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RECURRENCE OF RANDOM WALK TRACES[1]

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We show that the edges crossed by a random walk in a network form a recurrent graph a.s. In fact, the same is true when those edges are weighted by the number of crossings.

1. Introduction. Let $G = (\mathsf{V}, \mathsf{E})$ be a locally finite graph and let $c\colon \mathsf{E} \to [0, \infty)$ be an assignment of *conductances* to the edges. We call (G, c) a *network*. The associated random walk has transition probabilities $p(x, y) := c(x, y)/\pi(x)$, where $\pi(x) := \sum_y c(x, y)$. Assume that the network random walk is transient when it starts from some fixed vertex o. How big can the trace be, the set of edges traversed by the random walk? We show that they form a.s. a recurrent graph (for a simple random walk).

This fact is already known when G is a Euclidean lattice and $c \equiv 1$ since a.s. the paths there have infinitely many cut-times, a time when the past of the path is disjoint from its future; see [7] and [8]. From this, recurrence follows by the criterion of Nash-Williams [12]. By contrast, Lyons and Peres [9] constructed an example of a transient birth-and-death chain which a.s. has only finitely many cut-times.

A result of similar spirit to ours was proved by Morris [11], who showed that the components of the wired uniform spanning forest are a.s. recurrent. For another a.s. recurrence theorem (for distributional limits of finite planar graphs), see [5].

We expect that a Brownian analogue of the theorem is true, that is, a.s. parabolicity of the Wiener sausage, with reflected boundary conditions. For background on recurrence in the Riemannian context, see, for example, [6]. It would be interesting to prove similar theorems for other processes. For example, consider the trace of a branching random walk on a graph G.

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Then we conjecture that almost surely the trace is recurrent for a branching random walk with the same branching law. Perhaps a similar result holds for general tree indexed random walks. See Benjamini and Peres [3, 4] for definitions and background.

Perhaps one can strengthen our result as follows. Given a transient network (G,c), denote by T_n the trace of the first n steps of the network random walk. Let $R(n)$ be the maximal effective resistance on T_n between o and another vertex of T_n, where each edge has unit conductance. By our theorem, $R(n)\uparrow\infty$ a.s. [Note, of course, that $R(n)\uparrow\infty$ for growing subgraphs does not imply recurrence of their union, as balls in the binary tree show.] Is there a uniform lower bound over all transient networks for the rate at which $R(n)\uparrow\infty$? That is, does there exist a function f with $\lim_n f(n)=\infty$ such that for any transient network,

$$\limsup_n R(n)/f(n)>0 \qquad \text{a.s.?}$$

In particular, one can speculate that $f(n)=\log^2 n$ might work, which would arise from the graph $\mathbb{Z}^2$ (although $\mathbb{Z}^2$ is recurrent, it is on the border of transience). On the other hand, transient wedges in $\mathbb{Z}^3$ might allow one to prove that there is no such f.

2. Proof. Our proof will demonstrate the following stronger results.

Let $N(x,y)$ denote the number of traversals of the edge (x,y).

THEOREM 2.1. *The network $(G,\mathbf{E}[N])$ is recurrent. The networks (G,N) and $(G,\mathbf{1}_{\{N>0\}})$ are a.s. recurrent.*

We shall use some facts relating electrical networks to random walks. See [10] for more background.

Let $\mathcal{G}(x,y)$ be the Green function, that is, the expected number of visits to y for a network random walk started at x.

The *effective resistance* from a vertex o to infinity is defined to be the minimum energy $\frac{1}{2}\sum_{x\neq y}\theta(x,y)^2/c(x,y)$ of any unit flow θ from o to infinity. This also equals

$$\alpha:=\mathcal{G}(o,o)/\pi(o). \tag{2.1}$$

In particular, the effective resistance is finite iff the network random walk is transient. Its reciprocal, *effective conductance*, is given by Dirichlet's principle as the infimum of the Dirichlet energy $\frac{1}{2}\sum_{x\neq y}c(x,y)[F(x)-F(y)]^2$ over all functions $F:\mathsf{V}\to[0,1]$ that have finite support and satisfy $F(o)=1$. Since the functional $c\mapsto\sum_{x\neq y}c(x,y)[F(x)-F(y)]^2$ is linear for any given F, we see that effective conductance is concave in c. Thus, if the conductances $\langle\mathbf{E}[N(x,y)];\,(x,y)\in\mathsf{E}(G)\rangle$ give a recurrent network, then so a.s. do $\langle N(x,y);$

$(x, y) \in \mathsf{E}(G)\rangle$. Furthermore, Rayleigh's monotonicity principle implies that if (G, N) is recurrent, then so is $(G, \mathbf{1}_{\{N>0\}})$. (Of course, it follows that any finite union of traces, whether independent or not, is also recurrent a.s.)

Thus, it remains to prove that $(G, \mathbf{E}[N])$ is recurrent. We shall, however, also show how the proof that (G, N) is a.s. recurrent follows from a simpler argument. Another mostly probabilistic proof of this is due to Benjamini and Gurel-Gurevich [2].

The *effective resistance* from a finite set of vertices A to infinity is defined to be the effective resistance from a to infinity when A is identified to a single vertex, a. The *effective resistance* from an infinite set of vertices A to infinity is defined to be the infimum of the effective resistance from B to infinity among all finite subsets $B \subset A$. Its reciprocal, the *effective conductance* from A to infinity in the network (G, c), will be denoted by $\mathcal{C}(A, G, c)$. From the above, we have

$$\mathcal{C}(A, G, c) = \sup_B \inf \left\{ \frac{1}{2} \sum_{x \neq y} c(x, y)[F(x) - F(y)]^2; \right. \\ \left. F \restriction B \equiv 1,\ F \text{ has finite support} \right\}, \tag{2.2}$$

where the supremum is over finite subsets B of A.

Let the original voltage function be $v(\bullet)$ throughout this article, where $v(o) = 1$ and $v(\bullet)$ is 0 "at infinity." Then $v(x)$ is the probability of ever visiting o for a random walk starting at x.

Note that

$$\begin{aligned} \mathbf{E}[N(x, y)] &= \mathcal{G}(o, x)p(x, y) + \mathcal{G}(o, y)p(y, x) \\ &= (\mathcal{G}(o, x)/\pi(x) + \mathcal{G}(o, y)/\pi(y))c(x, y) \end{aligned}$$

and

$$\pi(o)\mathcal{G}(o, x) = \pi(x)\mathcal{G}(x, o) = \pi(x)v(x)\mathcal{G}(o, o).$$

Thus, we have [from the definition (2.1)]

$$\begin{aligned} \mathbf{E}[N(x, y)] &= \alpha c(x, y)[v(x) + v(y)] \\ &\leq 2\alpha \max\{v(x), v(y)\}c(x, y) \leq 2\alpha c(x, y). \end{aligned} \tag{2.3}$$

In a finite network (H, c), we write $\mathcal{C}(A, z; H, c)$ for the effective conductance between a subset A of its vertices and a vertex z. This is given by Dirichlet's principle as the infimum of the Dirichlet energy of F over all functions $F : \mathsf{V}(H) \to [0, 1]$ that satisfy $F \restriction A \equiv 1$ and $F(z) = 0$. Clearly, $A \subset B \subset \mathsf{V}$ implies that $\mathcal{C}(A, z; H, c) \leq \mathcal{C}(B, z; H, c)$. The function that minimizes the Dirichlet energy is the voltage function, v. The amount of current that flows from A to z in this case is defined as $\sum_{x\in A, y\notin A}[v(x)-v(y)]c(x,y)$; it equals $\mathcal{C}(A,z;H,c)$. The voltage function that is t on A instead of 1 has Dirichlet energy equal to $t^2\mathcal{C}(A,z;H,c)$ and gives a current that is t times as large as the unit-voltage current, which shows that $\mathcal{C}(A,z;H,c)$ is the amount of current that flows from A to z divided by the voltage on A.

LEMMA 2.1. *Let (H,c) be a finite network and $a,z\in \mathsf{V}(H)$. Let v be the voltage function that is 1 at a and 0 at z. For $0<t<1$, let A_t be the set of vertices x with $v(x)\geq t$. Then $\mathcal{C}(A_t,z;H,c)\leq \mathcal{C}(a,z;H,c)/t$. Thus, for every $A\subset \mathsf{V}(H)\setminus\{z\}$, we have*

$$\mathcal{C}(A,z;H,c)\leq \frac{\mathcal{C}(a,z;H,c)}{\min v\restriction A}.$$

PROOF. We subdivide edges as follows. If any edge (x,y) is such that $v(x)>t$ and $v(y)<t$, then subdividing the edge (x,y) with a vertex z by giving resistances

$$r(x,z):=\frac{v(x)-t}{v(x)-v(y)}r(x,y)$$

and

$$r(z,y):=\frac{t-v(y)}{v(x)-v(y)}r(x,y)$$

will result in a network such that $v(z)=t$ while no other voltages change. Doing this for all such edges gives a possibly new graph H' and a new set A'_t whose internal vertex boundary is a set W'_t on which the voltage is identically equal to t. We have $\mathcal{C}(A_t,z;H,c)=\mathcal{C}(A_t,z;H',c)\leq \mathcal{C}(A'_t,z;H',c)$. Now $\mathcal{C}(A'_t,z;H',c)=\mathcal{C}(a,z;H,c)/t$ since the amount of current that flows is $\mathcal{C}(a,z;H,c)$ and the voltage difference is t. Therefore, $\mathcal{C}(A_t,z;H,c)\leq \mathcal{C}(a,z;H,c)/t$, as desired.

For a general A, let $t:=\min v\restriction A$. Since $A\subset A_t$, we have $\mathcal{C}(A,z;H,c)\leq \mathcal{C}(A_t,z;H,c)$. Combined with the inequality just reached, this yields the final conclusion. $\square$

For $t\in(0,1)$, let $\mathsf{V}_t:=\{x\in\mathsf{V}; v(x)<t\}$. Let W_t be the external vertex boundary of V_t, that is, the set of vertices outside V_t that have a neighbor in V_t. Write G_t for the subgraph of G induced by $\mathsf{V}_t\cup W_t$.

We will refer to the conductances c as the *original* ones and the conductances $\mathbf{E}[N]$ as the *new* ones for convenience.

LEMMA 2.2. *The effective conductance from W_t to ∞ in the network $(G_t,\mathbf{E}[N])$ is at most 2.*

Proof. If any edge (x,y) is such that $v(x) > t$ and $v(y) < t$, then subdividing the edge (x,y) with a vertex z as in the proof of Lemma 2.1 and consequently adding z to W_t has the effect of raising the conductance of the edge (x,y) to $c(z,y) = c(x,y)[v(x) - v(y)]/[t - v(y)]$ and also, by (2.3), of raising its conductance in the new network from $\mathbf{E}[N(x,y)]$ to

$$\begin{aligned} \alpha c(z,y)[t + v(y)] &= \alpha c(z,y)[t - v(y) + 2v(y)] \\ &= \alpha c(x,y)[v(x) - v(y)] + 2\alpha c(z,y)v(y) \\ &> \alpha c(x,y)[v(x) - v(y)] + 2\alpha c(x,y)v(y) = \mathbf{E}[N(x,y)]. \end{aligned}$$

Since raising edge conductances clearly raises effective conductance, it suffices to prove the lemma in the case that $v(x) = t$ for all $x \in W_t$. Thus, we assume this case for the remainder of the proof.

Suppose that $\langle (H_n, c); n \geq 1 \rangle$ is an increasing exhaustion of (G, c) by finite networks that include o. Identify the boundary (in G) of H_n to a single vertex, z_n. Let v_n be the corresponding voltage functions with $v_n(o) = 1$ and $v_n(z_n) = 0$. Then $\mathcal{C}(o, z_n; H_n, c) \downarrow 1/\alpha$ and $v_n(x) \uparrow v(x)$ as $n \to \infty$ for all $x \in \mathsf{V}(G)$. Let A be a finite subset of W_t. By Lemma 2.1, as soon as $A \subset \mathsf{V}(H_n)$, we have that the effective conductance from A to z_n of H_n is at most $\mathcal{C}(o, z_n; H_n, c)/\min\{v_n(x); x \in A\}$. Therefore by Rayleigh's monotonicity principle, $\mathcal{C}(A, G_t, c) \leq \mathcal{C}(A, G, c) = \lim_{n\to\infty} \mathcal{C}(A, z_n; H_n, c) \leq 1/(\alpha t)$. Since this holds for all such A, we have

$$\mathcal{C}(W_t, G_t, c) \leq 1/(\alpha t). \tag{2.4}$$

By (2.3), the new conductances on G_t are obtained by multiplying the original conductances by factors that are at most $2\alpha t$. Combining this with (2.4), we obtain that the new effective conductance from W_t to infinity in G_t is at most 2. $\square$

When the complement of V_t is finite for all t, which is the case for "most" networks, this completes the proof by the following lemma (and by the fact that $\bigcap_{t>0} \mathsf{V}_t = \varnothing$):

Lemma 2.3. *If H is a transient network, then for all $m > 0$, there exists a finite subset $K \subset \mathsf{V}(H)$ such that for all finite $K' \supseteq K$, the effective conductance from K' to infinity is more than m.*

Proof. Let θ be a unit flow of finite energy from a vertex o to ∞. Since θ has finite energy, there is some $K \subset \mathsf{V}(G)$ such that the energy of θ on the edges with some endpoint not in K is less than $1/m$. That is, the effective resistance from K to infinity is less than $1/m$. $\square$

Even when the complement of V_t is not finite for all t, this is enough to show that the network (G, N) is a.s. recurrent: If X_n denotes the position of the random walk on (G,c) at time n, then $v(X_n) \to 0$ a.s. by Lévy's 0–1 law. Thus, the path is a.s. contained in V_t after some time, no matter the value of $t > 0$. By Lemma 2.3, if (G,N) is transient with probability $p > 0$, then $\mathcal{C}(B_n, G, N)$ tends in probability, as $n \to \infty$, to a random variable that is infinite with probability p, where B_n is the ball of radius n about o. In particular, this effective conductance is at least $6/p$ with probability at least $p/2$ for all large n. Fix n with this property. Let $t > 0$ be such that $\mathsf{V}_t \cap B_n = \varnothing$. Write D for the (finite) set of vertices in G incident to an edge $e \notin G_t$ with $N(e) > 0$. Then $\mathcal{C}(W_t, G_t, N) = \mathcal{C}(W_t \cup D, G, N) \geq \mathcal{C}(B_n, G, N)$. However, this implies that $\mathcal{C}(W_t, G_t, \mathbf{E}[N]) \geq \mathbf{E}[\mathcal{C}(W_t, G_t, N)] \geq 3$, which contradicts Lemma 2.2.

To complete the proof that $(G, \mathbf{E}[N])$ is recurrent in general, we show that although V_t may not separate the source o from infinity, its complement in the network is recurrent:

LEMMA 2.4. *The vertices $\mathsf{V} \setminus \mathsf{V}_t$ induce a recurrent network for the original and for the new conductances.*

PROOF. Condition that the original random walk on G returns to its starting point, o. Of course, the corresponding Doob-transformed Markov chain is recurrent. This corresponds to transformed transition probabilities $p(x,y)v(y)/v(x)$ for $x \neq o$, whence to transformed conductances $c'(x,y) := c(x,y)v(x)v(y)$. Rayleigh's monotonicity principle gives that when we delete V_t, we still have a recurrent network. But off of V_t, the conductances c' differ by a bounded factor from the original conductances and also from the new conductances. This means that the part remaining after we delete V_t is recurrent for both the original and new conductances. $\square$

PROOF OF THEOREM 2.1. The function $x \mapsto v(x)$ has finite Dirichlet energy for the original network, hence for the new (since conductances are multiplied by a bounded factor). Assume (for a contradiction) that the new random walk is transient. Then by Ancona, Lyons and Peres [1], $\langle v(X_n) \rangle$ converges a.s. for the new random walk. By Lemma 2.4, it a.s. cannot have a limit $> t$ for any $t > 0$, so it converges to 0 a.s.

This means that the unit current flow i for the new network (which is the expected number of signed crossings of edges) has total flow 1 through W_t into G_t for all $t > 0$. Thus, we may choose a finite subset A_t of W_t through which at least $1/2$ of the new current enters. With the notation $(d_t^* i)(x) := \sum_{y \in \mathsf{V}(G_t)} i(x,y)$, this means that $\sum_{x \in A_t} d_t^* i(x) \geq 1/2$. By Lemma 2.2, there is a function $F_t : \mathsf{V}_t \cup W_t \to [0,1]$ with finite support and with $F_t \equiv 1$ on A_t whose Dirichlet energy on the network $(G_t, \mathbf{E}[N])$ is at most 3. Write

$(dF_t)(x,y) := F_t(x) - F_t(y)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left[\sum_{x\neq y\in \mathsf{V}(G_t)} i(x,y)\, dF_t(x,y)\right]^2 &\leq \sum_{x\neq y\in \mathsf{V}(G_t)} i(x,y)^2/c(x,y) \\ &\qquad \times \sum_{x\neq y\in \mathsf{V}(G_t)} c(x,y)\, dF_t(x,y)^2 \\ &\leq 3 \sum_{x\neq y\in \mathsf{V}(G_t)} i(x,y)^2/c(x,y).\end{aligned}$$

On the other hand, summation by parts yields that

$$\sum_{x\neq y\in \mathsf{V}(G_t)} i(x,y)\, dF_t(x,y) = \sum_{x\in \mathsf{V}(G_t)} d_t^* i(x) F_t(x) \geq \sum_{x\in A_t} d_t^* i(x) \geq 1/2.$$

Therefore, $\sum_{x\neq y\in \mathsf{V}(G_t)} i(x,y)^2/c(x,y) \geq 1/12$, which contradicts $\bigcap_t \mathsf{V}(G_t) = \varnothing$ and the fact that i has finite energy. $\square$

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