NON-CONCENTRATION OF RETURN TIMES

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ABSTRACT. We show that the distribution of the first return time τ to the origin, v, of a simple random walk on an infinite recurrent graph is heavy tailed and non-concentrated. More precisely, if d_v is the degree of v then for any $t \geq 1$ we have

$$\mathbf{P}_v(\tau \geq t) \geq \frac{c}{d_v\sqrt{t}},$$

and

$$\mathbf{P}_v(\tau = t \mid \tau \geq t) \leq \frac{C\log(d_v t)}{t},$$

for some universal constants $c > 0$ and $C < \infty$. The first bound is attained for all t when the underlying graph is $\mathbb{Z}$, and as for the second bound, we construct an example of a recurrent graph G for which it is attained for infinitely many t's.

Furthermore, we show that in the *comb* product of G with $\mathbb{Z}$, two independent random walks collide infinitely many times almost surely. This answers negatively a question of Krishnapur and Peres [5] who asked whether any comb product of two infinite recurrent graphs has the finite collision property.

1. INTRODUCTION

1.1. **Return times.** In this paper we study the distribution of return times of a simple random walk X_t on an infinite connected graph $G = (V, E)$ with finite degrees. For $v \in V$, the *hitting time* of v by X, denoted τ_v, is defined by $\tau_v = \min\{t \geq 1 : X_t = v\}$. When X starts at v (that is, $X_0 = v$), we call τ_v the *return time* to v. As usual, the law of X when $X_0 = v$ is denoted by $\mathbf{P}_v$. Our main result is that on any graph these times are heavy tailed, with exponent at most 1/2, and non-concentrated.

Theorem 1.1. *Let $G = (V, E)$ be an infinite connected graph with finite degrees $\{d_v\}_{v \in V}$. There exists a universal constants $c > 0$ such that for any $t \geq 1$ we have*

$$\mathbf{P}_v(\tau_v \geq t) \geq \frac{c}{d_v\sqrt{t}},$$

Theorem 1.2. *Let $G = (V, E)$ be an infinite connected graph with finite degrees $\{d_v\}_{v\in V}$. There exists a universal constants $C < \infty$ such that for any $t \geq 1$ we have*

$$\mathbf{P}_v(\tau_v = t \mid \tau_v \geq t) \leq \frac{-C\log(d_v t)}{t}.$$

The proof of Theorem 1.1 uses electrical network and martingale arguments and the proof of Theorem 1.2 incorporates spectral decomposition of killed random walks. These two inequalities are sharp up to multiplicative constants. Indeed, for Theorem 1.1, it is easy to see that in a copy of $\mathbb{N}$, together with $d-1$ new vertices who are attached only to 0 we have $\mathbf{P}_0(\tau_0 \geq t) \approx \frac{c}{d\sqrt{t}}$.

Constructing a graph which saturates the inequality of Theorem 1.2 is harder and we perform this in section 4. The sharpness of Theorem 1.2 is perhaps more surprising since most natural examples exhibit an upper bound of order $1/t$. For example, in $\mathbb{Z}$ it is classical (see [3]) that $\mathbf{P}_v(\tau_v \geq t) \approx t^{-1/2}$ and $\mathbf{P}_v(\tau_v = t) \approx t^{-3/2}$. It is likely that if the distribution of τ_v is regular varying in some sense it is possible to prove a $1/t$ upper bound. Indeed, in the construction in Section 4 the rate of decay of $\mathbf{P}_v(\tau_v \geq t)$ has extremely different behavior at different scales of t.

It is a well known fact that $\mathbb{E}\tau_v = \infty$ for any infinite connected graph. This of course follows from Theorem 1.1, but a simpler way to see it is to consider the Green function

$$g(u) = \mathbb{E}_v \sum_{t=1}^{\tau_v} \mathbf{1}_{\{X_t = u\}},$$

that is, the expected number of visits to u before returning to v. It is easy to check that the vector $\{g(u)\}_{u\in G}$ is invariant under the random walk operator and that $g(v) = 1$. Hence, $g(u) = d_u/d_v$ for all u in the connected component of v. Furthermore, it is clear that $\sum_u g(u) = \mathbb{E}\tau_v$ and since G is connected and infinite we deduce that $\mathbb{E}\tau_v = \infty$.

1.2. **The finite collision property.** The construction of Section 4 is related to the finite collision property. Recall that an infinite graph G has the *finite collision property* if two simple random walks X_t and Y_t collide only finitely many times almost surely, i.e., the set $\{t : X_t = Y_t\}$ is almost surely finite. It is obvious that any bounded-degree transient graph have the finite collision property, and it is an easy exercise to check that $\mathbb{Z}$ and $\mathbb{Z}^2$ do not have the finite collision property. In fact, any transitive recurrent graph does not have the finite collision property (to see this, note that in a transitive graph the number of collisions has a geometric distribution, hence it is a.s. finite if and only if it has finite mean).

It is a surprising discovery of Krishanpur and Peres [5] that there exists recurrent graphs with the finite collision property. In these graphs, both random walks visit every vertex infinitely often, but only collide finitely many times. Their constructions involve the *comb* product of two graphs and is defined as follows. Given two graph G, H and a vertex $v \in H$ define $\mathrm{Comb}_v(G,H)$ to be the graph with vertex set $V(G) \times V(H)$ and edge set

$$\Big\{\{(x,w),(x,z)\} : \{w,z\} \in E(H)\ ,\ x \in V(G)\Big\} \cup \Big\{\{(x,v),(y,v)\} : \{x,y\} \in E(G)\Big\}\,.$$

Krishanpur and Peres prove in [5] that $\mathrm{Comb}_0(G,\mathbb{Z})$ and $\mathrm{Comb}_{(0,0)}(G,\mathbb{Z}^2)$ have the finite collision property whenever G is an infinite recurrent graph with bounded degrees. They asked (see first question of Section 4 of [5]) whether $\mathrm{Comb}_v(G,H)$ has the finite collision property whenever G and H are infinite recurrent graphs. Our next result answers their question negatively.

Theorem 1.3. *There exists a bounded-degree, connected, infinite graph H and a vertex $v \in H$ such that $\mathrm{Comb}_v(\mathbb{Z},H)$ does not have the finite collision property.*

We do not use Theorems 1.1 and 1.2 for the proof of Theorem 1.3, however, the graph for which (1.2) is saturated (see Section 4) is the graph H in the statement of the above theorem. The important property of this graph is that, roughly, at certain scales it behaves like a finite graph. This property is crucial both for showing the sharpness of Theorem 1.2) and for the proof of Theorem 1.3.

In fact, general results in this flavor have recently been obtained. Barlow, Peres and Sousi [1] give a general condition for a graph not to have the finite collision property. While this condition fails for the graph constructed in the proof of Theorem 1.3, they use it to show that various natural graphs with fractal geometry do not have the finite collision property.

1.3. **Extensions and questions.** Theorems 1.1 and 1.2 can be extended to the setting of finite graphs. Indeed, the proofs of both theorems can be extended so that they hold for any finite graph and any $t \le R^2$, where R is the effective resistance diameter of the graph $R = \max_{v,u} R_{\mathrm{eff}}(v \leftrightarrow u)$. These extensions to the proof are straightforward. In particular cases it is even possible to prove stronger assertions, see for example Lemma 3.1.

We cannot expect Theorems 1.1 and 1.2 to hold for general hitting times. Indeed, if u is a vertex such that its removal leaves v in a finite component (these are sometimes called *cutpoints*), then the distribution of τ_u started from v has exponential decay since as much as the distribution of τ_u started from v is concerned, the graph is finite. However, perhaps there is hope to prove similar estimates when u is not such a cutpoint.

To demonstrate that Theorem 1.2 does not hold for hitting times in general, consider the following example: The graph is simply the natural numbers, with 2^{2^n} edges between n and $n+1$. If a simple random walk starts at 0, there is a positive probability it will never take a step backward, that is, $X_i = i$ for all i. This means that $\mathbf{P}_0(\tau_n = n | \tau_n \geq n)$ does not decay to 0. Of course, this graph has unbounded degrees, so it remains to see whether a bounded degree example exists. Similar questions can be also asked about *commute times*, that is, the first time to hit some specific vertex and return to the origin. These retain some of the symmetry of return times and perhaps Theorem 1.1 and 1.2 can be extended to them.

Finally, is it true that for any $t \geq 1$ the graph $\mathbb{N}$ minimizes the quantity $\mathbf{P}_0(\tau_0 \geq t)$ of all connected infinite graphs with the origin having degree 1?

2. Proof of Theorems 1.1 and 1.2.

We begin with a few lemmas. For background about effective resistance we refer the reader to [6].

Lemma 2.1. *Let G be a finite graph. For any two vertices x, y and any $\varepsilon > 0$ we have*

$$\mathbf{P}_x(\tau_y \leq \varepsilon (R_{\text{eff}}(x \leftrightarrow y))^2) \leq \varepsilon$$

where $R_{\text{eff}}(x \leftrightarrow y)$ is the effective resistance between x and y, when G is considered as an electric network with unit resistances.

Proof. Let $f : G \to \mathbb{R}_+$ be the potential corresponding to a unit current flow of the electrical network between x and y. That is, f is the harmonic function on $G \setminus \{x, y\}$ with boundary values $f(x) = 0$ and $f(y) = R_{\text{eff}}(x \leftrightarrow y)$ (as G is finite f is uniquely determined). The associated unit current flow is an antisymmetric function on directed edges $i : E(G) \to \mathbb{R}$ such that

(i) $\sum_{u \sim x} i(xu) = 1$, and
(ii) For any $u \in G \setminus \{x, y\}$ we have $\sum_{v \sim u} i(uv) = 0$, and
(iii) For any oriented cycle $e_1, \ldots, e_m$ we have $\sum_{1 \leq j \leq m} i(e_j) = 0$.

Since we have unit edge resistances we get that $i(uv) = f(u) - f(v)$ for any edge uv. We first observe that f is Lipschitz, that is, for any edge uv we have $f(u) - f(v) \le 1$. Indeed, assume without loss of generality that $f(u) < f(v)$ and let $s > 0$ be a number such that $f(u) < s \le f(v)$. Consider the cut (S, S^c) defined by $S = \{u : f(u) < s\}$. The sum of the unit current flow i on edges leading from S to S^c is 1 and each edge receives non-negative flow, hence $f(v) - f(u) = i(u, v) \le 1$ (another way to see this is combining Proposition 2.2 and Exercise 2.31 of [6]). We deduce that

$$\mathbb{E}\big[f^2(X_t) - f^2(X_{t-1}) \,\big|\, X_{t-1}\big] = \mathbb{E}\big[(f(X_t) - f(X_{t-1}))^2 \,\big|\, X_{t-1}\big] \le 1\,,$$

when $X_{t-1} \neq y$ and hence$f^2(X_{t\wedge\tau_y}) - t\wedge\tau_y$ is a supermartingale. Put $T = \varepsilon R_{\text{eff}}(x \leftrightarrow y)^2$. Optional stopping yields that

$$\mathbb{E}_x[f^2(X_{T\wedge\tau_y})] \le \mathbb{E}_x[T\wedge\tau_y] \le T\,.$$

If $\tau_y < T$, then $f^2(X_{T\wedge\tau_y}) = R_{\text{eff}}(x \leftrightarrow y)^2$. Thus, by Markov's inequality we get

$$\mathbf{P}_x(\tau_y < T) \le \frac{T}{R_{\text{eff}}(x \leftrightarrow y)^2} \le \varepsilon\,,$$

concluding the proof of the lemma. ∎

Proof of Theorem 1.1. We prove the assertion with $c = \frac{1}{4}$. For $r > 0$ we write $B(v, r)$ for the ball of radius r in G according to the shortest path metric and write $\partial B(v, r)$ for its boundary, that is, $\partial B(v, r) = B(v, r) \setminus B(v, r-1)$. We consider the effective resistance $R_{\text{eff}}(v \leftrightarrow \partial B(v, r))$. Fix $t \ge 1$. If for all $r > 0$ we have that $R_{\text{eff}}(v \leftrightarrow \partial B(v, r)) \le 4\sqrt{t}$ (this can only happen in the transient case), then

$$\mathbf{P}_v(\tau_v \ge t) \ge \lim_{r\to\infty} \mathbf{P}_v\big(X_t \text{ hits } \partial B(v, r) \text{ before } v\big) \ge \frac{1}{4d_v\sqrt{t}}\,.$$

Otherwise, let r to be the first radius such that $R_{\text{eff}}(v \leftrightarrow \partial B(v, r)) \ge 4\sqrt{t}$. As in the proof of Lemma 2.1 let f be the harmonic function on $B(v, r)$ with $f(v) = 0$ and $f(\partial B(v, r)) = R_{\text{eff}}(v \leftrightarrow \partial B(v, r))$. Let S be the set of vertices $S = \{u : f(u) \le 2\sqrt{t}\}$. We saw in the proof of Lemma 2.1 that f is Lipschitz. Hence, any vertex $x \in N(S)$, where $N(S)$ denotes the neighbors of S which are not in S, has $2\sqrt{t} \le f(x) \le 2\sqrt{t} + 1$. We deduce that

$$2\sqrt{t} \le R_{\text{eff}}(v \leftrightarrow N(S)) \le 2\sqrt{t} + 1\,.$$

We have that

$$\mathbf{P}_v\big(\tau_{N(S)} < \tau_v\big) = \frac{1}{d_v R_{\text{eff}}(v \leftrightarrow N(S))} \ge \frac{1}{d_v(2\sqrt{t} + 1)} \tag{2.1}$$

where $\tau_{N(S)} = \min\{t \geq 1 \mid X_t \in N(S)\}$. The strong Markov property implies

$$\mathbf{P}_v(\tau_v \geq t) \geq \mathbf{P}_v\big(\tau_{N(S)} < \tau_v\big) \min_{u \in N(S)} \mathbf{P}_u\big(\tau_v \geq t\big).$$

To estimate the second probability on the right hand side we apply Lemma 2.1 with $\varepsilon = 1/4$. We deduce that this probability is at least $3/4$. This together with (2.1) gives that

$$\mathbf{P}_v(\tau_v \geq t) \geq \frac{3}{4d_v(2\sqrt{t}+1)} \geq \frac{1}{4d_v\sqrt{t}},$$

concluding our proof. ∎

Lemma 2.2. [Spectral decomposition] *Let $G = (V, E)$ be an infinite connected graph with finite degrees and let $v \in V$. Then there exists a finite measure μ on $[-1, 1]$ such that for all $t \geq 2$ we have*

$$\mathbf{P}_v(\tau_v = t) = \int_{-1}^{1} x^{t-2}\, d\mu.$$

Proof. By conditioning on the location of the random walk at time $\lceil t/2 \rceil$ and using the Markov property we get that

$$\mathbf{P}_v(\tau_v = t) = \sum_{u \neq v} \mathbf{P}_v\big(X_{\lceil t/2 \rceil} = u, \tau_v \geq \lceil t/2 \rceil\big)\mathbf{P}_u\big(\tau_v = \lfloor t/2 \rfloor\big).$$

Observe that by the reversibility property of the simple random walk we have

$$\mathbf{P}_v\big(X_{\lceil t/2 \rceil} = u, \tau_v \geq \lceil t/2 \rceil\big) = \frac{d_u}{d_v}\mathbf{P}_u\big(\tau_v = \lceil t/2 \rceil\big)$$

and hence

$$\mathbf{P}_v(\tau_v = t) = \frac{1}{d_v} \sum_{u \neq v} d_u \mathbf{P}_u\big(\tau_v = \lceil t/2 \rceil\big)\mathbf{P}_u\big(\tau_v = \lfloor t/2 \rfloor\big). \tag{2.2}$$

Consider the Hilbert space $\ell_2(G)$ of functions from $V \setminus \{v\}$ to $\mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle = \sum_{u \neq v} d_u f(u) g(u)$$

and the corresponding norm. Let $\mathbf{Q}$ be the random walk operator killed upon hitting v. That is,

$$\mathbf{Q}f(u) = \frac{1}{d_u} \sum_{w \sim u, w \neq v} f(w).$$

One can easily check that $\mathbf{Q}^t f = \mathbb{E}_u(f(X_t)\mathbf{1}_{\{\tau_v > t\}})$. Define the function $h(w) = \mathbf{P}_w(\tau_v = 1)$ (i.e. $h(w) = 1/d_w$ if $w \sim v$ and $h(w) = 0$ otherwise). We have that

$$\mathbf{Q}^{t-1} h(u) = \mathbf{P}_u(\tau_v = t).$$

Hence, we can rewrite equation 2.2 as

$$\mathbf{P}_v(\tau_v = t) = \frac{1}{d_v}\langle \mathbf{Q}^{\lceil t/2\rceil - 1} h, \mathbf{Q}^{\lfloor t/2 \rfloor - 1} h\rangle.$$

A simple calculation shows that

$$\langle \mathbf{Q} f, g\rangle = \sum_{u \neq v} \sum_{w \sim u, w \neq v} f(w) g(u) = \langle f, \mathbf{Q} g\rangle,$$

that is, $\mathbf{Q}$ is self-adjoint with respect to the inner product of $\ell_2(G)$. Hence, we may apply the spectral theorem (see [7] or [4]) and conclude that there exists a measure ν on $[-1,1]$ and a real function $\lambda : [-1,1] \to [-1,1]$ such that $\mathbf{Q}$ is isometrically equivalent to multiplication by λ. In particular,

$$\mathbf{P}_v(\tau_v = t) = \frac{1}{d_v}\langle \mathbf{Q}^{\lceil t/2\rceil - 1} h, \mathbf{Q}^{\lfloor t/2 \rfloor - 1} h\rangle = \frac{1}{d_v}\int_{-1}^{1} \lambda^{t-2}(x) \hat{h}^2(x) d\nu$$

where $\hat{h}$ is the image of h under the isometry. If we define μ to be the pull-back measure

$$\mu(A) = \frac{1}{d_v}\int_{\lambda^{-1}(A)} \hat{h}^2(x) d\nu,$$

for any Borel set $A \subset [-1,1]$, we get that

$$\mathbf{P}_v(\tau_v = t) = \int_{-1}^{1} x^{t-2} d\mu,$$

which concludes the proof. ■

Corollary 2.3. *Let $G = (V,E)$ be an infinite connected graph with finite degree and let $v \in V$, then the sequence $\{\mathbf{P}_v(\tau_v = 2t)\}_{t \geq 1}$ is decreasing in t.*

Proof of Theorem 1.2 We prove the assertion with $C = e^{10}$. We may assume $t \geq \max\{e^{10}/d_v, 4\}$, since otherwise we have that $e^{10} t^{-1} \log d_v t \geq 1$ and the assertion is trivial. Lemma 2.2 gives that $\mathbf{P}_v(\tau_v = t) = \int_{[-1,1]} x^{t-2} d\mu$ for some measure μ. Write $A \subset [-1,1]$ for the set

$$A = \left\{x : |x| \geq 1 - \frac{4 \log d_v t}{t}\right\}.$$

Assume first that t is even. In this case we may bound

$$\mathbf{P}_v(\tau_v \geq t) \geq \sum_{j \geq 0} \mathbf{P}_v(\tau = t + 2j) = \int_{[-1,1]} \frac{x^{t-2}}{1 - x^2} d\mu \geq \int_A \frac{x^{t-2}}{1-x^2} d\mu. \quad (2.3)$$

Thus,

$$\frac{\mathbf{P}_v(\tau_v = t)}{\mathbf{P}_v(\tau_v \geq t)} \leq \frac{\int_{A^c} x^{t-2} d\mu}{\mathbf{P}_v(\tau_v \geq t)} + \frac{\int_A x^{t-2} d\mu}{\int_A \frac{x^{t-2}}{1-x^2} d\mu}. \tag{2.4}$$

If $x \notin A$, then $x^{t-2} \leq (1 - \frac{4\log(d_v t)}{t})^{t-2} \leq e^{-2\log(d_v t)} \leq (d_v t)^{-2}$ since $t \geq 4$. We also have $\mu([-1,1]) \leq 1$ by putting $t = 2$ in Lemma 2.2. Hence, by Theorem 1.1 (recall that we proved it with $c = \frac{1}{4}$) we get that

$$\frac{\int_{A^c} x^{t-2} d\mu}{\mathbf{P}_v(\tau_v \geq t)} \leq \frac{4}{t}.$$

If $x \in A$, then $x^2 \geq 1 - 8t^{-1} \log d_v t$ hence

$$\frac{\int_A x^{t-2} d\mu}{\int_A \frac{x^{t-2}}{1-x^2} d\mu} \leq \frac{8 \log d_v t}{t}.$$

We put these two in (2.4) and get that $\mathbf{P}_v(\tau_v = t \mid \tau_v \geq t) \leq \frac{12 \log d_v t}{t}$ when t is even. When t is odd we first bound

$$\mathbf{P}_v(\tau_v = t) = \int_{[-1,1]} x^{t-2} d\mu \leq \int_{[-1,1]} x^{t-3} d\mu = \mathbf{P}_v(\tau_v = t-1).$$

By the assertion for even t's we get that

$$\mathbf{P}_v(\tau_v = t) \leq \frac{12 \log(d_v(t-1))}{t-1} \mathbf{P}_v(\tau_v \geq t-1).$$

Also, $\mathbf{P}_v(\tau_v \geq t-1) = \mathbf{P}_v(\tau_v \geq t) + \mathbf{P}_v(\tau_v = t-1)$ so

$$\mathbf{P}_v(\tau_v \geq t-1) \leq \left[1 - \frac{12\log(d_v(t-1))}{t-1}\right]^{-1} \mathbf{P}_v(\tau_v \geq t),$$

whenever $12\log(d_v(t-1))/(t-1) < 1$. Indeed, whenever $t \geq e^{10}/d_v$ we have that $12\log(d_v(t-1))/(t-1) < 1/2$ and we have that

$$\mathbf{P}_v(\tau_v = t) \leq \frac{24 \log t}{t} \mathbf{P}_v(\tau_v \geq t),$$

concluding our proof. ■

3. PRELIMINARIES ON EXPANDERS.

Recall that a family $\{G_n\}$ of d-regular graphs on n vertices is called an *expander* family if there is some uniform constant $\rho < 1$ such that the second largest eigenvalue in absolute value of the transition matrix $\lambda_2(n)$ satisfies $|\lambda_2(n)| \leq \rho$ for all n. It is a classical fact (see Theorem 6.9 in [6]) that if $\{X_t\}$ is a simple random walk on G_n then for any $v \in G_n$ and any integer t we have

$$\Big|\mathbf{P}(X_t = v) - \frac{1}{n}\Big| \le e^{-(1-\rho)t}. \tag{3.1}$$

Another useful fact (see [2]) is that if we put unit resistance on each edge of the expander, then there exists a constant $C = C(\rho) < \infty$ such that for any $u, v \in G_n$ the effective resistance satisfies

$$R_{\text{eff}}(u \leftrightarrow v) \le C. \tag{3.2}$$

In the following four lemmas we study the simple random walk on the graph G obtained by taking a d-regular expander and an arbitrary vertex of v and adding a new vertex v' together with the edge $\{v', v\}$. We consider d as fixed and $|G| = n$ tending to infinity (in all our applications taking $d = 3$ suffices).

Lemma 3.1. *There exist constants $\delta > 0$ such that for any $u \neq v'$*

$$\mathbf{P}_u(\tau_{v'} \ge \delta n) \ge \delta\,,$$

and

$$\mathbf{P}_u(\tau_{v'} \le n) \ge \delta\,.$$

Proof. We begin by proving a lower bound on $\mathbf{P}(\tau_{v'} \ge \delta n)$. Since the walker must visit v in order to visit v' it suffices to prove the assertion for $X_0 = v$. Since G has bounded degree, there exists a vertex $y \in G$ with graph distance from v at least $c \log n$. By (3.2) the effective resistance between v and y is bounded by a constant and hence with constant positive probability X_t hits y before v. We get that that for some constant $c > 0$ we have

$$\mathbf{P}(\tau_{v'} \ge c \log n) \ge c\,. \tag{3.3}$$

Furthermore, by (3.1) and the union bound we have that

$$\mathbf{P}\big(\exists\, t \in [c \log n, \delta n] \text{ with } X_t = v\big) \le \delta + \frac{e^{-(1-\rho)c \log n}}{1 - e^{-(1-\rho)}}\,,$$

where $\rho < 1$ is the uniform bound on the second eigenvalue. This together with (3.3) shows that $\mathbf{P}(\tau_{v'} \ge \delta n) \ge \delta$ for some constant $\delta > 0$.

To prove a lower bound $\mathbf{P}(\tau_{v'} \le n)$ we employ a second moment calculation. Write Y for the number of visits to v' in before time n. It is clear by (3.1) that $\mathbf{P}(X_t = v') \ge \frac{1}{2n}$ for any $t \ge C \log n$ so $\mathbb{E} Y \ge c$ for some $c > 0$.

On the other hand, if $t_2 > t_1$ and $X_{t_1} = v'$, then by (3.1) the probability of having $X_{t_2} = v'$ is at most $n^{-1} + e^{-c(t_2-t_1)}$ for some $c > 0$. This gives that $\mathbb{E}Y^2 \le C$ and we get that $Y > 0$ with some fixed probability by the inequality

$$\mathbf{P}(V > 0) \ge \frac{(\mathbb{E}V)^2}{\mathbb{E}V^2},$$

valid for any non-negative random variable V. This concludes the proof. ∎

Lemma 3.2. *There exists constants $C, c > 0$ such that for any vertex $u \ne v'$ there exists a set of vertices S_u such that $|S_u| = n - o(n)$ and for any $w \in S_u$ and any $C \log n \le t \le n$*

$$\mathbf{P}_u(X_t = w\,, \tau_{v'} \ge t) \ge \frac{c}{n}.$$

Proof. By (3.1) and Lemma 3.1, for any two vertices u, w and any $C \log n \le t \le n$ we have $\mathbf{P}_u(X_t = w) \le 2/n$ and $\mathbf{P}_u(\tau_{v'} \ge t) \ge c > 0$. Hence

$$\mathbf{P}_u(X_t = w \mid \tau_{v'} \ge t) \le \frac{C}{n}, \tag{3.4}$$

for some $C = C(\rho) > 0$. Furthermore, $\mathbf{P}_\pi(\tau_{v'} \le C \log n) = O(n^{-1} \log n)$, where π is the stationary distribution. This is because the expected number of visits to v' by time $C \log n$ is $O(n^{-1} \log n)$. Define

$$S = \left\{u : \mathbf{P}_u(\tau_{v'} \le C \log n) \le C n^{-1} \log^2 n\right\},$$

and we deduce that $|S| \ge n(1 - \log^{-1} n)$. We combine this with (3.4) to get that

$$\mathbf{P}_u(X_t \in S \mid \tau_{v'} \ge t) \ge 1 - \frac{C}{\log n}. \tag{3.5}$$

By definition of S and (3.1), for any $u \in S$ and any w we have

$$\mathbf{P}_u(X_{C \log n} = w \mid \tau_{v'} \ge C \log n) \le \frac{1 + o(1)}{n}.$$

Thus, by the Markov property, for any u and w

$$\mathbf{P}_u(X_{t + C \log n} = w \mid X_t \in S, \tau_{v'} \ge t + C \log n) \le \frac{1 + o(1)}{n},$$

and therefore there exists a set S_u such that $|S_u| = n - o(n)$ such that for every $w \in S_u$ we have

$$\mathbf{P}_u(X_{t + C \log n} = w \mid X_t \in S, \tau_{v'} \ge t + C \log n) \ge \frac{1}{n}(1 - o(1)).$$

This together with (3.5) shows that for any $w \in S_u$ and $2C\log n \le t \le n$ we have

$$\mathbf{P}_u(X_t = w\,,\, \tau_{v'} > t) \ge \frac{c}{n}\,, \tag{3.6}$$

concluding our proof. ■

Lemma 3.3. *There exist constants $C = C(\rho), c = c(\rho) > 0$, such that for every $C\log n \le t \le n$ and any $u \ne v'$ we have*

$$\mathbf{P}_u(\tau_{v'} = t) \ge \frac{c}{n}\,.$$

Proof. Reversibility of the simple random walk implies that

$$\mathbf{P}_u(\tau_{v'} = t) \ge \frac{1}{d}\sum_{w\in V}\mathbf{P}_u(X_{t/2} = w, \tau_{v'} \ge \lceil t/2\rceil)\mathbf{P}_{v'}(X_{t/2} = w, \tau_{v'} \ge \lfloor t/2\rfloor)\,,$$

and the assertion follows from plugging in Lemma 3.2 and summing. ■

Our last lemma about expanders concerns two independent simple random walks X_t and Y_t. We denote by $\mathbf{P}_{u_1,u_2}$ for the probability distribution generated when $X_0 = u_1$ and $Y_0 = u_2$. We denote τ^X_u for the hitting time of X_t of u and similarly for Y.

Lemma 3.4. *There exists a constant $c = c(\rho) > 0$ such that for any $u_1 \ne v'$ and $u_2 \ne v'$*

$$\mathbf{P}_{u_1,u_2}\Big(\exists t \le n \wedge \tau^X_{v'} \wedge \tau^Y_{v'} \text{ such that } X_t = Y_t\Big) \ge c\,.$$

In other words, the probability that X_t and Y_t collide before time n and before either of them hits v' is uniformly positive.

Proof. For any $C\log n \le t \le n$, by Lemma 3.2 there exists a constant $c > 0$ and a set S of size $|S| = n - o(n)$ such that for any $w \in S$

$$\mathbf{P}_{u_1}(X_t = w\,,\, \tau^X_{v'} \ge t) \ge \frac{c}{n} \qquad \mathbf{P}_{u_2}(Y_t = w\,,\, \tau^Y_{v'} \ge t) \ge \frac{c}{n}\,.$$

Hence

$$\sum_{w\in G}\mathbf{P}_{u_1}\big(X_t = w\,,\, \tau^X_{v'} \ge t\big)\mathbf{P}_{u_2}\big(Y_t = w\,,\, \tau^Y_{v'} \ge t\big) \ge \frac{c}{n}\,.$$

Let $N = |\{t \le n \wedge \tau^X_{v'} \wedge \tau^Y_{v'} : X_t = Y_t\}|$, then by the previous inequality and the independence of X_t and Y_t we learn that $\mathbb{E}N \ge c$. To bound the second moment of N by (3.1) we have $\mathbf{P}_v(X_t = u) \le n^{-1} + e^{-ct}$, for some $c > 0$. Also, by reversibility $\mathbf{P}_{u,u}(X_t = Y_t) = \mathbf{P}_u(X_{2t} = u)$. We deduce by the Markov property that for any $t_2 > t_1$ we have that

$$\mathbf{P}_{u_1,u_2}(X_{t_1} = Y_{t_1} \text{ and } X_{t_2} = Y_{t_2}) \le (n^{-1} + e^{-2c(t_2-t_1)})\mathbf{P}_{u_1,u_2}(X_{t_1} = Y_{t_1})\,.$$

Similar considerations give that $\mathbf{P}_{u_1,u_2}(X_{t_1} = Y_{t_1}) \le n^{-1} + e^{-ct_1}$ and so we have that

$$\mathbb{E}N^2 \le \sum_{t_1=1}^{n} \sum_{t_2=t_1}^{n} \mathbf{P}_{u_1,u_2}(X_{t_1} = Y_{t_1} \text{ and } X_{t_2} = Y_{t_2}) \le C\,,$$

for some constant $C > 0$ and the assertion of the lemma follows. ■

4. SHARPNESS

In this section we show that the estimate of Theorem 1.2 is sharp up to the multiplicative constant C. In order to elucidate the ideas of the construction we begin with a simple construction showing the sharpness of Theorem 1.2 for a single t. We then construct a more elaborate graph for which the theorem is sharp for an infinite sequence of t's. This graph will be useful later in Section 5 — it will be the base of the comb for the construction of Theorem 1.3.

4.1. A simple construction.

Given an integer t we construct the graph G_t as follows. Let $\delta > 0$ be the constant from Lemma 3.1 and take

$$n = \frac{3\log(1/\delta)\,t}{\delta \log t}\,.$$

The graph G_t is constructed by taking $\mathbb{N} = \{0, 1, \ldots\}$ with edges between consecutive numbers, and attaching to 0, by an edge, a 3-regular expander of size n (the degree of 0 is thus 2).

Theorem 4.1. *There exists $c > 0$ such that the simple random walk on G_t satisfies*

$$\mathbf{P}_0(\tau_0 = t \mid \tau_0 \ge t) \ge \frac{c\log t}{t}\,. \tag{4.1}$$

Proof. We abbreviate τ for τ_0 and write $\{X_t\}$ for the simple random walk on G_t starting at 0. Write $\mathscr{A}$ for the event that $X_1 = 1$, so $\mathbf{P}(\mathscr{A}) = 1/2$. It is a well known fact (see [3]) that the probability that a random walk on $\mathbb{N}$ does not return to the origin in t steps decays like $t^{-1/2}$, that is, $\mathbf{P}(\tau \ge t \mid \mathscr{A}) \approx t^{-1/2}$. Let $\delta > 0$ be the constant from Lemma 3.1. By iterating Lemma 3.1 using the Markov property we get that $\mathbf{P}(\tau \ge t \mid \mathscr{A}^c) \ge \delta^{t/\delta n}$. Thus, since $t \le \frac{\delta}{2\log(1/\delta)} n \log n$ we have that $\mathbf{P}(\tau \ge t \mid \mathscr{A}^c) \ge t^{-1/2}$ and so

$$\mathbf{P}(\tau \ge t) \approx \mathbf{P}(\tau \ge t \mid \mathscr{A}^c)\,.$$

Now, Lemma 3.3 together with the Markov property gives that

$$\mathbf{P}(\tau = t \mid \mathscr{A}^c) \geq \frac{c}{n}\mathbf{P}(\tau \geq t - \delta n \mid \mathscr{A}^c) \geq \frac{c'}{n}\mathbf{P}(\tau \geq t \mid \mathscr{A}^c) \geq \frac{c \log t}{t}\mathbf{P}(\tau \geq t)\,,$$

concluding our proof. ■

4.2. **The full construction.** We now construct a graph saturating inequality (4.1) for infinitely many t's. This graph will also be used in the next section as the base graph (a *tooth*) of the comb exhibiting almost sure infinitely many collisions. Let $\{h_i, n_i\}_{i\geq 0}$ be two increasing sequences of integers such that

$$h_i \gg n_{i-1}h_{i-1}^2 \text{ and } n_i = h_i^{15}\,, \tag{4.2}$$

and let $\{E_i\}_{i\geq 1}$ be a sequence of 3-regular expanders of sizes $|E_i| = n_i$. For each i let $v_i \in E_i$ be an arbitrary vertex. The graph $G = G(\{h_i, n_i\})$ consists of $\mathbb{N} = \{0, 1, \ldots\}$ with edges between consecutive numbers and we attach the expander E_i by adding an edge between v_i and h_i. The following is the main result of this section.

Theorem 4.2. *Consider the graph $G(\{h_i, n_i\})$ for $\{h_i, n_i\}$ satisfying (4.2) and put $t_i = c h_i n_i \log n_i$ for some small constant $c > 0$. We have*

$$\mathbf{P}_0(\tau_0 = t_i \mid \tau_0 \geq t_i) \geq \frac{c \log t_i}{t_i}\,.$$

We begin with some preparatory lemmas and observations leading to the proof of this theorem. In all of the statements below we are considering a simple random walk on $G(\{h_i, n_i\})$ for $\{h_i, n_i\}$ satisfying (4.2). For a vertex v of $G(\{h_i, n_i\})$ we write $h(v)$ for its *height*, that is, if $v \in E_i$, then $h(v) = h_i$ and if $v \in \mathbb{N}$, then $h(v) = v$.

Proposition 4.3. *We have*

1. *For any $h > 0$ we have $\mathbf{P}_0(\tau_h < \tau_0) = h^{-1}$, and*
2. *For any i and v such that $0 \leq h(v) < h_i$ we have $\mathbb{E}_v(\tau_{h_i} \wedge \tau_0) \leq 2h_i^2$.*

Proof. Part (1) is immediate since the effective resistance between vertices 0 and h is precisely h. Part (2) follows immediately by the commute time identity (see [6]). Indeed, the effective resistance between v and $\{0, h\}$ is at most $h/2 + C$ (where C is the constant from (3.2)) and the number of edges in the subgraph *between* 0 and h (that is, the subgraph spanned on all vertices v having $0 \leq h(v) < h$ together with h) is at most

$$h_i + \sum_{j=1}^{i-1} n_j \leq 2h_i\,,$$

by condition (4.2). ■

Proposition 4.4. *There exist a constant $\delta > 0$ such that for any $i \geq 1$ and $k \geq 1$ we have*

$$\mathbf{P}_0(\tau_0 \geq \delta k n_i h_i) \geq \frac{\delta^k}{h_i}\,.$$

Proof. By Lemma 3.1, at each visit to h_i, with uniform positive probability the random walks spends cn steps in the expander E_i. At each visit to h_i the probability of visiting 0 before returning to h_i is h_i^{-1} by Proposition 4.3. We deduce by this that there exists some constant $c > 0$ such that for any $k \geq 1$ we have

$$\mathbf{P}_0(\tau_0 \geq \tau_{h_i} + ckn_i h_i \mid \tau_{h_i} < \tau_0) \geq c^k\,.$$

By Proposition 4.3 the event $\tau_{h_i} < \tau_0$ occurs with probability h_i^{-1}, concluding the proof. ■

For the next step we define $\tau_0^{(m)}$ to be the m-th return time to 0. That is, $\tau_0^{(1)} = \tau_0$ and for $m > 1$

$$\tau_0^{(m)} = \min\{t > \tau_0^{(m-1)} : X_t = 0\}\,.$$

It will also be convenient to define $\tau_0^{(0)} = 0$.

Proposition 4.5. *There exist positive constants C, c such that for any $i \geq 1$ and any $k \geq 1$ we have*

$$\mathbf{P}_0\big(\tau_0^{(Ckh_i)} < kh_i n_i\big) \leq Ce^{-ck}\,.$$

Proof. Since

$$\tau_0^{(Ckh_i)} = \sum_{m=1}^{Ckh_i} (\tau_0^{(m)} - \tau_0^{(m-1)})\,,$$

we learn that $\tau_0^{(Ckh_i)}$ is a sum of Ckh_i i.i.d. random variables distributed as τ_0. By Proposition 4.4, the probability of each of these variables to be at least $n_i h_i$ is at least ch_i^{-1} for some small $c > 0$. Large deviation for binomial random variable immediately gives that for large enough $C > 0$ we have

$$\mathbf{P}\Big(\sum_{m=1}^{Ckh_i} (\tau_0^{(m)} - \tau_0^{(m-1)}) \leq kh_i n_i\Big) \leq Ce^{-c_1 k}\,,$$

for some constant $c_1 > 0$. ■

The following Lemma shows that the random walk on G spends most of its time inside the the appropriate expander.

Lemma 4.6. *Let t be an integer such that $h_i n_i \le t \le h_i^2 n_i$ for some $i \ge 1$. Then*

$$\mathbf{P}_0\big(X_t \in E_i\big) \ge 1 - Ch_i^{-2}.$$

Proof. For convenience we write h for h_i and n for n_i. Write $N_0(t)$ for the number of visits to 0 by time t. By Proposition 4.5 with $k = h$ we deduce that for some constants $C, c > 0$ we have

$$\mathbf{P}(N_0(t) \ge Ch^2) \le Ce^{-ch}.$$

In each such excursion from 0 to 0 the probability of reaching h^5 is h^{-5} by Proposition 4.3. We conclude that the probability the walker does not reach height h^5 before time t is at least $1 - Ch^{-3}$. Denote this event by $\mathscr{A}$.

Let $t_0 = t - h^{12}$ and define iteratively t_{k+1} to be

$$t_{k+1} = \min\{t > t_k : X_t = h\},$$

for any integer $k \ge 1$. For any $k \ge 1$, if $X_{t_k+1} \not\in E_i$, then $X_{t_k+1} \in \{h-1, h+1\}$. The expected time to hit either h or h^5 from any vertex between them is at most h^{10} by Proposition 4.3. Hence, if $X_{t_k+1} = h+1$, then after every $2h^{10}$ steps we hit either h or h^5 with probability at least $\frac{1}{2}$. By iterating, the probability that the walker does not hit h or h^5 in h^{11} steps is at most e^{-ch}. This event occurs if both $t_{k+1} - t_k \ge h^{11}$ and $\mathscr{A}$ occur. Since $\mathbf{P}(\mathscr{A}) \ge 1 - 2h^{-3}$ we have

$$\mathbf{P}_0(t_{k+1} - t_k \ge h^{11} | X_{t_k+1} = h+1) = O(h^{-3}). \tag{4.3}$$

In a similar fashion one gets that

$$\mathbf{P}_0(t_{k+1} - t_k \ge h^{11} | X_{t_k+1} = h-1) = O(h^{-3}). \tag{4.4}$$

On the other hand, if $X_{t_k+1} \in E_i$, then (3.1) gives that

$$\mathbf{P}_0\big(t_{k+1} - t_k \in [h^{11}, h^{12}] \,\big|\, X_{t_k+1} \in E_i\big) \le \frac{2h^{12}}{n} = O(h^{-3}), \tag{4.5}$$

by condition (4.2). Lemma 3.1 yields

$$\mathbf{P}_0(t_{k+1} - t_k \ge h^{12} | X_{t_k+1} \in E_i) > c,$$

for some constant $c = c(\rho) > 0$. Now, let K be the first k such that $t_{k+1} - t_k \ge h^{12}$. The distribution of K is geometric with parameter at least $c/3 > 0$, since $\mathbf{P}_0(X_{t_k+1} \in E_i) = 1/3$. Hence $\mathbf{P}_0(K \ge h) \le e^{-ch}$, for some $c = c(\rho) > 0$. Now, if $X_t \not\in E_i$, then either $K \ge h$, or for some $k \le h$ we have that $t_{k+1} - t_k \in [h^{11}, h^{12}]$, or for some $k \le h$ we have that $t_{k+1} - t_k \ge h^{12}$ and $X_{t_k+1} \not\in E_i$. By the union bound together with (4.3), (4.4) and (4.5) we get that this probability is at most $O(h^{-2})$, concluding our proof. ■

Proof of Theorem 4.2. Fix i and abbreviate $t = t_i$, $h = h_i$ and $n = n_i$. We have that

$$\mathbf{P}_0(\tau_0 \geq t) = \mathbf{P}_0(\tau_0 \geq t \text{ and } \tau_0 < \tau_h) + h^{-1}\mathbf{P}_0(\tau_0 \geq t \mid \tau_h < \tau_0)\,.$$

The first term is negligible since starting from any vertex v between 0 and h, we have $\mathbf{P}_v(\tau_0 \wedge \tau_h \geq 2h^2) \leq \frac{1}{2}$ by Proposition 4.3, and hence, by the Markov property, $\mathbf{P}_0(\tau_0 \wedge \tau_h \geq t) \leq e^{-ct/h^2}$. Theorem 1.1 gives that $\mathbf{P}_0(\tau_0 \geq t) \geq 4^{-1}t^{-1/2}$ and since $t \geq h^3$, we conclude that

$$\mathbf{P}_0(\tau_0 \geq t) = (1+o(1))h^{-1}\mathbf{P}_0(\tau_0 \geq t \mid \tau_h < \tau_0)\,. \tag{4.6}$$

Assuming the event $\tau_h < \tau_0$ occurred, let $T_0 = \tau_h$ and for $j \geq 1$ define

$$T_j = \min\{t > T_{j-1} : X_t = h\}\,,$$

to be the time of the j-th visit to h. Also, let $J = \max\{j : T_j < \tau_0\}$ be the index of the last visit to h before returning to 0. We define a sequence of random bits $\{b_j\}_{j\geq 0}$ in the following way. We set $b_j = 1$ if $X_t = 0$ for some $T_j < t < T_{j+1}$ and $b_j = 0$ otherwise. Conditioned on the history of the walk until T_j the probability of $b_j = 1$ is exactly $(3h)^{-1}$, since the walk need to take a step to $h-1$ and then the probability of hitting 0 before h is h^{-1}, by Proposition 4.3. Hence, the distribution of J is geometric with parameter $(3h)^{-1}$.

Observe that the distribution of the walk between T_J and τ_0 is that of a simple random walk started at h and conditioned to hit 0 before returning to h and is independent of the walk until time T_J. In particular, T_J is independent of $\tau_0 - T_J$. We may now bound $\mathbf{P}_0(\tau_0 = t)$ from below by

$$\mathbf{P}_0(\tau_0 = t) \geq h^{-1}\mathbf{P}_0(T_J = t - (\tau_0 - T_J) \mid \tau_h < \tau_0)\,.$$

Since T_J is independent of $\tau_0 - T_J$ we may condition on the event $\tau_0 - T_J = t - s$ and get that

$$\mathbf{P}_0(T_J = t - (\tau_0 - T_J) \mid \tau_h < \tau_0) = \sum_s \mathbf{P}(\tau_0 - T_J = t - s \mid \tau_h < \tau_0)\mathbf{P}(T_J = s \mid \tau_h < \tau_0)\,. \tag{4.7}$$

When starting a simple random walk at $h-1$, the expected hitting time of h or 0 is at most $2h^2$ by Proposition 4.3. Hence,

$$\mathbf{P}_0(\tau_0 - T_J > 4h^3 \mid \tau_h < \tau_0) \leq \frac{1}{2}\,.$$

Therefore, it is enough to show that for any s satisfying $t-4h^3 \le s \le t$ we have

$$\mathbf{P}_0(T_J = s \mid \tau_h < \tau_0) \ge \frac{c}{hn}\mathbf{P}_0(\tau_0 \ge t \mid \tau_h < \tau_0)\,, \tag{4.8}$$

since then by (4.6) and (4.7) we get that

$$\mathbf{P}_0(\tau_0 = t) \ge \frac{c}{hn}\mathbf{P}_0(\tau_0 \ge t) = \Theta\Big(\frac{\log t}{t}\Big)\mathbf{P}_0(\tau_0 \ge t)\,.$$

To show (4.8) we take some small δ and bound

$$\mathbf{P}_0(T_J = s \mid \tau_h < \tau_0) \ge \mathbf{P}_0(T_J = s, X_{s-\delta n} \in E_i, \tau_0 \ge s \mid \tau_h < \tau_0)\,.$$

By the Markov property the last probability is at least

$$\mathbf{P}_h(\tau_0 < \tau_h) \min_{u \in E_i} \mathbf{P}_u(\tau_h = s)\mathbf{P}_0(X_{s-\delta n} \in E_i, \tau_0 \ge s-\delta n \mid \tau_h < \tau_0)\,. \tag{4.9}$$

Lemma 3.3 and Proposition 4.3 gives that the product of the first two probabilities is at least $c(hn)^{-1}$. By Lemma 4.6 we have that

$$\mathbf{P}_0(X_{s-\delta n} \not\in E_i) \le Ch^{-2}\,,$$

and since $s = (1+o(1))chn\log n$ Proposition 4.4 with $k = c\log n$ gives that

$$\mathbf{P}_0(\tau_0 \ge s-\delta n) \ge \delta^{c\log n} h^{-1} \ge h^{-1.5}\,,$$

as long as $c > 0$ is chosen to be small enough. We conclude that

$$\mathbf{P}_0(X_{s-\delta n} \in E_i, \tau_0 \ge s-\delta n \mid \tau_h < \tau_0) \ge c\mathbf{P}_0(\tau_0 \ge s-\delta n \mid \tau_h < \tau_0) \ge c\mathbf{P}_0(\tau_0 \ge t)\,,$$

which together with (4.9) shows (4.8) and the proof is concluded. ∎

5. Combs

Recall the definition of the comb product of two graphs and of the finite collision property in Section 1.2. In this section we prove that the graph $G = G(\{h_i, n_i\})$, for $\{h_i, n_i\}$ satisfying (4.2), is such that $\mathrm{Comb}_0(\mathbb{Z}, G)$ does not have the finite collision property. We begin with a sketch to illustrate the idea of the proof.

Let $i \ge 1$ and write h, n for h_i, n_i respectively. Our goal is to get the two walkers inside the same expander E_i since then they collide with positive probability by Lemma 3.4. Starting from the base of the comb, the probability of reaching height h before returning to the base is h^{-1}. If this happens, the random walk has positive probability of being "swallowed" in the expander E_i and stay in it for n steps. At each visit to the tip of the expander, that is, the vertex h, the probability of getting back to the base of the comb is h^{-1}. The other expanders, above and below E_i are either too small or too far away to matter. We deduce that by time hn the typical behavior of the walker is to walk about h steps on the base of the comb, then rise to height h, have about h excursions of length n inside the expander and finally return to the base of the comb.

Thus, after $h^2 n$ steps, each random walker has performed about h^2 steps on the base of the comb (this is a simple random walk on $\mathbb{Z}$) and in about h of them it performs excursions of length hn in which it spends most of the time in the expander E_i. The base points on $\mathbb{Z}$ of these h long excursions are roughly h uniform points in $\{-h,\dots,h\}$, so the probability that in at least one of them the two walkers are in the same base point is uniformly positive. We conclude that by time $h^2 n$ the two walkers have positive probability of colliding. This occurs in all scales, that is, for all $i \geq 0$. Each scale has no influence on what occurs in the next scale hence we get the required result.

We now make this heuristic precise. Given a simple random walk X_t on $\mathrm{Comb}(\mathbb{Z}, G)$ we write $X_t^{(1)}$ and $X_t^{(2)}$ for its first and second coordinates respectively. Note that $X_t^{(1)}$ is a time change of a simple random walk on $\mathbb{Z}$ and $X_t^{(2)}$ is distributed precisely as simple random walk on $G(\{h_i, n_i\})$ equipped with extra two loops at 0. One can easily check that the estimates of Section 4 are valid for this graph as well. Put $T_0 = 0$ and $T_i = T_{i-1} + n_i h_i^2$. For any $i \geq 1$ and $k = 1, \dots, h_i$ define the random variables

$$I_k = \{X^{(1)}_{T_{i-1}+kh_i n_i} = Y^{(1)}_{T_{i-1}+kh_i n_i} \text{ and } X^{(2)}_{T_{i-1}+kh_i n_i} \in E_i \text{ and } Y^{(2)}_{T_{i-1}+kh_i n_i} \in E_i\}.$$

The following lemma is the key step for proving Theorem 1.3.

Lemma 5.1. *There exists a constant $c > 0$ such that for all $i \geq 1$ we have*

$$\mathbf{P}\Big(\bigcup_{k=1}^{h_i} I_k \mid X_{T_{i-1}}, Y_{T_{i-1}}\Big) \geq c.$$

Proof of Theorem 1.3. Write $\mathscr{A}_i$ for the event that X_t and Y_t collide in the time interval $[T_{i-1}, T_i]$. Lemma 5.1 together with Lemma 3.4 shows that

$$\mathbf{P}(\mathscr{A}_i \mid X_{T_{i-1}}, Y_{T_{i-1}}) \geq c,$$

for some constant $c > 0$. We deduce that $\mathscr{A}_i$ occurs infinitely many times with probability 1, concluding the proof. ■

We will prove Lemma 5.1 using a second moment argument, however, we require two additional preparatory lemmas about the random walk on a single copy of $G(\{h_i, n_i\})$, the base of the comb.

Lemma 5.2. *Let X_t be a simple random walk on G and let $h = h_i$. Then there exists positive constants C, c, independent of i, such that for any vertex v satisyfing* $0 \le h(v) \le h^4$ *we have*

$$\mathbb{E}_v\left[e^{cn^{-1}(\tau_0 \wedge \tau_h \wedge \tau_{h^4})}\right] \le C.$$

Proof. For any vertex v of G between 0 and h^4 we have that $\mathbb{E}_v(\tau_0 \wedge \tau_h \wedge \tau_{h^4}) \le Cn$. To see this observe that there are three cases: if v is in the expander E_h the expected hitting time of h is of order $O(n)$ by the commute time identity and (3.2). If $h(v) > h$, then the expected hitting time at h or h^4, by the commute time identity, is at most Ch^8, which is $o(n)$ (there are no expanders between h and h^4 by (4.2)). Similarly, if $h(v) < h$ then the expected hitting time at 0 or h is $o(n)$. Hence for any such v we have

$$\mathbf{P}_v\big(\tau_0 \wedge \tau_h \wedge \tau_{h^4} \ge 2Cn\big) \le \frac{1}{2},$$

hence

$$\mathbf{P}_v\big(\tau_0 \wedge \tau_h \wedge \tau_{h^4} \ge Bn\big) \le e^{-cB},$$

and the (ii) follows by integration. ■

Lemma 5.3. *Let X_t be a simple random walk on G and let $h = h_i$ and $n = n_i$. There exists constants $C, c > 0$, independent of i, such that*

$$\mathbf{P}_0(\tau_0 \wedge \tau_{h^4} \ge Bnh) \le \frac{2e^{-cB}}{h},$$

hence

$$\mathbb{E}_0 e^{c(nh)^{-1}(\tau_0 \wedge \tau_{h^4})} \le 1 + \frac{C}{h}.$$

Proof. Let N_h denote the number of visits to h before time $\tau_0 \wedge \tau_{h^4}$. We have that

$$\mathbf{P}_0(N_h \ge k) \le \frac{1}{h}\Big(1 - \frac{1}{3h}\Big)^{k-1} \le \frac{e^{-(k-1)/3h}}{h},$$

since reaching to h before 0 has probability h^{-1}, and given that, at each visit to h the probability of visiting 0 before returning to h is precisely $(3h)^{-1}$. By this bound it suffices to prove that

$$\mathbf{P}_0\big(\tau_0 \wedge \tau_{h^4} \ge Bnh \text{ and } N_h \le cBh\big) \le \frac{e^{-cB}}{h}, \tag{5.1}$$

for some small $c > 0$. Let γ_m for $m = 1, \dots, cBh$ be i.i.d. random variables distributed as the stopping time $\tau_0 \wedge \tau_h \wedge \tau_{h^4}$ for the random walk starting at h. Then the event on the left hand side of (5.1) implies that

$$\sum_{m=1}^{cBh} \gamma_m \geq Bnh .$$

By Lemma 5.2 we have that there exists some $c_2 > 0$ such that $\mathbb{E}_0 e^{c_2 n^{-1} \gamma_m} \leq C$. Hence, by independence and Markov's inequality we get that

$$\mathbf{P}_0\Big(\sum_{m=1}^{cBh} \gamma_m \geq Bnh \Big) \leq \frac{C^{cBh}}{e^{c_2 Bh}} ,$$

which is of order e^{-cBh} if $c = c(c_2, C) > 0$ is chosen small enough compared with c_2. This proves (a stronger assertion than) (5.1) and concludes the proof. ∎

Consider now the random walk X_t on Comb$(\mathbb{Z}, G)$ and fix some $i \geq 1$. Write $X_t^{(2)}$ for the second coordinate of X_t and let $\ell(t)$ denote the random variable

$$\ell(t) = \Big| \big\{ j \in [T_{i-1}, T_{i-1} + t] : X_{j-1}^{(2)} = X_j^{(2)} = 0 \big\} \Big| .$$

In other words, $\ell(t)$ counts the number of times $j \in [T_{i-1}, T_{i-1} + t]$ in which X_j walked on the $\mathbb{Z}$ base of the comb.

Lemma 5.4. *Let X_t be a simple random walk on G and let $h = h_i$ and $n = n_i$. There exists constants $C, c > 0$, independent of i, such that for any $k = 1, \ldots, h$ we have*

$$\mathbf{P}\big(\ell(khn) \geq Ckh \big) \leq Ce^{-ck} ,$$

and

$$\mathbf{P}\big(\ell(khn) \leq C^{-1} kh \big) \leq \frac{1}{h^2} + Ce^{-ck} .$$

Proof. Part (i) of the lemma is equivalent to Proposition 4.5. For $m \geq 1$ write t_m for the time in which X_t takes the m-th step on $\mathbb{Z}$. That is, $t_0 = 0$ and for $m \geq 1$ we have

$$t_m = \min \big\{ j > t_{m-1} : X_{j-1}^{(2)} = X_j^{(2)} = 0 \big\} .$$

To prove the second assertion of the lemma, note that the event $\ell(khn) \leq C^{-1} kh$ is equivalent to

$$\sum_{m=1}^{C^{-1}kh} (t_m - t_{m-1}) \geq khn . \tag{5.2}$$

For each m let A_m be the event that X_t visited h^4 between times t_{m-1} and t_m and write $\bar{A}_m$ for the complement of A_m. By Proposition 4.3 we have that $\mathbf{P}(A_m) = h^{-4}$. Thus, the probability that A_m occurs for some $m = 1, \dots, C^{-1}kh$ is at most h^{-2} since $k \le h$. We get that

$$\mathbf{P}\Big(\sum_{m=1}^{C^{-1}kh} (t_m - t_{m-1}) \ge khn\Big) \le \frac{1}{h^2} + \mathbf{P}\Big(\sum_{m=1}^{C^{-1}kh} (t_m - t_{m-1}) \mathbf{1}_{\{\bar{A}_m\}} \ge khn\Big).$$

To bound the last term of this inequality observe that

$$(t_m - t_{m-1}) \mathbf{1}_{\{\bar{A}_m\}} \overset{(d)}{\le} \tau_0 \wedge \tau_{h^4}.$$

By Lemma 5.3 there exists some $C_2 > 0$ such that

$$\mathbb{E} e^{c(nh)^{-1}(t_m - t_{m-1})\mathbf{1}_{\{\bar{A}_m\}}} \le 1 + \frac{C_2}{h},$$

and by independence and Markov's inequality we deduce that

$$\mathbf{P}\Big(\sum_{m=1}^{C^{-1}kh} (t_m - t_{m-1}) \mathbf{1}_{\{\bar{I}_m\}} \ge khn\Big) \le \frac{\big(1 + \frac{C_2}{h}\big)^{C^{-1}kh}}{e^{ck}},$$

which is at most Ce^{-ck} if $C = C(c, C_2) > 0$ is chosen large enough. This concludes the proof. ∎

Lemma 5.5. *For any $k = 1, \dots, h$*

$$\mathbf{P}(I_k | X_{T_{i-1}}, Y_{T_{i-1}}) \approx \frac{1}{\sqrt{kh}}.$$

Proof. Lemma 5.4 implies that for some positive constants C, c we have

$$\mathbf{P}\big(C^{-1}kh \le \ell(khn) \le Ckh\big) \ge 1 - Ce^{-ch} - Ch^{-2}.$$

So with this probability, this holds for both walks X_t and Y_t. Clearly $X^{(1)}_{T_{i-1}}$ and $Y^{(1)}_{T_{i-1}}$ are of distance at most T_{i-1} away from the origin Comb$(\mathbb{Z}, G)$, and $T_{i-1} \ll \sqrt{h}$ by (4.2). Thus, the local CLT for the simple random walk on $\mathbb{Z}$ implies that the probability that at time $T_{i-1} + khn$ the two walkers are in the same copy of G is at least $c(kh)^{-1/2}$ and at most $C(kh)^{-1/2}$. This shows $\mathbf{P}(I_k) \le C(kh)^{-1/2}$. Furthermore, by Lemma 4.6 the probability that at that time the walks are not inside the expander E_h is at most h^{-2}. The lower bound $\mathbf{P}(I_k) \ge c(kh)^{-1/2}$ follows. ∎

Lemma 5.6. *For any $k_1 < k_2$ in $\{1, \dots, h\}$ we have that*

$$\mathbf{P}(I_{k_1} I_{k_2} \mid X_{T_{i-1}}, Y_{T_{i-1}}) \le \frac{1}{h\sqrt{k_1(k_2 - k_1)}}.$$

Proof. By Lemma 5.5 we have that $\mathbf{P}(I_{k_1}) \approx (k_1 h)^{-1/2}$. Conditioned on I_{k-1} we have that at time $T_{i-1} + k_1 hn$ the two random walks are in the same expander E_i, and in particular in the same copy of G. Another application of Lemma 5.5 then gives that

$$\mathbf{P}(I_{k_2} \mid I_{k_1}) \leq \frac{C}{\sqrt{h(k_2 - k_1)}},$$

concluding the proof. ∎

Proof of Lemma 5.1. Lemma 5.5 gives that

$$\sum_{k=1}^{h} \mathbf{P}(I_k) \geq c,$$

and Lemma 5.6 yields that

$$\sum_{k_1=1}^{h} \sum_{k_2=1}^{h} \mathbf{P}(I_{k_1} I_{k_2}) \leq C.$$

The lemma follows by the inequality $\mathbf{P}(X > 0) \geq (\mathbb{E}X)^2 / \mathbb{E}X^2$ valid for any non-negative random variable X. ∎

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