14b Utility Theory without the Completeness Axiom: A Correction

1. IN [1] WE STATED (p. 265) and allegedly proved (p. 272) the following:

THEOREM B: *Let E be a convex polyhedron in a partially ordered mixture space X, and let $x \in E$. Then x is maximal in E under the partial order $\succsim$, if and only if there is a utility u on X such that x maximizes u over E.* (For definitions of the terms, see [1].)

The "only if" statement in this "theorem" is false. The trouble with the proof is that D need not be polyhedral. For a counter-example, let the space be R^3, and let the order be pure, the cone T of vectors preferred to 0 being obtained as follows: Let U be the cone whose intersection with the plane $x^3 = 1$ is given in Figure 1; T is then obtained from U by removing the x^3-axis (which is perpendicular to the plane of Figure 1 and meets it at the intersection of the axes in that figure). Let E be the line segment on the x^3-axis with end points $(0, 0, -1)$ and $(0, 0, 1)$. Then $(0, 0, 0)$ is maximal but no utility maximizes it

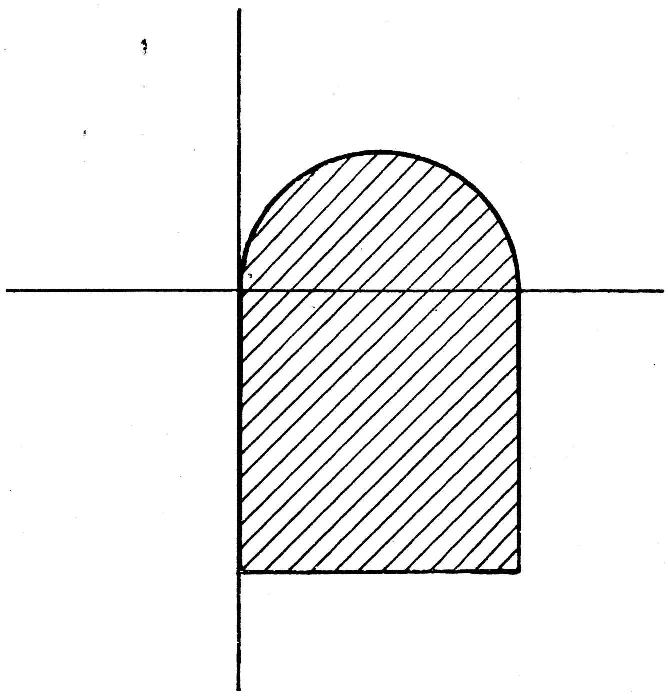

Fig. 1

over E. Requiring that T be regular [1, Section 6] does not help; a counterexample can then be given in four dimensions.

2. The theorem is true if T is assumed to be the intersection of a *finite number* of open or closed half-spaces; an equivalent requirement is that the preference order be "defined" by a finite number of linear functions $f_1, \ldots, f_k, g_1, \ldots, g_m$, in the sense that $x \succ y$ if and only if

$$f_i(x) \geqq f_i(y) \quad \text{for} \quad i = 1, \ldots, k\,,$$

and

$$g_j(x) > g_j(y) \quad \text{for} \quad j = 1, \ldots, m\,.$$

To prove this revised version of Theorem B, assume without loss of generality that the underlying space is R^n and that the given order $\succsim$ is pure. Define a (possibly impure) auxiliary order $\succsim'$ by $x \succsim' y$ if and only if $f_i(x) \geqq f_i(y)$ for $i = 1, \ldots, k$ and $g_j(x) \geqq g_j(y)$ for $j = 1, \ldots, m$. Note that

(1) $\quad x \succsim y \;\text{ implies }\; x \succsim' y$

and

(2) $\quad z \succsim' w \;\text{ and }\; w \succ x \;\text{ imply }\; z \succ x\,.$

Let D be the set of points dominated with respect to $\succsim'$ by members of E, i.e.,

$$D = \{y : \exists z \in E \text{ such that } z \succsim' y\}\,.$$

This time D really is a polyhedral set; let D^r be the unique r-dimensional face of D whose (relative) interior contains x. Let H be a supporting hyperplane for D that meets D precisely in D^r. Then if u is the normal to H, we have $u(x - y) \geqq 0$ for all $y \in D$, with equality only if $y \in D^r$. If $x \succsim y$, then by (1) $x \succsim' y$; so $y \in D$ and hence $u(x - y) \geqq 0$. We claim that $u(x - y) > 0$ whenever $x \succ y$. If not, there would be a $y \in D^r$ such that $x \succ y$. But since x is in the relative interior of D^r, there would also be a $w \in D^r$ (of the form $w = x + \delta(x - y)$, $\delta > 0$) such that $w \succ x$. By the definition of D, there is a $z \in E$ such that $z \succsim' w$; applying (2), we deduce $z \succ x$, contradicting the maximality of x in E with respect to $\succ$. Thus our claim is substantiated, and it follows that u is a utility.

The strong and weak orders both satisfy the conditions of the revised theorem, as do the orders "between" the strong and the weak—i.e., strong in some coordinates, weak in others. Any preference order that is "generated" by a finite number of preference statements of the form $x^i \succsim y^i$—in the sense that all preferences are implied by this finite number of preferences (cf. [1, p. 266])—is admissible under the revised theorem. Thus the practical application of Theorem B quoted on p. 266 of [1] is still valid. The revised theorem remains applicable when the finite number of "generating" preference statements are allowed to take the form of specifying the interval $\{\gamma : \gamma x^i + (1 - \gamma)z^i \succsim y^i\}$, for each of a finite number of x^i, y^i, and z^i in X.

3. Theorem B is also correct if *either* of the strengthened forms of the archimidean axiom that are given on p. 264 of [1] hold. These two forms are

(4.1) $\quad \{\gamma : \gamma x + (1 - \gamma)z \succsim y\}$ is closed;

and

(4.2) $\{\gamma : \gamma x + (1 - \gamma)z \succ y\}$ is open.

For the proof, we again restrict ourselves to $X = R^n$ and pure orders, and assume also that $x = 0$. If (4.2) holds, then the set $T = \{y: y \succ 0\}$ is open, and because 0 is maximal in E, T does not intersect E. A hyperplane (weakly) separating E from T is therefore an open support for T, and so provides a utility that is maximized over E at 0.

If (4.1) holds, then $S = \{y: y \succsim 0\}$ is closed. Since the order is pure, $T = \{y: y \succ 0\} = S\backslash\{0\}$. Let $\|y\|$ be the euclidean norm $(y_1^2 + \ldots + y_n^2)^{\frac{1}{2}}$. Let u be a utility, and let G be the hyperplane $\{y: uy = 1\}$. Then $G \cap T$ generates T, in the sense that $T = \{\alpha y: y \in G \cap T, \alpha > 0\}$. Also $G \cap T = G \cap S$ and $G \cap T$ is therefore closed. Suppose now that $y_1, y_2, \ldots, \in G \cap T$ is a sequence with unbounded norm, then $u(y_i/\|y_i\|) = 1/\|y_i \to 0$. Since $y_i/\|y_i\|$ is on the unit sphere, it has a limit point y_∞ such that $\|y_\infty\| = 1$; we may assume that it is actually the limit. Since u is linear, it is continuous, and hence $u(y_\infty) = 0$; but, since S is closed, $y_\infty \in S$; and, since $y_\infty \neq 0$, it follows that $y_\infty \in T$, so that $u(y_\infty) > 0$. This contradiction shows that $G \cap T$ is bounded and therefore compact. Next, let F be the cone generated by E, i.e., $F = \{\alpha y: y \in E, \alpha \geq 0\}$. Since E is a polyhedron containing 0, F is closed. Since 0 is maximal in E, $E \cap T = \phi$, therefore $F \cap T = \phi$, and therefore also $(G \cap F) \cap (G \cap T) = \phi$. Now $G \cap F$ is closed and $G \cap T$ is compact; so $G \cap T$ has a convex open neighborhood U in G that does not intersect F (for example, the set of all points with distance $< \delta$ from $G \cap T$, where $\delta > 0$ is smaller than the distance of $G \cap T$ from $G \cap F$). If T' is the cone generated by U, then T' does not intersect F. We may define an auxiliary pure order $\succsim'$ on R^n by $\{y: y \succ' 0\} = T'$; then $\succsim'$ satisfies (4.2), and 0 is maximal in F, and hence also in E, with respect to $\succsim'$. So there is a utility u' for $\succsim'$ such that 0 maximizes u' over E. But since $U \supset G \cap T$ and $G \cap T$ generates T, it follows that $T' \supset T$, and hence u' is a utility also for $\succsim$. This completes the proof.

4. In addition to Theorem B, Theorem C of [1], concerning equilibrium points of games, is false as it stands. To make it true, one of the restrictions mentioned in the foregoing must be placed on the preference orders.

Note that the "if" half of Theorem B (and of Theorem C) remains true as it stands. Though trivial to prove, it is probably more important for the applications than the "only if" half; for, together with the existence theorem for utilities (Theorem A of [1]) it gives a method for obtaining at least *some* solution for each maximization problem. In particular, it is true that every game (either 2-person or n-person) with partially ordered outcome spaces (cf. Section 5 of [1]) possesses an equilibrium point.

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REFERENCE

[1] AUMANN, R. J.: "Utility Theory without the Completeness Axiom," *Econometrica*, Vol. 30 (1962), pp. 445–62 [Chapter 14a].