambda}}$, (18.2), and (18.3).

First we show that $p(\boldsymbol{\lambda})$ is continuous in $\boldsymbol{\lambda}$. Let $\boldsymbol{\lambda}_k\to\boldsymbol{\lambda}_0$. Let $r_{\boldsymbol{\lambda}_k}(T)$ be attained at $\boldsymbol{x}_k$, and let $\boldsymbol{x}_0$ be a limit point of $\boldsymbol{x}_k$, which w.l.o.g. we may take to be the limit; then $r_{\boldsymbol{\lambda}_0}(T)$ is attained at $\boldsymbol{x}_0$. For each i, there is a t such that $\boldsymbol{x}_0^i(t)>0$; this follows from $\int\boldsymbol{x}_0=\int e>0$. Hence for sufficiently large k, which w.l.o.g. we may take to be all k, we have $\boldsymbol{x}_k^i(t)>0$. Define

(18.9)
$$p_k^i=\boldsymbol{\lambda}_k(t)u_t^i(\boldsymbol{x}_k(t))$$

for such t; note that $\boldsymbol{\lambda}_k(t)$ cannot vanish, since $\boldsymbol{x}_k(t)=0$ if it does. Then by (9.5), p_k is an efficiency price vector for $\boldsymbol{x}_k$ for $k=0,1,2,\cdots$, i.e.

(18.10)
$$p(\boldsymbol{\lambda}_k)=p_k\Big/\sum p_k,$$

where $\Sigma p_k=\Sigma_{i=1}^l p_k^i$. From (18.9), $\boldsymbol{x}_k\to\boldsymbol{x}_0$, $\boldsymbol{\lambda}_k\to\boldsymbol{\lambda}_0$, and the continuity of the derivative u_t^i, we deduce $p_k\to p_0$. Hence by (18.10), $p(\boldsymbol{\lambda}_k)\to p(\boldsymbol{\lambda}_0)$, which proves that $p(\boldsymbol{\lambda})$ is continuous in $\boldsymbol{\lambda}$.

[31] For a similar argument, see [4, p. 190].

From (18.8), it follows that $r_\lambda^p(S)$ is simultaneously continuous as a function of λ and p (recall that $p > 0$ always). Hence $r_\lambda^{p(\lambda)}(S)$ is continuous in λ, hence so is $q_\lambda^{p(\lambda)}(S)$, and hence so is $\psi_\lambda(t) = \phi q_\lambda^{p(\lambda)}(\{t\})$. To prove (18.2), note that on the right side we have $r_\lambda(T)$, which is the same as $r_\lambda^{p(\lambda)}(T)$ because $p(\lambda)$ is an efficiency price vector for the x at which $r_\lambda(T)$ is attained. On the left side we have $\Sigma_{t\in T}(\phi q_\lambda^{p(\lambda)})(\{t\}) = (\phi q_\lambda^{p(\lambda)})(T)$, which because of the efficiency condition (2.3) for values, $= q_\lambda^{p(\lambda)}(T) = r_\lambda^{p(\lambda)}(T)$. Thus (18.2) is proved. (18.3) follows by a monotonicity argument exactly as in the previous case.

From Proposition 18.1 we then deduce (18.4). Taking into account the definition (18.5) of C, we find that (18.4) says that there is a generalized comparison function λ and an allocation x such that

$$(18.11) \qquad (\phi q_\lambda^{p(\lambda)})(\{t\}) = \lambda(t)u_t(x(t))/|T|$$

for all t in T.

We next claim that the λ obeying (18.11) is in fact a comparison function, i.e. $\lambda(t) > 0$ for all t. This proof is exactly the same as in the previous case. We must only replace u_t by $u_t^{p(\lambda)}$ and e by $p(\lambda)e$; since $p^i(\lambda)$ never vanishes, we have $p(\lambda)e = 0$ if and only if $e = 0$, and so the proof goes through as before.

Finally, arguing as in the previous case — but in $M^{p(\lambda)}$ rather than M — we find that the existence of a comparison function λ and an allocation x satisfying (18.11) implies that the Harsanyi coalitional form $v_\lambda^{p(\lambda)}$ of the redistribution game $\Gamma(M^{p(\lambda)})$ is defined, and that $(\phi v_\lambda^{p(\lambda)})(S) = \int_S \lambda u(x)$ for all S. Since $p(\lambda)$ is an efficiency price vector for x, we have $u(x) = u^{p(\lambda)}(p(\lambda)x)$. Hence $(\phi v_\lambda^{p(\lambda)})(S) = \int_S \lambda u^{p(\lambda)}(p(\lambda)x)$ for all S, and so by Proposition 11.12, $p(\lambda)x$ is a tax allocation in $M^{p(\lambda)}$. Since $p(\lambda)$ is an efficiency price vector for x, it follows that $(x, p(\lambda))$ is a competitive tax equilibrium, and hence x is an income tax allocation. This completes the proof of Theorem A.

19. Non-equivalence for finite T

When there are finitely many agents and at least 2 commodities, the commodity tax allocations are in general not the same as the income tax allocations. We present an example, from which it will be clear that for finite T, equivalence is the exception rather than the rule.

Define M by $T = \{1,2\}$, $e(1) = (8,0)$, $e(2) = (0,27)$, and $u_1(x) = u_2(x) = f((\sqrt[3]{x^1} + \sqrt[3]{x^2})^3)$, where f is an increasing C^1 concave bounded function that is differentiable and is the identity on the "relevant" part of the line ($f(s) = s$ for $s \leqq 125$ is sufficient). We need f only to satisfy the boundedness condition on the u_t.

The example is relatively transparent because of the homothetic preferences. Any Pareto optimal allocation — including any commodity tax allocation and any income tax allocation — consists exclusively of bundles lying on the "diagonal," i.e. the straight line segment connecting the origin to the aggregate endowment $(8, 27)$ of the economy (see Fig. 2). Therefore the efficiency price vector associated with any Pareto optimal allocation must be orthogonal to the indifference curve at $(8, 27)$, which means that it is proportional to $(9, 4)$.

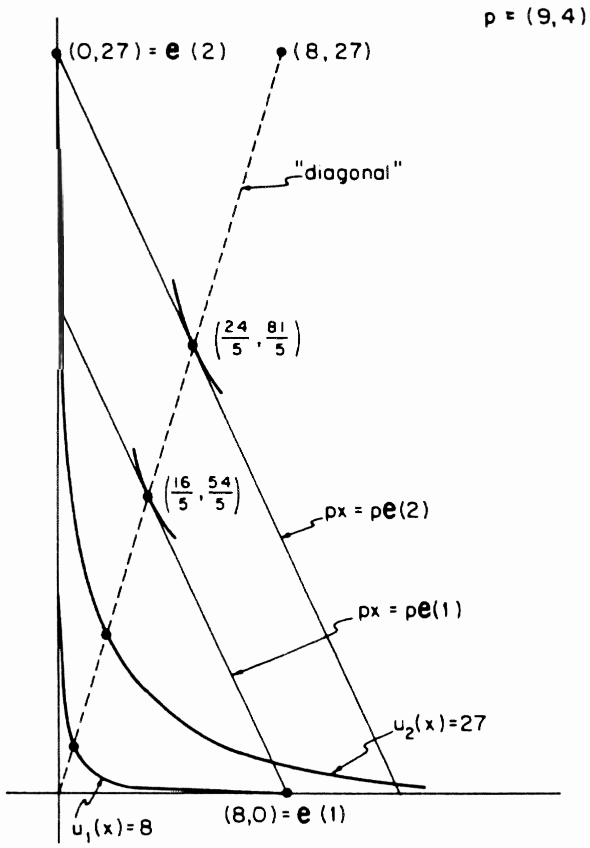

Fig. 2.

Let us first calculate the commodity tax allocations. From $|T| = 2$ it follows that $q_\lambda = r_\lambda$, and hence by Proposition 11.11 and Corollary 11.9 that $\phi v_\lambda = \phi r_\lambda$. Thus we must find an allocation $\boldsymbol{x}$ and a comparison function $\boldsymbol{\lambda}$ for which $\boldsymbol{\lambda}(t)u_t(\boldsymbol{x}(t)) = (\phi r_\lambda)(\{t\})$ for $t = 1, 2$. By a remark of Shapley [14], this is equivalent to solving a Bargaining Problem in the sense of Nash [11]; namely, the problem in which the disagreement payoffs are $r(\{1\}) = u_1(\boldsymbol{e}(1)) = 8$ and $r(\{2\}) = u_2(\boldsymbol{e}(2)) = 27$, and the set of feasible contracts is the set of all utility pairs $(u_1(\boldsymbol{x}(1)), u_2(\boldsymbol{x}(2)))$, where $\boldsymbol{x}$ ranges over all allocations. We have already seen that all Pareto optimal allocations consist of bundles on the diagonal; furthermore, the utility functions are equal to each other and are homogeneous of degree 1, hence linear on the diagonal. Hence the Pareto optimal surface of the feasible set is the line $u_1 + u_2 = 125$. In this case the Nash solution is to "split the surplus," the surplus being $125 - (8 + 27) = 90$. Hence agents 1 and 2 end up with utilities of $8 + 45 = 53$ and $27 + 45 = 72$ respectively. There is exactly one commodity tax allocation, and it is given by $\boldsymbol{x}(1) = (8 \cdot 53/125, 27 \cdot 53/125) \approx (3.4, 11.4)$ and $\boldsymbol{x}(2) = (8 \cdot 72/125, 27 \cdot 72/125) \approx (4.6, 15.6)$ (see Fig. 2).

We turn next to the competitive tax equilibria $(\boldsymbol{x}, p)$. Here the price vector p is an efficiency price vector for the income tax allocation $\boldsymbol{x}$ (see (7.2)); therefore by the homotheticity, we may take $p = (9, 4)$, as we saw above. In the market M^p, each trader t can guarantee to himself $u_t^p(pe(t))$. The maximum of u_t on the line $px = pe(t)$ is taken on when that line crosses the "diagonal" (see Fig. 2), which is at $(16/5, 54/5)$ for $t = 1$ and at $(24/5, 81/5)$ for $t = 2$. Hence $u_1^p(pe(1)) = 50$ and $u_2^p(pe(2)) = 75$. Reasoning as before but in the market M^p, we obtain a Nash Problem with disagreement payoffs of 50 and 75 and a Pareto optimal line given by $u_1 + u_2 = 125$. Here, therefore, there is no surplus, so the agents end up with utilities of 50 and 75 respectively. There is exactly one income tax allocation, given by $\boldsymbol{x}(1) = (16/5, 54/5) = (3.2, 10.8)$ and $\boldsymbol{x}(2) = (24/5, 81/5) = (4.8, 16.2)$.

The reason for the discrepancy between the commodity tax allocation and the income tax allocation is, of course, that though the feasible payoffs in M and M^p are the same, the disagreement payoffs are quite different and in fact larger in M^p than in M.

References

1. R. J. Aumann, *Values of markets with a continuum of traders*, Econometrica **43** (1975), 611–646 [Chapter 51].

2. R. J. Aumann and M. Kurz, *Power and taxes*, Econometrica **45** (1977), 1137–1161 [Chapter 52].

3. R. J. Aumann and M. Perles, *A variational problem arising in economics*, J. Math. Anal. Appl. **11** (1965), 488–503 [Chapter 70].

4. R. J. Aumann and L. S. Shapley, *Values of Non-Atomic Games*, Princeton University Press, Princeton, 1974.

5. G. Debreu, *Theory of Value*, John Wiley and Sons, New York, 1959.

6. J. C. Harsanyi, *A bargaining model for the cooperative n-person game*, in *Contributions to the Theory of Games IV* (R. D. Luce and A. W. Tucker, eds.), Princeton University Press, Princeton, 1959, pp. 325–355.

7. W. Hildenbrand, *Pareto optimality for a measure space of economic agents*, International Economic Review **10** (1969), 363–372.

8. W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, 1974.

9. Y. Kannai, *Values of games with a continuum of players*, Israel J. Math. **4** (1966), 54–58.

10. J. W. Milnor and L. S. Shapley, *Values of Large Games II: Oceanic Games*, Math. Oper. Res. **3** (1978), 290–307.

11. J. F. Nash, *The bargaining problem*, Econometrica **18** (1950), 155–162.

12. J. F. Nash, *Two person cooperative games*, Econometrica **21** (1953), 128–140.

13. L. S. Shapley, *A value for n-person games*, in *Contributions to the Theory of Games II* (H. W. Kuhn and A. W. Tucker, eds.), Princeton University Press, Princeton, 1953, pp. 307–317.

14. L. S. Shapley, *Utility comparisons and the theory of games*, in *La Décision*, Editions du Centre National de la Recherche Scientifique, Paris, 1969, pp. 251–263.

15. L. S. Shapley and M. Shubik, *Quasi-cores in a monetary economy with nonconvex preferences*, Econometrica **34** (1966), 805–827.

16. J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. **50** (1949), 401–485.

17. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1944.

DEPARTMENT OF MATHEMATICS
HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL

AND

DEPARTMENT OF ECONOMICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA, USA