The Biham-Middleton-Levine traffic model for a single junction

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September 1, 2007

Abstract

In the Biham-Middleton-Levine traffic model cars are placed in some density p on a two dimensional torus, and move according to a (simple) set of predefined rules. Computer simulations show this system exhibits many interesting phenomena: for low densities the system self organizes such that cars flow freely while for densities higher than some critical density the system gets stuck in an endless traffic jam. However, apart from the simulation results very few properties of the system were proven rigorously to date. We introduce a simplified version of this model in which cars are placed in a single row and column (a junction) and show that similar phenomena of self-organization of the system and phase transition still occur.

1 The BML traffic model

The Biham-Middleton-Levine (BML) traffic models was first introduced in [3] published 1992. The model involves two types of cars: "red" and "blue". Initially the cars are placed in random with a given density p on the $N \times N$ torus. The system dynamics are as follows: at each turn, first all the red cars try to move simultaneously a single step to the right in the torus. Afterwards all blue cars try to move a single step upwards. A car succeeds in moving as long as the relevant space above/beside it (according to whether it is blue/red) is vacant.

The basic properties of this model are described in [3] and some recent more subtle observations due to larger simulations are described in [4]. The main and most interesting property of the system originally observed in simulations is a *phase transition*: for some critical density p_c one observes, that while filling the torus randomly with cars in density $p < p_c$ the system self organizes such that after some time all cars flow freely with no car ever running into another car (see figure 1), by slightly changing the density to some $p > p_c$ not only does system not reach a free flow, but it will eventually get stuck in a configuration in which no car can ever move (see figure 2).

Very little of the above behaviour is rigorously proven for the BML model. The main rigorous result is that of Angel, Holroyd and Martin [2], showing that for some

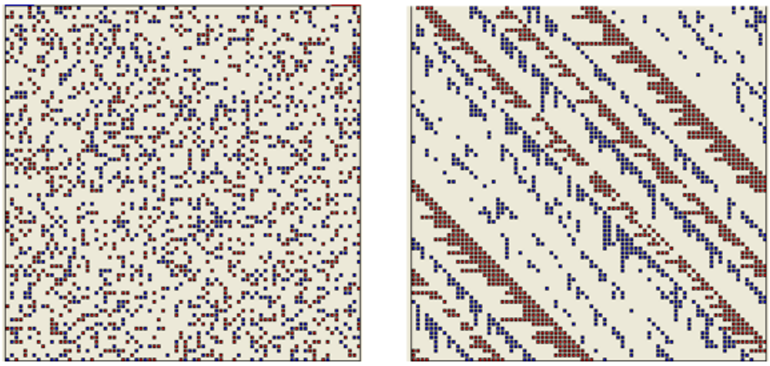

Figure 1: Self organization for $p = 0.3$: the initial configuration in the left organizes after 520 steps to the "diagonal" configuration on the right which flows freely

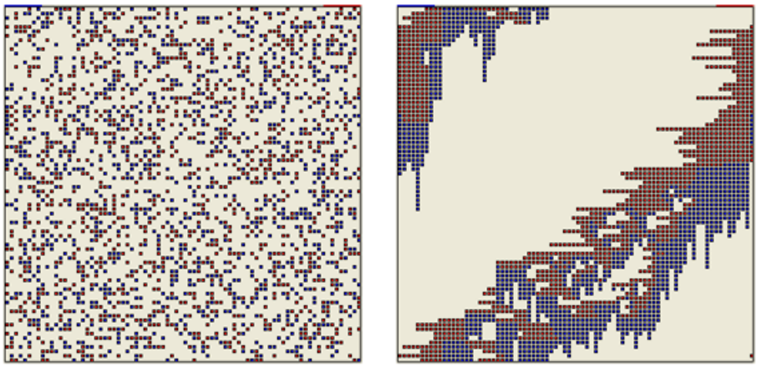

Figure 2: Traffic jam for $p = 0.35$: the initial configuration on the left gets to the stuck configuration on the right after 400 steps, in which no car can ever move

fixed density $p < 1$, very close to 1, the probability of the system getting stuck tends to 1 as $N \to \infty$. In [1] one can find a study of the very low density regime (when $p = O(\frac{1}{N})$).

First, we introduce a slight variant of the original BML model by allowing a car (say, red) to move not only if there is a vacant place right next to it but also if there is a red car next to it that moves. Thus, for sequence of red cars placed in a row with a single vacant place to its right - all cars will move together (as oppose to only the rightmost car in the sequence for the original BML model). Not only does this new variant exhibits the same phenomena of self-organization and phase transition, they even seem to appear more quickly (i.e. it takes less time for the system to reach a stable state). Actually the demonstrated simulations (figures 1, 2) were performed using the variant model. Note that the results of [2] appear to apply equally well to the variant model.

In the following, we will analyze a simplified version of the BML model: BML on a single junction. Meaning, we place red cars in some density p on a single row of the torus, and blue cars are placed in density p on a single column. We will show that for all p the system reaches *optimal speed*[1], depending only on p. For $p < 0.5$ we will show the system reaches speed 1, while for $p > 0.5$ the speed cannot be 1, but the system will reach the same speed, regardless of the initial configuration. Moreover, at $p = 0.5$ the system's behaviour undergoes a phase transition: we will prove that while for $p < 0.5$ the stable configuration will have linearly many sequences of cars, for $p > 0.5$ we will have only $O(1)$ different sequences after some time. We will also examine what happens at a small window around $p = 0.5$.

Note that in the variant BML model (and unlike the original BML model) car sequences are never split. Therefore, the simplified version of the variant BML model can be viewed as some kind of 1-dimensional coalescent process.

Much of the proofs below rely on the fact that we model BML on a symmetric torus. Indeed, the time-normalization of section 2.2 would take an entirely different form if the height and width were not equal. We suspect that the model would exhibit similar properties if the height and width had a large common denominator, e.g. if they had a fixed proportion. More importantly, we believe that for height and width with low common denominator (e.g. relatively prime), we would see a clearly different behaviour. As a simple example, note that in the relatively prime case, a speed of precisely 1 cannot be attained, no matter how low is p, in contrast with corollary 3.14. This dependence on the arithmetic properties of the dimensions is also apparent in [4].

2 The Junction model

2.1 Basic model

We start with the exact definition of our simplified model. On a cross shape, containing a single horizontal segment and a single vertical segment, both of identical length N, red cars are placed in exactly pN randomly (and uniformly) chosen locations along the row, and blue cars are similarly placed in pN locations along the column. p will be called the density of the configuration. For simplicity and consistency with the general BML model, we refer to the cars placed on the horizontal segment as red and those on the vertical segment as blue. For simplicity we may assume that the *junction*, i.e. the single location in the intersection of the segments, is left unoccupied. The segments are cyclic - e.g. a red car located at the rightmost position $(N-1)$ that moves one step to the right re-emerges at the leftmost position (0).

At each turn, all the red cars move one step to the right, except that the red car that is just left of the junction will not move if a blue car is in the junction (i.e. blocking it), in which case also all the red cars immediately following it will stay still. Afterwards, the blue cars move similarly, with the red and blue roles switched. As in the original BML, we look at the asymptotic speed of the system, i.e. the

[1] The speed of the system is the asymptotic average rate in which a car moves - i.e. number of actual moves per turn.

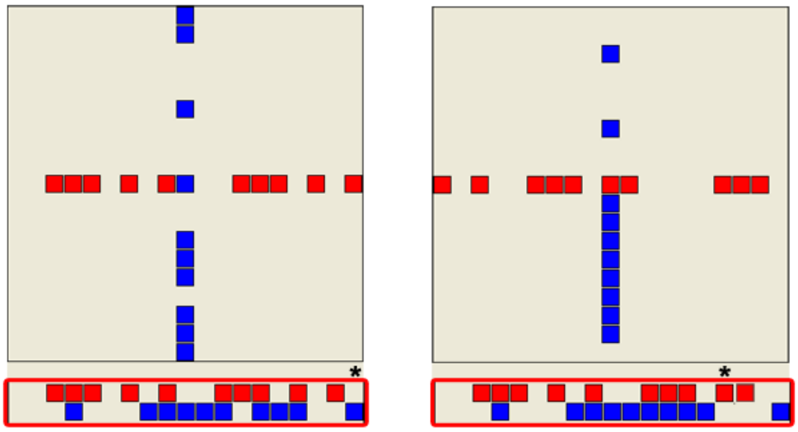

Figure 3: On the left: a junction configuration and the analogous configuration beneath it. The junction is marked with an asterisk. On the right: same configuration after 3 turns in both views

(asymptotic) average number of steps a car moves in one turn. It is easily seen that this speed is the same for all blue cars and all red cars, since the number of steps any two cars of the same color have taken cannot differ by more than N. It is somewhat surprising, perhaps, that these two speed must also be the same. For instance, there is no configuration for which a blue car completes 2 cycles for every 1 a red car completes.

2.2 Time-normalized model

Though less natural, it will be sometimes useful to consider the equivalent description, in which two rows of cars - red and blue - are placed one beneath the other, with a "special place" - the junction - where at most one car can be present at any time. In every step first the red line shifts one to the right (except cars immediately to the left of the junction, if it contains a blue car) and then the blue line does the same. Furthermore, instead of having the cars move to the right, we can have the junction move to the left, and when a blue car is in the junction, the (possibly empty) sequence of red cars immediately to the left of the junction moves to the left, and vice verse. Figure 3 illustrates the correspondence between these models.

From the discussion above we get the following equivalent system, which we will call the time-normalized junction:

1. Fix $S = N - 1 \in \mathbb{Z}_N$ and fix some initial configuration $\{R_i\}_{i=0}^{N-1}, \{B_i\}_{i=0}^{N-1} \in \{0,1\}^N$ representing the red and blue cars respectively, i.e. $R_i = 1$ iff there's a red car at the i-th place. We require that $\sum R_i = \sum B_i = p \cdot N$, and at place S $(= N - 1)$ there is at most one car in both rows [2].

2. In each turn:

[2] The last requirement is that the junction itself can contain only one car at the beginning

- If place S contains a blue car, and place $S-1$ contains a red car (if $B_S = R_{S-1} = 1$), push this car one step to the left. By pushing the car we mean also moving all red cars that immediately followed it one step to the left, i.e. set $R_{S-1} = 0$, $R_{S-i} = 1$ for $i = \min_{j \geq 1}[R_{S-j} = 0]$.
- If place S does not contain a blue car and place $S-1$ contains both a red and a blue car (if $B_S = 0$ and $R_{S-1} = B_{S-1} = 1$), push the blue car at $S-1$ one step to the left (set $B_{S-1} = 0$ and $B_{S-i} = 1$ for $i = \min_{j \geq 1}[B_{S-j} = 0]$).
- set $S = S - 1$

Note that indeed the dynamics above guarantee that after a turn place $(S-1)$ - the new junction - contains at most one car. Generally, as long as cars flow freely, the time-normalized system configuration does not change (except for the location of the junction). Cars in the time-normalized system configuration actually move only when some cars are waiting for the junction to clear (in the non-time-normalized system).

3 Analysis of an (N, p) junction

Our analysis of the junction will argue that for any p, regardless of the initial configuration, the system will reach some optimal speed, depending only on p. First let us state what is this optimal speed:

Theorem 3.1 *For a junction with density p, the maximal speed for any configuration is* $\min(1, \frac{1}{2p})$.

Proof Obviously, the speed cannot exceed 1. Observe the system when it reaches its stable state, and denote the speed by s. Thus, at time t, a car has advanced $ts(1+o(ts))$ steps (on average) which means that it has passed the junction $ts/N(1+o(1))$ times. As only one car can pass the junction at any time, the total number of cars passing the junction at time t is bounded by t. Therefore, we have $2pN \times ts/N \leq t$ which implies $s \leq 1/2p$. ■

We will now show that the system necessarily reaches these speeds from any starting configuration.

3.1 The case $p < 0.5$

We begin by proving that for $p < 0.5$ the junction will eventually reach speed very close to 1. A system in a stable state is said to be *free-flowing* if no car is ever waiting to enter the junction.

Lemma 3.2 *A junction is free-flowing iff the time-normalized junction satisfies:*
(1) For all $0 \leq i \leq N-1$ there is only one car in place i in both rows.
(2) For all $0 \leq i \leq N-1$ if place i contains a blue car, place $(i-1) \mod N$ does not contain a red car.

Proof Obviously, this is just a reformulation of free-flowing. ■

We will now turn to show that for $p < 0.5$, following the system dynamics of the time-normalized junction will necessarily bring us to a free flowing state, or at worst an "almost" free flowing state, meaning a state for which the system speed will be arbitrarily close to 1 for large enough N.

For this let us consider some configuration and look at the set of "violations" to lemma 3.2, i.e. places that either contain both a blue and a red car, or that contain a blue car and a red car is in the place one to the left. As the following lemma will show, the size of the set of violations is very closely related to the system speed, and posses a very important property: it is non-increasing in time.

More formally, For a configuration R, B we define two disjoint sets for the two types of violations:

$$V_B = \{0 \leq i \leq N-1 : R_{i-1} = B_{i-1} = 1, B_i = 0\}$$

$$V_R = \{0 \leq i \leq N-1 : R_{i-1} = B_i = 1\}$$

Also, let $V = V_B \cup V_R$ be the set of all violations. It will be sometimes useful to refer to a set of indicators $X = \{X(i)\}_{i=0}^{N-1}$, where each $X(i)$ is 1 if $i \in V$ and 0 otherwise, thus $|V| = \sum_{i=0}^{N-1} X(i)$.

For a junction with some initial configuration R, B, let R^t, B^t be the system configuration at time t, and let $V_t = V_{B^t} \cup V_{R^t}$ be the set of violations for this configuration, and X^t be the corresponding indicator vector. Similarly, let S^t denote the junction's position at time t.

Lemma 3.3

1. $|V_{t+1}| \leq |V_t|$
2. *For any t, the system speed is at least* $(1 + \frac{|V_t|}{N})^{-1} \geq 1 - \frac{|V_t|}{N}$

Proof Property (1) follows from the system dynamics. To see this, examine the three possible cases for what happens at time t:

1. If in turn t place S^t does not contain a violation then the configurations do not change during the next turn, i.e. $R^{t+1} = R^t$, $B^{t+1} = B^t$ hence clearly $|V_{t+1}| = |V_t|$.
2. If $S^t \in V_{B^t}$, then the configuration B^{t+1} changes in two places:
   (a) $B^t_{S^t-1}$ is changed from 1 to 0. Thus, place S^t is no longer in V_{B^t}, i.e. $X^t(S^t)$ changes from 1 to 0.
   (b) $B^t_{S^t-i}$ is changed from 0 to 1 for $i = \min_{j\geq 1}(B^t_{S^t-j} = 0)$. This may affect $X^{t+1}(S^t - i)$, and $X^{t+1}(S^t - i + 1)$. However, for place $S^t - i + 1$, by changing $B^t_{S^t-i}$ from 0 to 1 no new violation can be created (since by definition of i, $B^{t+1}_{S^t-i+1} = 1$, so $X^{t+1}(S^t - i + 1) = 1$ iff $R^{t+1}_{S^t-i} = 1$ regardless of $B^{t+1}_{S^t-i}$).

For other indices $X^{t+1}(i) = X^t(i)$ since R, B do not change, so between times t and $t+1$, we have that $X^t(S^t)$ changes from 1 to 0, and at worst only one other place - $X^t(S^t - i + 1)$ changes from 0 to 1, so $|V_{t+1}| = \sum_{i=0}^{N-1} X^{t+1}(i) \leq \sum_{i=0}^{N-1} X^t(i) = |V_t|$.

3. Similarly, if place $S^t \in V_{R^t}$ then the configuration R^{t+1} changes in two places:
   (a) $R^{t+1}_{S^t-1}$ is changed from 1 to 0. Thus, $X^t(S^t)$ changes from 0 to 1.
   (b) R_{S^t-i} is changed from 0 to 1 for $i = \min_{j\geq 1}(R^t_{S^t-j} = 0)$, affecting $X^{t+1}(S^t-i)$, $X^{t+1}(S^t-i+1)$. However for place S^t-i changing R_{S^t-i} does not affect whether this place is a violation or not, so at worst $X^t(S^t-i+1)$ changed from 0 to 1.

   By the same argument we get $|V_{t+1}| \leq |V_t|$.

Therefore, $|V_{t+1}| \leq |V_t|$.

For property (2) we note that in the time-normalized system, following a specific car in the system, its "current speed" is $\frac{N}{N+k} = (1 + \frac{k}{N})^{-1}$ where k is the number of times the car was pushed to the left during the last N system turns. We note that if a car at place j is pushed to the left at some time t, by some violation at place S^t, this violation can reappear only to the left of j, so it can push the car again only after S^t passes j. Hence any violation can push a car to the left only once in a car's cycle (i.e. N moves). Since by (1) at any time from t onwards the number of violations in the system is at most $|V_t|$, then each car is pushed left only $|V_t|$ times in N turns, so its speed from time t onwards is at least $(1 + \frac{|V_t|}{N})^{-1} > 1 - \frac{|V_t|}{N}$ as asserted. ■

With lemma 3.3 at hand we are now ready to prove system self organization for $p < 0.5$. We will show that for $p < 0.5$, after $2N$ system turns $|V_t| = O(1)$, and hence deduce by part (2) of lemma 3.3 the system reaches speed $1 - O(\frac{1}{N}) \to 1$ (as $N \to \infty$).

As the junction advances to the left it pushes some car sequences, thus affecting the configuration to its left. The next lemma will show that when $p < 0.5$, for some $T < N$, the number of cars affected to the left of the junction is only a constant, independent of N.

Lemma 3.4 *Consider a junction with density $p < 0.5$. There exists some constant $C = C(p) = \frac{p}{1-2p}$, independent of N, for which:*

From any configuration R, B with junction at place S there exist some $0 < T < N$ such that after T turns:

(1) For $i \in \{S - T, \ldots, S\}$, $X^T(i) = 0$ (i.e. there are no violations there)

(2) For $i \in \{S+1, \ldots, N-1, 0, \ldots, S-T-C\}$, $R^T_i = R^0_i$ and $B^T_i = B^0_i$ (R, B are unchanged there)

Proof First let us consider $T = 1$. For $T = 1$ to not satisfy the lemma conclusions, there need to be a car sequence (either red or blue) of length exceeding C, which is pushed left by the junction as it moves. Progressing through the process, if, for some $T < N - C$, the sequence currently pushed by the junction is shorter then C, then this is the T we seek. Therefore, for the conclusions *not* to hold, the length of the car sequence pushed by the junction must exceed C for all $0 < T < N$. If this is the case, then leaving the junction we see alternating red and blue sequences, all of lengths exceeding C and one vacant place after any blue sequence and before any red one.

However, if this is the case for all $0 < T \leq T'$ then the average number of cars per location in $\{S - T', \ldots, S\}$ at time T' must be at least $\frac{2C}{2C+1}$ (at least $2C$ cars between vacant places). Therefore, the total number of cars in $\{S - T', \ldots, S\}$ at time T' is more than $\frac{2C}{2C+1}T' = 2pT'$ (Recall that $C = \frac{2p}{1-2p}$).

Since there are only $2pN$ cars in the system, this cannot hold for all T up to N. Thus, there must be some time for which the conclusions of the lemma are satisfied.■

We are now ready to easily prove the main result for this section.

Theorem 3.5 *A junction of size N with density $p < 0.5$ reaches speed of $1 - \frac{C(p)}{N}$ from any initial configuration.*

Proof Let R, B be some initial configuration, with $S = N - 1$ and let V the corresponding set of violations and X the matching indicators vector. By lemma 3.4 there exist $T_0 > 0$ for which $X^{T_0}(i) = 0$ for $i \in [N-1-T_0, N-1]$. Now starting at R^{T_0}, B^{T_0} and $S = N-1-T_0$ reusing the lemma there exist $T_1 > 0$ s.t. $X^{T_0+T_1}(i) = 0$ for $i \in [N - 1 - T_1, N - 1 - T_0]$, and also, as long as $N - 1 - T_0 - T_1 > C(p) = \frac{2p}{1-2p}$, $X^{T_0+T_1}(i) = X^{T_0}(i) = 0$ for $i \in [N - 1 - T_0, N - 1]$ as well.

Proceeding in this manner until $T = \sum T_i \geq N$ we will get that after T turns, $X^T(i) = 0$ for all but at most $C(p)$ places, hence by lemma 3.3 the system speed from this time onward is at least $1 - \frac{C(p)}{N}$. ■

We remark one cannot prove an actual speed 1 (rather than $1 - o(1)$) for the case $p < \frac{1}{2}$ since this is definitely not the case. Figure 4 (a) demonstrates a general construction of a junction with speed $1 - \frac{1}{N}$ for all N. Finally we remark that a sharper constant $C'(p)$ can be provided, that will also meet a provable lower bound for the speed (by constructions similar to the above). This $C'(p)$ is exactly half of $C(p)$. We will see proof of this later in this paper as we obtain theorem 3.5 as a corollary of a different theorem, using a different approach.

3.2 Number of segments for $p < 0.5$

The proof in the previous section were combinatorial in nature and showed that for a density $p < 0.5$ the system can slow-down only by some constant number independent of N regardless of the configuration. As we turn to examine the properties of the stable configuration of course we can not say much if we allow any initial configuration. There are configurations in which the number of different segments in

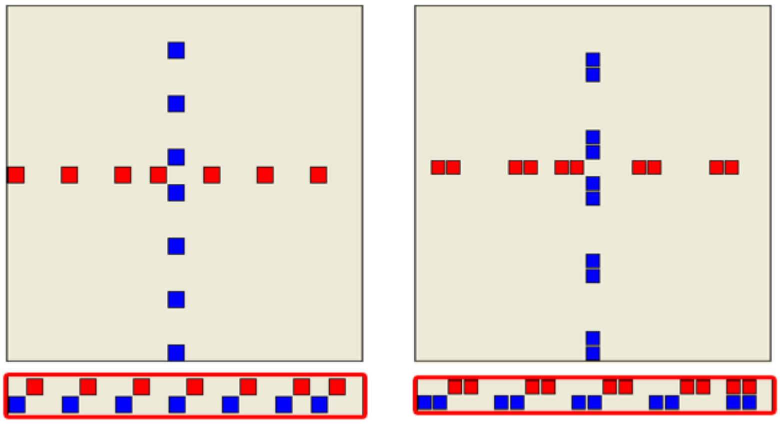

Figure 4: (a) A junction configuration with density $p = \frac{1}{3}$ ($\frac{p}{1-2p} = 1$) not reaching speed 1 (speed is $1 - \frac{1}{N}$). (b) A similar construction for $p = 0.4$ ($\frac{p}{1-2p} = 2$) reaching speed of only $1 - \frac{2}{N}$

each row will be $\Theta(N)$ while clearly if the cars are arranged in a single red sequence and a single blue sequence in the first place, we will have only one sequence of each color at any time.

However, we will show that for a random initial configuration, the system will have linearly many different segments of cars with high probability.

Theorem 3.6 *A junction of size N with density $p < 0.5$, started from a random initial configuration, will have $\Theta(N)$ different segments at all times with high probability (w.r.t. N).*

Proof As we already seen in the proof of 3.4, as the system completes a full round, since every place (but a constant number of places) contains at most a single car there must be $(1-2p)N$ places in which no car is present. Each two such places that are not adjacent must correspond to a segment in the cars configuration.

It is evident by the system dynamics, that the number of places for which $R_i = B_i = R_{i-1} = B_{i-1} = 0$ is non-increasing. More precisely, only places for which $R_i = B_i = R_{i-1} = B_{i-1} = 0$ in the initial configuration can satisfy this in the future. In a random initial configuration with density p, the initial number of these places is expected to be $(1-p)^4 N$, and by standard CLT, we get that with high probability this number is at most $((1-p)^4 + \varepsilon)N$. Thus, the number of different segments in the system configuration at any time is at least $((1-2p) - (1-p)^4 - \varepsilon)N$. However for p very close to $\frac{1}{2}$ this bound may be negative, so this does not suffice.

To solve this we note that similarly also for any fixed K, the number of consecutive K empty places in a configuration is non-increasing by the system dynamics, and w.h.p. is at most $((1-p)^{2K} + \varepsilon)N$ for an initial random state. But this guarantees at least $\frac{(1-2p)-(1-p)^{2K}-\varepsilon}{K-1}N$ different segments in the system. Choosing $K(p)$

(independent of N) sufficiently large s.t. $(1-p)^{2K} < (1-2p)$ we get a linear lower bound on the number of segments from a random initial configuration. ■

3.3 The case $p > 0.5$

The proofs of speed optimality and segment structure for $p > 0.5$ will rely mainly of a the combinatorial properties of a *stable configuration*. A *stable configuration* for the system is a configuration that re-appears after running the system for some M turns. Since for a fixed N the number of possible configurations of the system is finite, and the state-transitions (traffic rules) are time independent, the system will necessarily reach a stable configuration at some time regardless of the starting configuration.

We will use mainly two simple facts that must hold after a system reached a stable configurations: (a) $|V_t|$ cannot change - i.e. no violation can disappear. This is clear from lemma 3.3; (b) Two disjoint segments of car cannot merge to one (i.e. one pushed until it meets the other), since clearly the number of segments in the system is also non-increasing in time.

These two facts alone already provide plenty of information about the stable configuration for $p > 0.5$. We begin with the following twin lemmas on the stable state.

Lemma 3.7 *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$, $s_R, s_B \geq 1$. Then:*

1. *$B_i = 0$ for $i \in [N - s_R - s_B - 1, N - s_B - 1]$*
2. *$R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$*
3. *$R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$*

Proof Since it is easy to lose the idea in all the notations, a visual sketch of the proof is provided in figure 5.

By the assumptions:
- $B_i = 1$ for $i \in [N - s_B, N - 1]$, $B_{N-s_B-1} = 0$
- $R_i = 1$ for $i \in [N - s_R, N - 1]$, $R_{N-s_R-1} = 0$

To get (1), we note that with the assumption $B_0 = 0$, the blue sequence will be pushed to the left in the next s_R turns, so by restriction (b), $B_i = 0$ for $i \in [N - s_B - s_R - 1, N - s_B - 1]$ since otherwise 2 disjoint blue segments will merge while the sequence is pushed.

Thus following the system, after s_R turns we will get: $B_i = 1$ for $i \in [N - s_R - s_B, N - s_R - 1]$ and $B_i = 0$ for $i \in [N - s_R, N - 1]$, and B_i not changed left to $N - s_R - s_B$, and R unchanged.

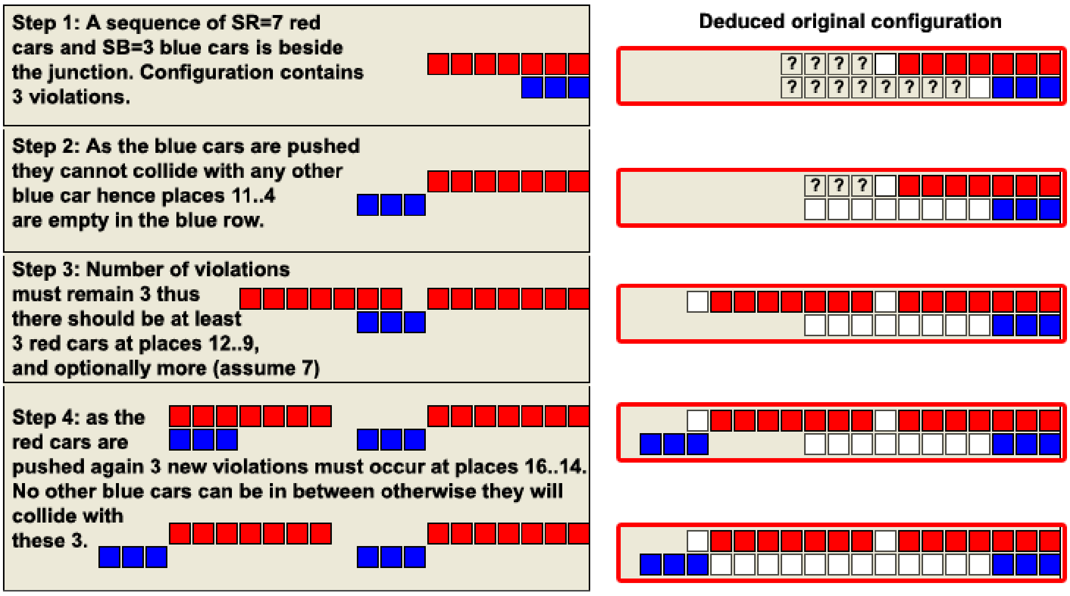

Figure 5: Sketch of proof ideas for lemmas 3.7 (steps 1-3) and 3.8 (step 4)

Note that originally R, B contained $\min(s_R, s_B)$ consecutive violations in places $[N - \min(s_R, s_B), 0]$ which all vanished after s_R turns. Possible violations at places $[N-\max(s_R, s_B), N-\min(s_R, s_B)]$ remained as they were. From here we get that we must have $R_i = 1$ for $\min(s_R, s_B)$ places within $[N-s_R-s_B-1, N-\max(s_R, s_B)-1]$. Since for place $N-\max(s_R, s_B)-1$ either R or B are empty by the assumption, we must therefore have $R_i = 1$ for $i \in [N - s_B - s_R - 1, N - \max(s_R, s_B) - 2]$, giving (2).

If we follow the system for s_B more steps we note that any red car in $[N - \max(s_R, s_B) - 1, N - s_R - 1]$ will be pushed left until eventually hitting the red car already proven to be present at $N - \max(s_R, s_B) - 2]$, thus $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$ giving (3). ■

The following lemma is completely analogous when reversing the roles of R, B, and can be proven the same way.

Lemma 3.8 *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 1$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B + 1, 0]$, $s_R, s_B \geq 1$. Then:*

1. $R_i = 0$ *for* $i \in [N - s_B - s_R - 1, N - s_R - 1]$
2. $B_i = 1$ *for* $i \in [N - s_B - s_R, N - \max(s_B, s_R) - 1]$
3. $B_i = 0$ *for* $i \in [N - \max(s_B, s_R), N - s_B]$

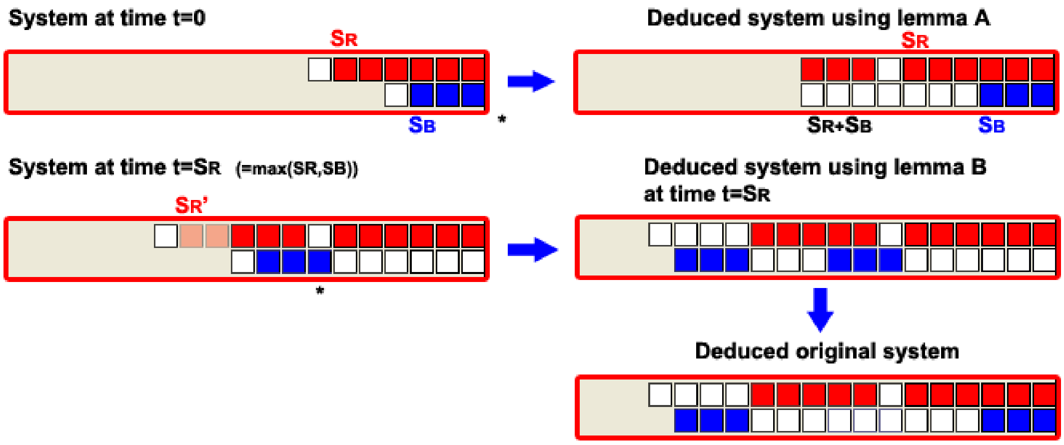

Figure 6: Lemmas 3.7 (A) and 3.8 (B) combined together yield lemma 3.9

Putting lemmas 3.7, 3.8 together we get the following characterization for stable configurations:

Lemma 3.9 *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$. Denote $M = \max(s_B, s_R)$ Then*

1. *There are no additional cars are at $[N - M, N - 1]$.*
2. *Place $i = N - M - 1$ is empty - i.e. $R_i = B_i = 0$*
3. *Starting at $N - M - 2$ there is a sequence of $K_1 \geq \min(s_B, s_R)$ places for which $R_i = 1; B_i = 0; (i \in [N - M - K_1 - 1, N - M - 2])$*
4. *Starting $N - M - K_1 - 2$ (i.e. right after the red sequence) there is a sequence of $K_2 \geq \min(s_B, s_R)$ places for which $B_i = 1; R_i = 0; (i \in [N - M - K_1 - K_2 - 1, N - M - K_1 - 2])$.*

Proof Figure 6 outlines the proof, without the risk of getting lost in the indices. For the full proof, first get from lemma 3.7 that:

1. $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - s_B - 1]$
2. $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$
3. $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$

From (1),(3) we get in particular, $B_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_B - 1]$, and $R_i = 0$ for $i \in [N - \max(s_R, s_B) - 1, N - s_R - 1]$, so indeed no additional cars are at $[N - \max(s_B, s_R), N - 1]$, and place $N - \max(s_R, s_B) - 1$ is empty, proving claims 1,2 in the lemma.

From (2),(3) we get $B_i = 0$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$ and $R_i = 1$ for $i \in [N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$, thus places $[N - s_R - s_B - 1, N - \max(s_R, s_B) - 2]$ contain a sequence of length $\min(s_R, s_B)$ of red cars with no blue cars in parallel to it. This sequence is possibly a part of a larger sequence of length $s'_R \geq \min(s_R, s_B)$, located at $[N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - 2]$.

Now running the system for $\max(s_R, s_B)$ turns, we will have the junction at place $S = N - \max(s_R, s_B) - 1$, $B_S = 1$, followed by sequences of s'_R reds and $s'_B = \min(s_B, s_R)(\leq s'_R)$ blues. Applying lemma 3.8 for the system (rotated by $\max(s_R, s_B)$ i.e. for $N' = N - \max(s_R, s_B) - 1$):

1. $R_i = 0$ for $i \in [N' - s'_B - s'_R - 1, N' - s'_R - 1] = [N - \max(s_R, s_B) - \min(s_R, s_B) - s'_R - 2, N - \max(s_R, s_B) - s'_R - 2] = [N - s_R - s_B - s'_R - 2, N - \max(s_R, s_B) - s'_R - 2]$

2. $B_i = 1$ for $i \in [N' - s'_B - s'_R, N' - \max(s'_B, s'_R) - 1] = [N - s_R - s_B - s'_R - 1, N - \max(s_R, s_B) - \max(s'_B, s'_R) - 2] = [N - s_R - s_B - s'_R - 1, N - \max(s_R, s_B) - s'_R - 2]$

3. $B_i = 0$ for $i \in [N' - \max(s'_B, s'_R), N' - s'_B] = [N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - \min(s_R, s_B) - 1] = [N - \max(s_R, s_B) - s'_R - 1, N - s_R - s_B - 1]$

In particular from (3) we get that no blue cars are in parallel to the entire red segment in $[N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - 2]$: We were previously assured this is true up to place $N - s_R - s_B - 1$, and for places $[N - \max(s_R, s_B) - s'_R - 1, N - s_R - s_B - 2] \subseteq [N - \max(s_R, s_B) - s'_R - 1, N - \max(s_R, s_B) - s'_B - 1]$ this holds by (3).

Furthermore by (2) we get that a sequence of blue cars which begins from place $N - \max(s_R, s_B) - s'_R - 2$ with no red cars in parallel to it by (1). Note that $N - \max(s_R, s_B) - s'_R - 2$ is exactly to the left of $N - \max(s_R, s_B) - s'_R - 1$ where the red sequence ended. Now clearly choosing $K_1 = s'_R; K_2 = \min(s_R, s_B)$ we get claims 3,4,5 in the lemma. ■

Putting it all together we can now get a very good description of a stable state.

Theorem 3.10 *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$. Denote $M = \max(s_B, s_R)$.*

Then no additional cars are at $[N - M, N - 1]$, and at places $[0, N - M - 1]$ the configurations R, B satisfies:

1. *Each place contains at most one type of car, red or blue.*

2. *Place $N - M - 1$ is empty. Each empty place, is followed by a sequence of places containing red cars immediately left to it, which is followed by a sequence of places containing blue cars immediately left to it.*

3. *Any sequence of red or blue cars is of length at least $\min(s_R, s_B)$*

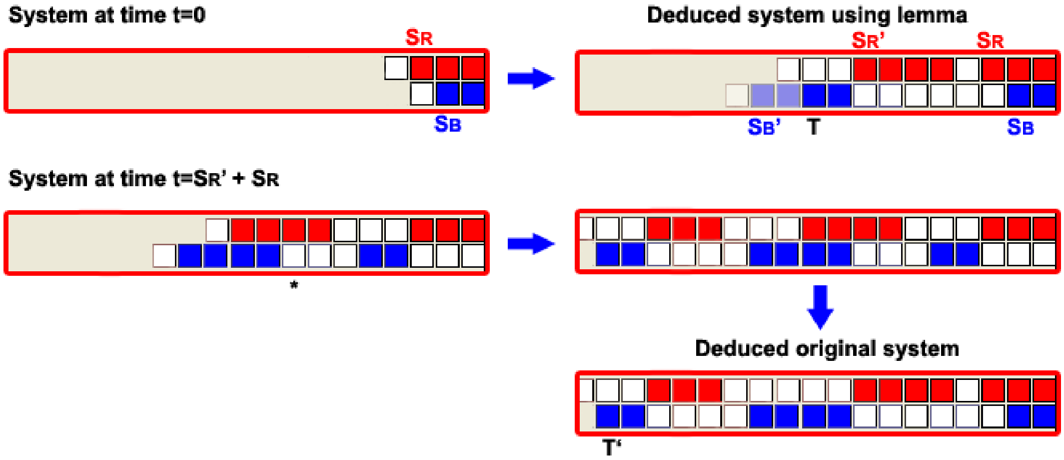

Figure 7: Repetitively applying lemma 3.9, we unveil a longer segment in the configuration, $[T, N-1]$, for which properties of theorem 3.10 hold

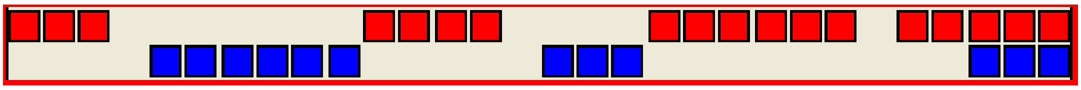

Figure 8: A typical stable configuration

Proof This is merely applying lemma 3.9 repeatedly. Applying lemma 3.9 we know that there exist $K_1, K_2 \geq \min(s_R, s_B)$ such that: Place $N-M-1$ is empty, followed by K_1 consecutive places with only red cars and K_2 consecutive places with only blue cars left to it, thus the assertion holds for the segment $[T, N-M-1]$ for $T = N-M-1-K_1$.

Now we completely know R, B in $[N-M-1-K_1, N-M-1]$, and this is enough to advance the system for K_1+M turns. The s_B blue segment is pushed left s_R places, further pushing the K_1 red sequence $\min(s_R, s_B)$ places to the left such that its last $\min(s_R, s_B)$ cars now overlap with the K_2 blue sequence.

So after K_1+M turns the system evolves to a state where $S = N-K_1-M$, $B_S = 0$, and left to S there are $K_2' > K_2$ consecutive blue cars and exactly $\min(s_R, s_B)$ consecutive red cars. Noting that this time $M' = max(K_2', \min(s_R, s_B)) = K_2'$, once again we can deduce from lemma 3.9 that: there are no additional cars in $[N-M-K_1-K_2', N-M-K_1-1]$ (thus we are assured that the entire blue segment of length K_2' does not have red cars parallel to it), Place $N-M-K_1-K_2'-1$ is empty, followed by some K_3 consecutive places with only red cars and K_4 consecutive places with only blue cars left to it, for $K_3, K_4 \geq \min(K_1, K_2) = \min(s_R, s_B)$ thus the assertion holds for the segment $[T', N-M-1]$ for $T' = N-M-K_1-K_2'-K_3-1 < T$.

Repeatedly applying lemma 3.9 as long as $T > 0$, we repeatedly get that the assertion holds for some $[T, N-M-1]$ for T strictly decreasing, so the assertion holds in $[0, N-M-1]$ ■

We have worked hard for theorem 3.10, but it will soon turn to be worthwhile. Let us obtain some useful corollaries.

Corollary 3.11 *Let R, B be a stable configuration with junction at $S = 0$ and $B_0 = 0$. Assume that there is a sequence of exactly s_R consecutive red cars at places $[N - s_R, N - 1]$ and s_B blue cars at places $[N - s_B, N - 1]$. Denote $m = \min(s_B, s_R)$ Then:*

1. *The number of blue segments in the system equals the number of red segments.*
2. *System speed is at least $(1 + \frac{m}{N})^{-1}$*
3. *Total number of cars in the system is at most $N + m$*
4. *Total number of cars in the system is at least $\frac{2m}{2m+1}N + m$*
5. *Total number of segments in the system is at most $\frac{N}{m} + 1$*

Proof (1) Follows by the structure described in 3.10, since there is exactly one red and one blue sequence in $[N - M, N - 1]$ (as always $M = \max(s_R, s_B)$) and an equal number of reds and blues in $[0, N - M - 1]$ since any red sequence is immediately proceeded by a blue sequence.

For (2) we note that by 3.10 the configuration R, B contains exactly m violations : the m overlapping places of the segments s_R, s_B in $[N - M, N - 1]$ are the only violations in the configuration since any places in $[0, N - M - 1]$ contains a single car, and no red car can be immediately to the left of a blue car. By 3.3 the system speed is hence at least $(1 + \frac{m}{N})^{-1}$.

By 3.10 any place in $[0, N - m - 1]$ contains at most one car, or it is empty, and places $[N - m, N - 1]$ contain both a red car and a blue car. Thus the total number of cars in the system is at most $N - m + 2m = N + m$ giving (3).

On the other hand, Since any sequence of a red car or a blue car is of length at least m, an empty place in $[0, N - m]$ can occur only once in $2m + 1$ places, and other places contain one car. Thus the number of cars is lower bounded by $\frac{2m}{2m+1}(N - m) + 2m = \frac{2m}{2m+1}N + \frac{2m^2+2m}{2m+1} \geq \frac{2m}{2m+1}N + m$ giving (4).

The last property follows the fact that any sequence of cars is of length at least m, and by (3) total number of cars is at most $N + m$, thus number of different sequences is at most $\frac{N+m}{m} = \frac{N}{m} + 1$. ■

Theorem 3.12 *All cars in the system have the same asymptotic speed.*

Proof As we have seen, when it reaches a stable state, the system consists of alternating red and blue sequences of cars. Obviously, the order of the sequences cannot change. Therefore, the difference between the number of steps two different cars have taken cannot be more then the length of the longest sequence, which is less then N. Thus, the asymptotic (w.r.t. t) speed is the same for all cars. ■

With these corollaries we can now completely characterize the stable state of a junction with $p > 0.5$, just by adding the final simple observation, that since the number of cars in the model is greater than N, there are violations at all times, including after reaching a stable state. Now let us look at some time when the junction reaches a violation when the system is in stable state. At this point the conditions of theorem 3.10 are satisfied, thus:

Theorem 3.13 *A junction of size N and density $p > 0.5$ reaches speed of $\frac{1}{2p} - O(\frac{1}{N})$ (i.e. arbitrarily close to the optimal speed of $\frac{1}{2p}$, for large enough N), and contains at most a bounded number (depending only on p) of car sequences.*

Proof We look at the system after it reached a stable state. Since $2pN > N$ at some time after that conditions of theorem 3.10 are satisfied for some $s_R, s_B \geq 1$. Let $m = \min(s_R, s_B)$ at this time. Using claims (3),(4) in corollary 3.11 we get:

$$\frac{2m}{2m+1}N + m \leq 2pN \leq N + m$$

From here we get

$$(*) \quad (2p-1)N \leq m \leq (2p - \frac{2m}{2m+1})N = (2p-1)N + \frac{N}{2m+1}$$

and reusing $m \geq (2p-1)N$ on the left hand size we get:

$$(2p-1)N \leq m \leq (2p-1)N + \frac{1}{4p-1}$$

For $C = \frac{1}{4p-1}$ a constant independent of N ($C = O(1)$). So $m = (2p-1)N + K$, for $K \leq C$. Now by claim (2) in 3.11, system speed is at least

$$(1 + \frac{m}{N})^{-1} = (1 + \frac{(2p-1)N + K}{N})^{-1} = (2p + \frac{K}{N})^{-1} \geq_{(2p>1)} \frac{1}{2p} - \frac{K}{N}$$

But by theorem 3.1 system speed is at most $\frac{1}{2p}$, thus system speed is exactly $\frac{1}{2p} - \frac{K'}{N}$ for some $0 \leq K' \leq K \leq C = O(1)$, thus $K' = O(1)$ proving the first part of the theorem.

By (5) we get total number of segments in the system is at most $\frac{N}{m} + 1$, applying $m \geq (2p-1)N$ we get the number of segments is bounded by:

$$\frac{N}{m} + 1 \leq \frac{1}{2p-1} + 1 = \frac{2p}{2p-1} = O(1)$$

Thus the second part proven. ∎

3.4 $p < 0.5$ revisited

The characterization in 3.10 can be also proven useful to handle $p < 0.5$. Actually the main result for $p < 0.5$, theorem 3.5, can be shown using similar technique, and even sharpened.[3]

Corollary 3.14 *For $p < 0.5$ the junction reaches speed of at least $1 - \frac{C(p)}{N}$, for $C(p) = \left\lfloor \frac{p}{1-2p} \right\rfloor$. In particular, for $p < \frac{1}{3}$ the junction reaches speed 1, for any initial configuration.*

Proof Let R, B be any initial configuration. Looking at the configuration after it reached the stable state, if the system reached speed 1 we have nothing to prove. Assume the speed is less than 1. Since in this case violations still occur, at some time the stable configuration will satisfy theorem 3.10. As before, letting $m = \min(s_R, s_B)$ at this time, by claim (4) in corollary 3.11 we have:

$$\frac{2m}{2m+1}N + m \leq 2pN \ \Rightarrow m \leq (2p - 1 + \frac{1}{2m+1})N$$

In particular $2p - 1 + \frac{1}{2m+1} > 0$, rearranging we get $m < \frac{p}{1-2p}$, and since $m \in \mathbb{Z}$, $m \leq \left\lfloor \frac{p}{1-2p} \right\rfloor = C(p)$. m must be positive, thus for $p < \frac{1}{3}$, having $C(p) = \left\lfloor \frac{p}{1-2p} \right\rfloor = 0$ we get a contradiction, thus the assumption (speed < 1) cannot hold. For $p \geq \frac{1}{3}$ by claim (2) in corollary 3.11 the system speed is at least $(1 + \frac{C(p)}{N})^{-1} \geq 1 - \frac{C(p)}{N}$. ■

3.5 The critical $p = 0.5$

Gathering the results so far we get an almost-complete description of the behaviour of a junction. For junction of size N and density p:

- If $p < 0.5$ the junction will reach speed $1 - o(1)$ (asymptotically optimal), and contain linearly many different segments in the stable state.
- If $p > 0.5$ the junction will reach speed $\frac{1}{2p} - o(1)$ (asymptotically optimal), and contain constant many segments in the stable state.

From the description above one sees that the junction system goes through a sharp phase transition at $p = 0.5$, as the number of segments of cars as the system stabilizes drops from being linear to merely constant. The last curiosity, is what happens at $p = 0.5$. Once again by using the powerful theorem 3.10 we can deduce:

Theorem 3.15 *A junction of size N with $p = 0.5$ reaches speed of at least $1 - \frac{1}{\sqrt{N}}$ and contains at most $\sqrt{N}$ different segments.*

[3]To theorem 3.5 defence, we should note that key components of its proof, such as lemma 3.3, were also used in the proof of 3.10

Proof For $p = 0.5$ we have exactly N cars in the system. As we reach stable state, violations must still occur (since a system with exactly N cars must contain at least one violation), thus at some time theorem 3.10 is satisfied. For $m = \min(s_R, s_B)$ at this time, by claim (4) in corollary 3.11 we have:

$$\frac{2m}{2m+1}N + m \leq N \;\Rightarrow m(2m+1) \leq N$$

Thus $m < \sqrt{N}$. From here by claim (2) the system speed is at least $(1 + \frac{\sqrt{N}}{N})^{-1} \geq 1 - \frac{1}{\sqrt{N}}$.

If S is the number of segments, then by theorem 3.10 we can deduce that the total number of cars, N, is: $2m$ cars in places $[N-m, N-1]$, and $N-m-S$ cars in places $[0, N-M-1]$ (since each place contains one car, except transitions between segments that are empty).

$$N = (N-m-S) + 2m = N + m - S \;\Rightarrow S = m \leq \sqrt{N}$$

Thus the configuration contains at most $\sqrt{N}$ segments. ■

Simulation results show that these bounds are not only tight, but typical, meaning that a junction with a random initial configuration with density p indeed has $O(\sqrt{N})$ segments in the stable state, with the largest segment of size near $n^{1/2}$. This suggests that the system undergoes a second order phase transition.

4 Simulation results

Following are computer simulation results for the junction for critical and near critical p, demonstrating the phase transition. The columns below consist of N, the average asymptotic speed, the average number of car segments in the stable state, the average longest segment in the stable state, and the average number of segments divided by N and $\sqrt{N}$.

4.1 $p = 0.48$

For large N, system reaches speed 1 and the average number of segments is linear (approx. $0.037N$).

N	*Speed*	*No. segs*	*Longest*	*No. segs*$/N$	*No. segs*$/\sqrt{N}$
1000	0.99970	38.7	6.8	0.0387	1.2238
5000	1.00000	186.4	8.5	0.0373	2.6361
10000	1.00000	369.8	7.5	0.0370	3.6980
50000	1.00000	1850.6	8.2	0.0370	8.2761

4.2 $p = 0.52$

For large N, system reaches speed $0.961 = \frac{1}{2p}$ and the average number of segments is about constant.

N	*Speed*	*No. segs*	*Longest*	*No. segs*$/N$	*No. segs*$/\sqrt{N}$
1000	0.95703	5.7	76.7	0.0057	0.1802
5000	0.96041	6.9	330.0	0.0014	0.0976
10000	0.96091	7.3	416.1	0.0007	0.0730
50000	0.96142	7.2	3207.1	0.0001	0.0322

4.3 $p = 0.5$

At criticality, the speed is approaching 1 like $1 - \frac{C}{\sqrt{N}}$ and the average number of segments is around $0.43 \cdot \sqrt{N}$.

N	*Speed*	*No. segs*	*Longest*	*No. segs*$/N$	*No. segs*$/\sqrt{N}$
1000	0.98741	13.4	38.4	0.0134	0.4237
5000	0.99414	30.0	82.8	0.0060	0.4243
10000	0.99570	43.8	142.1	0.0044	0.4375
50000	0.99812	95.0	248.4	0.0019	0.4251

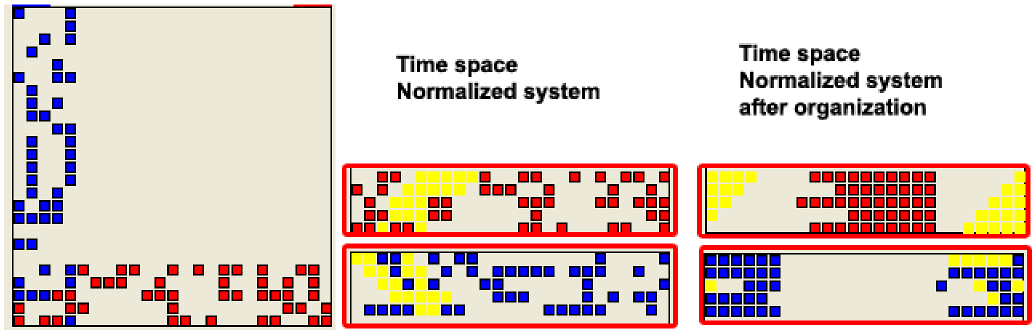

Figure 9: K-Lanes junction in a time-space normalized view, before and after organization. Junction is marked in yellow and advances left each turn

5 Summary

The fascinating phenomena observed in the BML traffic model are still far from being completely understood. In this paper we showed a very simplified version of this model, which, despite its relative simplicity, displayed very similar phenomena of phase transition at some critical density and of self-organization, which in our case both can be proven and well understood.

We used two approaches in this paper: The first one was to use some sort of "independence" in the way the system evolves, as in the way we handle $p < 0.5$. We showed that the system self-organizes "locally" and with a bounded affect on the rest of the configuration, thus it will eventually organize globally. The second approach is the notion of the stable configuration, i.e. we characterize the combinatorial structure of any state that the system can "preserve" and use it to show it is optimal (in a way saying, that as long as we are not optimal the system must continue to evolve).

Can these results be extended to handle more complicated settings than the junction? Possibly yes. For example, considering a k-lanes junction (i.e. k consecutive red lines meeting k consecutive blue rows), one can look at a time-space-normalized version of the system, as shown in figure 9, with the junction now being a $k \times k$ parallelogram traveling along the red and blue lines, and "propagating violations" (which now have a more complicated structure depending on the $k \times k$ configuration within the junction). Stable states of this configurations seem to have the same characteristics of a single junction, with a red (blue) car equivalent to some red (blue) car in any of the k red (blue) lines. Thus a zero-effort corollary for a k-lanes junction is that it reaches speed 1 for $p < \frac{1}{2k}$, but for k nearing $O(N)$ this bound is clearly non-significant. It is not surprising though, since combinatorics alone cannot bring us too far, at least for the complete BML model – even as little as $2N$ cars in an N^2 size torus can be put in a stuck configuration - i.e. reach speed 0.

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