# Giant components in biased graph processes

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### Abstract

A random graph process, $\mathcal{G}_1(n)$, is a sequence of graphs on $n$ vertices which begins with the edgeless graph, and where at each step a single edge is added according to a uniform distribution on the missing edges. It is well known that in such a process a giant component (of linear size) typically emerges after $(1 + o(1))\frac{n}{2}$ edges (a phenomenon known as "the double jump"), i.e., at time $t = 1$ when using a timescale of $n/2$ edges in each step.

We consider a generalization of this process, $\mathcal{G}_K(n)$, proposed by Itai Benjamini in order to model the spreading of an epidemic. This generalized process gives a weight of size 1 to missing edges between pairs of isolated vertices, and a weight of size $K \in [0, \infty)$ otherwise. This corresponds to a case where links are added between $n$ initially isolated settlements, where the probability of a new link in each step is biased according to whether or not its two endpoint settlements are still isolated.

Combining methods of [13] with analytical techniques, we describe the typical emerging time of a giant component in this process, $t_c(K)$, as the singularity point of a solution to a set of differential equations. We proceed to analyze these differential equations and obtain properties of $\mathcal{G}_K$, and in particular, we show that $t_c(K)$ strictly decreases from $\frac{3}{2}$ to 0 as $K$ increases from 0 to $\infty$, and that $t_c(K) = \frac{4}{\sqrt{3K}}(1 + o(1))$, where the $o(1)$-term tends to 0 as $K \to \infty$. Numerical approximations of the differential equations agree both with computer simulations of the process $\mathcal{G}_K(n)$ and with the analytical results.

## 1   Introduction

### 1.1   The Achlioptas problem and the biased process

The random graph process on $n$ vertices, $\mathcal{G}_1 = \mathcal{G}_1(n)$, introduced by Erdős and Rényi, is a sequence of $\binom{n}{2} + 1$ graphs, $(\mathcal{G}_1{}^0, \ldots, \mathcal{G}_1{}^{\binom{n}{2}})$, where the $\mathcal{G}_1{}^0$ is the edgeless graph on $n$ vertices, and $\mathcal{G}_1{}^m$ is

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obtained by adding a random edge to $\mathcal{G}_1{}^{m-1}$, chosen uniformly over all missing edges.

A classical result of Erdős and Rényi ([9]) states that if $T = C\frac{n}{2}$ and $C < 1$, then typically every connected component of $\mathcal{G}_1{}^T$ is of size $O(\log n)$, and if $C > 1$ then typically there is a single *giant component* of size $\Theta(n)$ and every other component is of size $O(\log n)$. Thus, there is a phase transition (the "double jump") after $(1+o(1))n/2$ edges. For further discussion of this phenomenon, see, e.g., [2].

A well known problem, introduced by Achlioptas, discusses a scenario where two randomly chosen edges are presented at each step, out of which a single edge is chosen by some algorithm $\mathcal{A}$. The goal of the algorithm is to postpone the emerging time of the giant component as much as possible. This was first examined by Bohman and Frieze in [3]. For additional results both on this problem (as an off-line and as an online problem), as well as on the converse problem of creating a giant component ahead of time, see [4],[5],[6],[7]. In [13], the authors describe a generic approach to analyzing the performance of algorithms for the mentioned Achlioptas problem. After applying Wormald's differential equation method for graph processes [16], the emerging time of the giant component is expressed as a singularity point to a differential equation. Using this method, the authors are able to provide bounds for the performance of several algorithms.

In this paper, we study a natural generalization of the Erdős-Rényi random graph process, proposed by Itai Benjamini in order to model the spreading of an epidemic. This process of Benjamini lets "infected" clusters have either a larger or a smaller probability of increasing their size, depending on the value of an external continuous parameter $K$. This is achieved by embedding basic degree information (namely, whether or not a site is currently isolated) into the probability distribution over the missing edges. Note that the given model is the most natural of its kind with respect to the dependency of this probability distribution on the degrees of the vertices. Results on other, more complicated, models may be obtained via methods similar to the ones presented here.

We combine methods of [13] together with analytical methods, with the same motivation of determining the critical time in which a giant component emerges in this process. As we mentioned, the generalized process we consider is a parameterized version, dependent on some $K \in [0, \infty)$, which modifies the probability of each edge according to whether or not its endpoints are already connected. In the original Achlioptas problem, different algorithms can postpone the phase transition or create it ahead of time, where the biased choice at each step is between precisely two randomly chosen edges. The process we study considers all missing edges when making its biased choice, and the phase transition is presented as a function of the continuous parameter $K$.

By applying the powerful differential equation method, we were able to derive properties of our model directly from the system of coupled non-linear ordinary differential equations (ODEs). While some of these properties can be proved by relatively simple combinatorial arguments, calculating the precise asymptotic behavior of $t_c(K)$ for $K \gg 1$ via combinatorial arguments seems challenging. Indeed, while at first glance this result appears as though it can be obtained by probabilistic arguments (e.g., using monotone coupling to variants of the model, tracking the structure

of the connected components throughout certain time intervals, etc.), we are not aware of any such derivation at the present time. In particular, the only way to obtain the $4/\sqrt{3K}$ behavior of the blowup time is through the careful asymptotic analysis of the derived coupled differential equations (see, e.g., [1] for more on asymptotic analysis of ODEs).

The generalized process can be efficiently implemented. In order to efficiently randomize the next edge, one needs to maintain the sets of isolated and non-isolated vertices, along with the set of edges already chosen. Our implementation runs in time $O(n \log n)$ and requires $O(n)$ memory. The computer simulations show an excellent agreement with the numerical solutions of the ODEs on the one hand, and with the analytical results concerning $t_c(K)$ on the other hand.

## 1.2 Notations and main results

The *biased graph process* on $n$ vertices, $\mathcal{G}_K$, is the following generalization of the random graph process: as before, the initial graph $\mathcal{G}_K{}^0$ is the edgeless graph on $n$ vertices, and $\mathcal{G}_K{}^m$ is obtained by adding a single edge to $\mathcal{G}_K{}^{m-1}$. The newly added edge is selected according to the following distribution on the missing edges: each edge between two isolated vertices is assigned a weight of 1, and the weight $K$ is assigned to all the remaining edges. Once there are less than 2 isolated vertices, the distribution on the missing edges is uniform. We extend the definition of $\mathcal{G}_K{}^m$ to $m > \binom{n}{2}$ by setting $\mathcal{G}_K{}^m = K_n$ for every such $m$, where $K_n$ denotes the complete graph on $n$ vertices. Furthermore, we use the notation $\mathcal{G}_K{}^T|_H$, where $H$ is a graph on $n$ vertices, to denote the biased graph process after $T$ steps, starting from the initial graph $H$ instead of the edgeless graph.

Let $\mathcal{G}_K(t)$ denote the biased process after scaling its time line by a factor of $n/2$:

$$\mathcal{G}_K(t) = \mathcal{G}_K{}^{\lfloor tn/2 \rfloor} \ .$$

Since a choice of $K = 1$ is equivalent to the Erdős-Rényi random graph process $\mathcal{G}_1$, the appearing time of the giant component in $\mathcal{G}_1$ is typically at $t = 1$. We study the effect that modifying $K$ has on this critical point, $t_c(K)$, keeping in mind that, intuitively, decreasing the value of $K$ should postpone the emerging point of a giant component and vice versa.

Throughout the paper, we say that a random graph on $n$ vertices satisfies some property *with high probability*, or *almost surely*, or that *almost every* graph process on $n$ vertices satisfies a property, if the corresponding event has a probability which tends to 1 as $n$ tends to infinity.

Let $G = (V, E)$ be a graph on $|V| = n$ vertices. Let $\mathcal{C} = \mathcal{C}(G)$ denote the set of connected components of $G$, and let $\mathcal{C}_v = \mathcal{C}_v(G)$ denote the connected component of a given vertex $v \in V$. We denote by $\mathcal{C}_0$ the set of isolated vertices in $G$, and by $I(G)$ the fraction of isolated vertices:

$$I(G) = \frac{|\mathcal{C}_0|}{n} \ .$$

Notice that, if the graph after $T$ steps (that is, once $T$ edges have been added) contains $|\mathcal{C}_0|$ isolated

vertices, then the probability that the next edge is a specific one between two isolated vertices is

$$\left[\binom{|\mathcal{C}_0|}{2} + \left(\binom{n}{2} - \binom{|\mathcal{C}_0|}{2} - T\right)K\right]^{-1},$$

and the probability that it is a specific new edge between two vertices that are not both isolated is $K$ times that value.

The *susceptibility* of $G$, $S(G)$, is defined to be the expected size of a connected component of a uniformly chosen vertex $v \in V$:

$$S(G) = \frac{1}{n}\sum_{v\in V}|\mathcal{C}_v| = \frac{1}{n}\sum_{C\in\mathcal{C}(G)}|C|^2 .$$

The susceptibility of $G$ is closely coupled with the existence of a giant component in $G$. Indeed, the existence of a giant component of size $\alpha n$ for some $\alpha > 0$ implies that $S(G) \geq \alpha^2 n$, and conversely, if $S(G) = \Omega(n)$ then at least one connected component is of linear size. Following the ideas of [13], we characterize the behavior of the susceptibility along $\mathcal{G}_K$, and in the process obtain the required results on $t_c(K)$. In order to do so, we need to examine the typical behavior of the number of isolated vertices along $\mathcal{G}_K$.

The fraction of isolated vertices in $\mathcal{G}_K$, $I(\mathcal{G}_K(t))$ has a value of 1 at $t = 0$, and decreases to 0 over time. The following theorem summarizes the behavior of $I(\mathcal{G}_K(t))$:

**Theorem 1.1.** *For every $K > 0$ and every $C > 0$, almost every biased process on $n$ vertices $\mathcal{G}_K$ satisfies $|I(\mathcal{G}_K(t)) - y(t)| = O(n^{-1/4})$ for every $0 \leq t \leq C$, where $y$ is the solution to the differential equation:*

$$\begin{cases} y' &= \dfrac{(1-y)K}{y^2 + (1-y^2)K} - 1 \\ y(0) &= 1 \end{cases}. \tag{1}$$

*In the special case $K = 0$, the above holds for $C = 1$.*

The following proposition analyzes the differential equation whose solution will prove to be a good approximation of the susceptibility along the biased process:

**Proposition 1.2.** *Let $z = z(t)$ denote the solution for the following differential equation:*

$$\begin{cases} z' &= \dfrac{K}{y^2 + (1-y^2)K}\left(z^2 - 1\right) + 1 \\ z(0) &= 1 \end{cases}, \tag{2}$$

*where $y$ is the solution to the differential equation (1). For every $K > 0$ there exists a singularity point $t_c = t_c(K) > 0$, such that $z(t)$ is continuous on $[0, t_c)$, and $\lim_{t\to t_c^-} z(t) = \infty$. Furthermore, there exists some constant $M > 0$, independent of $K$, such that $t_c(K) \in (0, M)$ for every $K > 0$.*

The next theorem implies that the singularity point of $z(t)$, $t_c = t_c(K)$, is the typical time at which a giant component emerges in a biased process:

**Theorem 1.3.** *For every $\varepsilon > 0$ and every $K > 0$, almost every biased process on $n$ vertices $\mathcal{G}_K$ satisfies the following:*

1. *Subcritical phase: $|S(\mathcal{G}_K(t)) - z(t)| = o(1)$ for every $t \in [0, t_c - \varepsilon]$, where $z(t)$ is the solution for the differential equation (2), and $t_c$ is its singularity point as defined in Proposition 1.2. Furthermore, for every $t \in [0, t_c - \varepsilon]$, the largest component of $\mathcal{G}_K(t)$ is of size $O(\log n)$.*

2. *Supercritical phase: $S(\mathcal{G}_K(t_c + \varepsilon)) = \Omega(n)$.*

*Altogether, the appearance of a giant component in $\mathcal{G}_K$ is almost surely at time $t_c$. In the special case $K = 0$, the above holds when replacing $z(t)$ with the function $\hat{z}(t) = \begin{cases} 1 + t & \text{if } t \leq 1 \\ \frac{1}{3/2 - t} & \text{if } t \geq 1 \end{cases}$.*

Finally, the behavior of $t_c(K)$, the typical point of the phase transition in $\mathcal{G}_K$, is characterized by the following theorem:

**Theorem 1.4.** *Let $t_c(K)$ denote the singularity point of the solution to the ODE (2), as defined in Proposition 1.2. Then $t_c(K)$ is continuous and strictly monotone decreasing as a function of $K$ and satisfies:*
$$\begin{cases} t_c(0) & = & \frac{3}{2} \\ t_c(K) & = & \frac{4}{\sqrt{3K}}\left(1 + o(1)\right) \end{cases} ,$$
*where the $o(1)$-term tends to $0$ as $K \to \infty$.*

The rest of the paper is organized as follows: in Section 2 we study general properties of the distribution of $\mathcal{G}_K$. Namely, we study the relation between $\mathcal{G}_K$ and $\mathcal{G}_1$, and describe an approximated process, $\widetilde{\mathcal{G}_K}$, which is easier to analyze.

In Section 3 we prove Theorem 1.1 and analyze the solution to equation (1), which characterizes the behavior of the isolation ratio throughout the biased process. Proposition 1.2 and Theorem 1.3 are both proved in Section 4, where computer simulations are also included.

In Section 5 we analyze the asymptotic behavior of $t_c(K)$ for large values of $K$, and prove Theorem 1.4. The final section 6 is devoted to open problems and concluding remarks.

## 2 Dominating and approximate distributions for $\mathcal{G}_K$

### 2.1 The relation between $\mathcal{G}_K$ and $\mathcal{G}_1$

A main tool in understanding and analyzing the biased process $\mathcal{G}_K$ is the stochastic domination relation between $\mathcal{G}_K$ and time-stretched versions of the Erdős-Rényi process $\mathcal{G}_1$. In order to formalize and prove this relation, we consider a wider family of graph processes, defined as follows:

**Definition.** *Let $M \in \mathbb{N}$. An **M-bounded weighted graph process** on $n$ vertices, $\mathcal{H} = \mathcal{H}(n)$, is an infinite sequence of graphs on $n$ vertices, $(\mathcal{H}^0, \mathcal{H}^1, \ldots)$, where $\mathcal{H}^0$ is some fixed initial graph, and $\mathcal{H}^t$ is generated from $\mathcal{H}^{t-1}$ by adding one edge at random, as follows: the probability of adding the edge $e$ to $\mathcal{H}^{t-1}$ is proportional to some weight function $W_t(e)$, satisfying:*

$$\max_{e \notin \mathcal{H}^{t-1}} W_t(e) \leq M \min_{e \notin H^{t-1}} W_t(e) \ .$$

*If for some $\nu \geq 0$ $\mathcal{H}^\nu = K_n$, we define $\mathcal{H}^t = \mathcal{H}^\nu = K_n$ for every $t > \nu$.*

Clearly, the biased process $\mathcal{G}_K$ is an $M$-bounded weighted graph process which starts from the edgeless graph, where $M = \lceil \max\{K, \frac{1}{K}\} \rceil$. We are interested in the relation between the probability that $\mathcal{G}_1$ satisfies some monotone graph property $\mathcal{A}$ (a set of graphs closed under isomorphism and under the addition of edges) and the corresponding probability of $\mathcal{G}_K$. The following theorem formalizes the stochastic domination of the original graph process $\mathcal{G}_1$ on $M$-bounded weighted graph processes:

**Theorem 2.1.** *Let $\mathcal{H}$ denote an $M$-bounded weighted graph process on $n$ vertices, and let $\mathcal{A}$ denote a monotone increasing property of graphs on $n$ vertices. The following statements hold for any $t \in \mathbb{N}$:*

$$\Pr[\mathcal{H}^t \in \mathcal{A}] \ \leq \ \Pr[\mathcal{G}_1{}^{Mt}|_{\mathcal{H}^0} \in \mathcal{A}] \ , \tag{3}$$

$$\Pr[\mathcal{G}_1{}^t|_{\mathcal{H}^0} \in \mathcal{A}] \ \leq \ \Pr[\mathcal{H}^{Mt} \in \mathcal{A}] \ . \tag{4}$$

We need the following lemma, which was first proved in [14] in a slightly different setting. For the sake of completeness, we include a short proof of the lemma using the Max-Flow-Min-Cut Theorem (a relation which was first observed in [12]):

**Lemma 2.2.** *Let $U, V$ be two finite sets, and let $R \subset U \times V$ denote a relation on $U, V$. Let $\mu$ and $\nu$ denote probability measures on $U$ and $V$ respectively, such that the following inequality holds for every $A \subset U$:*

$$\mu(A) \leq \nu(\{y \in V : xRy \text{ for some } x \in A\}) \ . \tag{5}$$

*Then there exists a coupling $\varphi$ of $\mu, \nu$ whose support is contained in $R$. That is, there is a joint distribution $\varphi$ on $U, V$ satisfying the following two properties:*

1. *The marginal distributions of $\varphi$ on $A$ and $B$ are $\mu$ and $\nu$ respectively.*

2. *For every $(x, y) \in U \times V$ such that $\neg(xRy)$, $\varphi(x, y) = 0$.*

*Furthermore, the distribution $\varphi$ can be found in time polynomial in $|U| + |V|$.*

*Proof.* Consider a directed weighted graph $G$ on the vertex set $U \cup V \cup \{\bar{s}, \bar{t}\}$, with the following set of edges and capacities (see Figure 1):
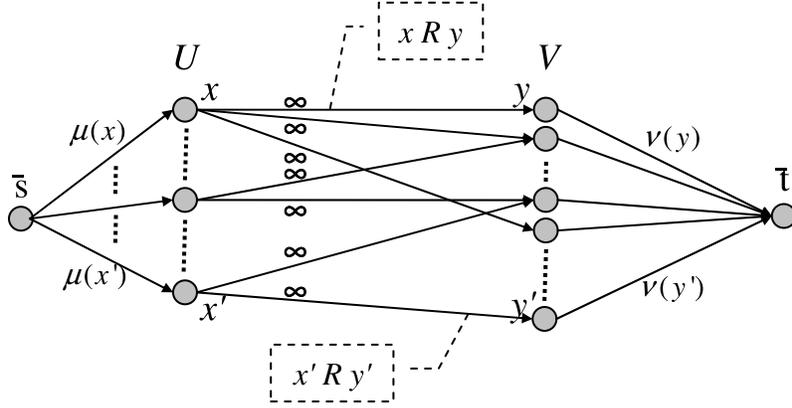
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Figure 1: The network graph $G$ in Lemma 2.2.

1. For every $x \in U$ and $y \in V$ such that $xRy$, place the edge $(x, y)$ with a capacity of $\infty$.

2. For every $x \in U$, place the edge $(\bar{s}, x)$ with a capacity of $\mu(x)$.

3. For every $y \in V$, place the edge $(y, \bar{t})$ with a capacity of $\nu(y)$.

Note that a flow of 1 from $\bar{s}$ to $\bar{t}$ in the network defined above provides the desired distribution $\varphi$, by defining $\varphi(x, y) = f(x, y)$ where $f$ is such a flow. The flow value of 1 implies the first property required from $\varphi$. The second property follows from the fact that, if $\neg(xRy)$ for some $x \in U$ and $y \in V$, then the edge $(x, y)$ is not in $G$. Altogether, if indeed $G$ has a flow of 1, then a joint distribution $\varphi$ satisfying the above properties can be computed in time polynomial in $|U| + |V|$.

The fact $G$ has a flow of 1 follows essentially from the proof of Hall's Theorem using the Max-Flow-Min-Cut Theorem (see, e.g., [10]), where inequality (5) replaces Hall's criteria for a maximal matching. Let $f(G)$ denote the maximal flow from $\bar{s}$ to $\bar{t}$ in $G$. By the Max-Flow-Min-Cut Theorem, $f$ corresponds to a minimal $(S, T)$ cut, where $\bar{s} \in S$ and $\bar{t} \in T$. Hence, if we define $A = S \cap U$ and $B = S \cap V$, the choice of the capacity $\infty$ for the edges between $U, V$ implies that $N(A) \cap V \subset B$, where $N(A)$ is the set of neighbors of $A$ in the graph $G$. Hence:

$$f = \mu(U \setminus A) + \nu(B) \geq \mu(U \setminus A) + \nu(N(A) \cap V) \geq \mu(U \setminus A) + \mu(A) = 1 \ ,$$

where the last inequality is by the assumption (5). ∎

*Proof of Theorem 2.1.* Let $\mathcal{P}, \mathcal{Q}$ denote two graph processes starting from the same initial graph. We wish to prove that, under certain conditions, the following inequality holds for every $t \in \mathbb{N}$:

$$\Pr[\mathcal{P}^t \in \mathcal{A}] \leq \Pr[\mathcal{Q}^{Mt} \in \mathcal{A}] \ . \tag{6}$$

The proof will follow from a coupling of the two processes $\mathcal{P}$ and $\mathcal{Q}$, such that for every instance of the coupling and every $t \in \mathbb{N}$, $\mathcal{P}^t$ is a subset of $\mathcal{Q}^{tM}$. More precisely, we define a joint distribution on the processes $(\mathcal{P}, \mathcal{Q})$ in the following manner: at each step $t \in \mathbb{N}$, we generate $\mathcal{P}^t$ according to

its usual law, then generate the pairs $\mathcal{Q}^{(t-1)M+1}, \ldots, \mathcal{Q}^{tM}$ according to some distribution, such that $E(\mathcal{P}^t) \subset E(\mathcal{Q}^{tM})$ and the marginal distribution of $\mathcal{Q}^{(t-1)M+1}, \ldots, \mathcal{Q}^{tM}$ is the correct one. Clearly, such a construction immediately proves (6).

For every $t \in \mathbb{N}$ and $i \in \{1, \ldots, M\}$, let $X_t^i$ and $Y_t$ denote the sets of missing edges of $\mathcal{Q}^{(t-1)M+i-1}$ and $\mathcal{P}^{t-1}$ respectively. Let $e_t \in Y_t$ denote the $t$-th edge added to $\mathcal{P}$, and let $f_t^i \in X_t^i$ denote the $((t-1)M+i)$-th edge added to $\mathcal{Q}$.

Recalling that, by definition, $\mathcal{P}^0 = \mathcal{Q}^0$, let $t \geq 1$ and assume that we have constructed the above coupling up to the point $(\mathcal{P}^{t-1}, \mathcal{Q}^{(t-1)M})$. In particular, $E(\mathcal{P}^{t-1}) \subset E(\mathcal{Q}^{(t-1)M})$, and hence $X_t^1 \subset Y_t$.

Note that in the special case $M \geq |X_t^1|$, in which $\mathcal{Q}^{tM}$ is by definition the complete graph, we can clearly generate $\mathcal{Q}^{(t-1)M+1}, \ldots, \mathcal{Q}^{(t-1)M+|X|}$ according to the usual law of $\mathcal{Q}$ and maintain the properties required from the coupling. Assume therefore that $M < |X_t^1|$.

We now wish to extend the coupling to $(\mathcal{P}^{tM}, \mathcal{Q}^t)$ using the joint distribution $\varphi$ of Lemma 2.2. To do so, take:

$$
\begin{cases}
U & = \ Y_t \\
V & = \ (X_t^1)_M \\
\mu(e) & = \ \Pr[e_t = e] \ \text{ for every } e \in Y_t \\
\nu(F) & = \ \Pr[(f_t^1, \ldots, f_t^M) = F] \ \text{ for every } F \in (X_t^1)_M \\
R & = \ \{(e, F) \in Y_t \times (X_t^1)_M \ : \ e \in (Y_t \setminus X_t^1) \text{ or } e \in F\}
\end{cases}
, \qquad (7)
$$

where $(X_t^1)_M$ denotes the set of ordered subsets of $M$ edges of $X_t^1$.

If inequality (5) holds for every $A \subset U$, then $\varphi$ associates every edge $e \in Y_t$ with a distribution on $\mathcal{Q}^{(t-1)M+1}, \ldots, \mathcal{Q}^{tM}$, which ensures that $e \in \mathcal{Q}^{tM}$, by the second property of the lemma. The choice of $\nu$, along with the first property of the lemma, ensures that $\mathcal{Q}^{(t-1)M+1}, \ldots, \mathcal{Q}^{tM}$ will have the correct marginal distributions.

Clearly, (5) holds for every $A \subset U$ such that $A \cap (Y_t \setminus X_t^1) \neq \emptyset$, since by the definition of $R$, the right hand side of (5) equals 1 in this case. It remains to show that inequality (5) indeed holds for every $A \subset X_t^1$ when $\mathcal{P}$ and $\mathcal{Q}$ play the roles of the processes $\mathcal{G}_1|_{\mathcal{H}^0}$ and $\mathcal{H}$. This is proved in the following two claims:

**Claim 2.3.** *Let $\mathcal{P} = \mathcal{H}$ and $\mathcal{Q} = \mathcal{G}_1|_{\mathcal{H}^0}$, and define $U, V, \mu, \nu, R$ as in (7). Then inequality (5) holds for every $A \subset U$.*

*Proof.* To simplify the notations, let $X = X_t^1$ and let $Y = Y_t$. By the above choices for $\mathcal{P}$ and $\mathcal{Q}$ and by the definition of the $M$-bounded weighted process, we have:

$$
\nu(F) = \Pr[(f_t^1, \ldots, f_t^M) = F] = \frac{1}{|(X)_M|} \ \text{ for every } F \in (X)_M ,
$$

$$
\mu(e) = \Pr[e_t = e] = \frac{W_t(e)}{\sum_{e' \in Y} W_t(e')} \ \text{ for every } e \in Y , \qquad (8)
$$

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where, without loss of generality:

$$\min_{e \in Y} W_t(e) = 1 \ , \ \max_{e \in Y} W_t(e) \leq M \ .$$

Substituting the values of $\mu, \nu$, inequality (5) takes the form: for every $A \subset X$,

$$\Pr[e_t \in A] \leq 1 - \frac{\binom{|X|-|A|}{M}}{\binom{|X|}{M}} \ . \tag{9}$$

Take $A \subset X$, and assume that $A \neq \emptyset, X$ (otherwise (9) trivially holds). By (8), we have:

$$\Pr[e_t \in A] = \sum_{e \in A} \Pr[e_t = e] = \frac{\sum_{e \in A} W_t(e)}{\sum_{e \in Y} W_t(e)} \leq \frac{|A|M}{|Y| + (M-1)|A|} \ ,$$

where the last inequality is by the fact that $1 \leq W_t(e) \leq M$ for every $e \in Y$, along with the inequality $\frac{a+c}{b+c} > \frac{a}{b}$ for every $0 < a < b$ and $c > 0$. Recalling that $X \subset Y$, we get:

$$\Pr[e_t \in A] \leq \frac{|A|M}{|X| + (M-1)|A|} \ . \tag{10}$$

Combining (10) with the fact that:

$$1 - \frac{\binom{|X|-|A|}{M}}{\binom{|X|}{M}} = 1 - \prod_{j=0}^{M-1} \frac{|X| - |A| - j}{|X| - j} \geq 1 - \left(1 - \frac{|A|}{|X|}\right)^M \ , \tag{11}$$

we obtain the following sufficient condition for (9):

$$\left(1 - \frac{|A|}{|X|}\right)^M \leq \frac{|X| - |A|}{|X| - |A| + M|A|} \ . \tag{12}$$

Set $\beta = 1 - \frac{|A|}{|X|}$, and recall that $0 < \beta < 1$. Inequality (12) takes the following form:

$$\beta^M \leq \frac{\beta}{\beta + (1 - \beta)M} \ ,$$

or equivalently:

$$\beta^{1-M} - (1 - M)\beta - M \geq 0 \ . \tag{13}$$

Defining $g(x) = x^{1-M} - (1 - M)x - M$, it is easy to verify that $g'(x) \leq 0$ for $0 < x < 1$ provided that $M \geq 1$, and that $g(1) = 0$. Hence, inequality (13) indeed holds, and inequality (9) follows, as required. ∎

**Claim 2.4.** *Let $\mathcal{P} = \mathcal{G}_1|_{\mathcal{H}^0}$ and $\mathcal{Q} = \mathcal{H}$, and define $U, V, \mu, \nu, R$ as in (7). Then inequality (5) holds for every $A \subset U$.*

*Proof.* Following the notation of the previous claim, define $X = X_t^1$ and $Y = Y_t$. As $\mu$ is the uniform distribution on $Y$, inequality (5) takes the following form: for every $A \subset X$,

$$\frac{|A|}{|Y|} \leq 1 - \Pr[f_t^1, \ldots, f_t^M \notin A] \ . \tag{14}$$

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Let $A \subset X$, and we may again assume $A \neq \emptyset, X$ otherwise (14) trivially holds. For $i = 1, \ldots, M$, let $W_t^i$ denote the weight function $W_{(t-1)M+i}$ on the set of edges $X_t^i$. According to this notation, the right hand side of (14) satisfies:

$$1 - \Pr[f_t^1, \ldots, f_t^m \notin A] = 1 - \prod_{i=1}^{M} \Pr[f_t^i \notin A \mid f_t^1, \ldots, f_t^{i-1} \notin A] =$$

$$= 1 - \prod_{i=1}^{M} \left( 1 - \frac{\sum_{e \in A} W_t^i(e)}{\sum_{e \in X_t^i} W_t^i(e)} \right) .$$

By the same argument used in Claim 2.3, we reduce the expression $\dfrac{\sum_{e \in A} W_t^i(e)}{\sum_{e \in X_t^i} W_t^i(e)}$ by assigning the value 1 to the weights of $A$ and $M$ to the rest. Hence,

$$1 - \Pr[f_t^1, \ldots, f_t^m \notin A] \geq 1 - \prod_{i=1}^{M} \left( 1 - \frac{|A|}{|X_t^i|M + |A|(1-M)} \right) \geq 1 - \left( \frac{|X|M - |A|M}{|X|M - |A|M + |A|} \right)^M .$$

Recalling that $X \subset Y$ and setting $\beta = 1 - \frac{|A|}{|X|}$ $(0 < \beta < 1)$, the claim then follows from the next inequality:

$$1 - \beta \leq 1 - \left( \frac{\beta}{\beta M + (1 - \beta)} \right)^M , \tag{15}$$

which holds for every $\beta \geq 0$ whenever $M \geq 1$. ∎

Combining Claim 2.3 with the arguments preceding it and Lemma 2.2, yields that it is possible to extend the coupling to $(\mathcal{P}^{tM}, \mathcal{Q}^t)$ when $\mathcal{P} = \mathcal{H}$ and $\mathcal{Q} = \mathcal{G}_1|_{\mathcal{H}^0}$, thereby completing the induction argument. Therefore, (3) holds for every $t \in \mathbb{N}$. Similarly, combining Claim 2.4 with the above arguments yields that (4) holds for every $t \in \mathbb{N}$. This completes the proof of the theorem. ∎

**Remark 2.5:** Theorem 2.1 required that one of the processes, $\mathcal{H}$, chooses each edge according to an $M$-bounded distribution, whereas the other chooses each edge according to a uniform distribution. It is not difficult to construct an example showing that this requirement cannot be replaced by the condition, that the maximal ratio between the weights of the two processes at each step is at most $M$.

Applying Theorem 2.1 on the biased graph process $\mathcal{G}_K$ gives the following immediate corollary:

**Corollary 2.6.** *Let $\mathcal{A}$ denote a monotone increasing property of graphs on $n$ vertices, and let $t \geq 0$. Then for every $K > 0$, the following two statements hold:*

1. *If $\mathcal{G}_K(t)$ almost surely satisfies $\mathcal{A}$, then $\mathcal{G}_1(\lceil \max\{K, \frac{1}{K}\} \rceil t)$ almost surely satisfies $\mathcal{A}$.*

2. *If $\mathcal{G}_1(t)$ almost surely satisfies $\mathcal{A}$, then $\mathcal{G}_K(\lceil \max\{K, \frac{1}{K}\} \rceil t)$ almost surely satisfies $\mathcal{A}$.*

## 2.2   The approximate biased graph process

In order to simplify the proofs of Theorems 1.1 and 1.3, we consider a variant of the biased process on $n$ vertices, $\widetilde{\mathcal{G}_K}$, which we dub an *approximate* biased process. At each step, a random ordered pair of vertices $(u, v) \in V^2$ (where $V$ is the set of vertices) is chosen out of the $n^2$ possible pairs, according to the following distribution: a pair of isolated vertices has a weight of 1, whereas all other pairs have weights of $K$. If the chosen pair corresponds to a (self) loop or to an edge which already exists in $\widetilde{\mathcal{G}_K}$, no edge is added in this step.

The following claim implies that it is sufficient to prove Theorems 1.1 and 1.3 for the approximate model:

**Claim 2.7.** *Let $C > 0$, and let $\{\mathcal{A}_t : 0 \leq t \leq C\}$ denote a family of properties of graphs on $n$ vertices. If for almost every approximate biased process $\widetilde{\mathcal{G}_K}$, $\widetilde{\mathcal{G}_K}(t)$ satisfies $\mathcal{A}_t$ for every $0 \leq t \leq T$, then for almost every biased process $\mathcal{G}_K$, $\mathcal{G}_K(t)$ satisfies $\mathcal{A}_t$ for every $0 \leq t \leq T$ as well.*

*Proof.* Let $B_j$ denote the event that the ordered pair, chosen in the $j$-th step, was not added to $\widetilde{\mathcal{G}_K}$, either being a loop or already belonging to $\widetilde{\mathcal{G}_K}$. Fix $C > 0$, and let $T = C\frac{n}{2}$. The probability of the event $B_j$ ($1 \leq j \leq T$) satisfies:

$$\Pr[B_j] \leq \frac{1}{n} + \frac{2(j-1)}{n^2} \leq \frac{C+1}{n} \ ,$$

and, as these events are independent, we apply the well known bound $1 - x \geq e^{-x/(1-x)}$ for $0 \leq x < 1$, and obtain:

$$\Pr[\wedge_{j=1}^T \overline{B_i}] \geq \left(1 - \frac{C+1}{n}\right)^T \geq \exp\left(-\frac{\frac{C+1}{n}T}{1 - o(1)}\right) \geq \exp(-C(C+1)) \ .$$

Notice that if we condition on the event $\wedge_{j=1}^T \overline{B_i}$, then the two graph sequences $(\widetilde{\mathcal{G}_K}^1, \ldots, \widetilde{\mathcal{G}_K}^T)$ and $(\mathcal{G}_K^1, \ldots, \mathcal{G}_K^T)$ have the same joint distribution. Since there is a fixed lower bound on the probability for this event, any statement on $(\widetilde{\mathcal{G}_K}^1, \ldots, \widetilde{\mathcal{G}_K}^T)$ which holds almost surely, also holds almost surely for $(\mathcal{G}_K^1, \ldots, \mathcal{G}_K^T)$. The result follows. $\blacksquare$

**Remark 2.8:**   In the above claim we used the low probability for a step to get omitted in order to show that statements that hold almost surely for $\mathcal{G}_K$ can be derived from such results on $\widetilde{\mathcal{G}_K}$. However, it is worth noting that the two processes are much closer than that; if we condition that $\widetilde{\mathcal{G}_K}^T$ has $M$ edges, then clearly it is distributed as $\mathcal{G}_K^M$. It is not difficult to show that, by the low probability for omitting a step, $\widetilde{\mathcal{G}_K}(t)$ has the same distribution as $\mathcal{G}_K((1 + o(1))t)$.

# 3  The behavior of the isolation ratio

## 3.1  Proof of Theorem 1.1

We begin with the special case $K = 0$, which we prove directly on $\mathcal{G}_K$. In this case, the differential equation (1) takes the simple form:

$$y' = -1 \ , \ y(0) = 1 \ ,$$

and hence its unique solution is $y(t) = 1 - t$. Notice that, as $K$ equals 0, the biased process connects two isolated vertices at each step with probability 1, as long as two such vertices exist. Hence, at time $0 \le t \le 1$, there are $\lfloor tn/2 \rfloor$ edges which are vertex disjoint in pairs. Thus the ratio of isolated vertices, $I(\mathcal{G}_K(t)) = |\mathcal{C}_0(\mathcal{G}_K(t))|/n$, satisfies:

$$|I(\mathcal{G}_K(t)) - (1 - t)| \le \frac{2}{n} \ ,$$

where the $\frac{2}{n}$-term is the rounding error.

We are left with the case $K > 0$. Fix $C > 0$, and let $G \sim \widetilde{\mathcal{G}_K}(t)$ denote a graph at some point $t < C$ along the approximate biased process $\widetilde{\mathcal{G}_K}$. We examine the effect that a single step of $\widetilde{\mathcal{G}_K}$ has on the ratio isolated vertices, $I(G) = |\mathcal{C}_0(G)|/n$.

Set $I = I(G)$; the total of the weights assigned to all ordered pairs is $(I^2 + K(1 - I^2))n^2$. Hence, with probability $\frac{I(I - \frac{1}{n})}{I^2 + K(1 - I^2)}$ the chosen edge is between two formerly isolated vertices, and with probability $\frac{2I(1 - I)K}{I^2 + K(1 - I^2)}$ precisely one end point of the chosen edge was formerly isolated. Let $G'$ denote the graph after performing the above step. The expected change in the isolation ratio between $G$ and $G'$ satisfies:

$$
\begin{aligned}
\mathbb{E}\left(I(G') - I(G)\right) &= \left(-\frac{2}{n}\right)\frac{I(I - \frac{1}{n})}{I^2 + K(1 - I^2)} + \left(-\frac{1}{n}\right)\frac{2I(1 - I)K}{I^2 + K(1 - I^2)} = \\
&= -\frac{2}{n}\left(1 - \frac{(1 - I)K}{I^2 + K(1 - I^2)} - \frac{I/n}{I^2 + K(1 - I^2)}\right) \ .
\end{aligned}
\tag{16}
$$

Define:

$$\mathrm{err}_y(I, K) = \frac{I/n}{I^2 + K(1 - I^2)} \ , \tag{17}$$

and notice that the denominator of $\mathrm{err}_y(I, K)$ lies between 1 and $K$ for every value of $I$. Hence, as $K > 0$:

$$\mathrm{err}_y(I, K) \le \frac{I/n}{\min\{1, K\}} = O(1/n) \ . \tag{18}$$

As $G'$ corresponds to $\widetilde{\mathcal{G}_K}(t + \frac{1}{n/2})$, we choose $\Delta t = 2/n$ and rewrite (16) in the following form:

$$\mathbb{E}\frac{I(\widetilde{\mathcal{G}_K}(t + \Delta t)) - I(\widetilde{\mathcal{G}_K}(t))}{\Delta t} = \frac{(1 - I)K}{I^2 + K(1 - I^2)} - 1 + \mathrm{err}_y(I, K) \ . \tag{19}$$

Notice that the left hand side of (19) resembles $\frac{dI}{dt}$, suggesting that the expected change in the isolation ratio $I(\widetilde{\mathcal{G}_K}(t))$ is linked with the solution to the differential equation (1). Notice that $y'$

is $C^\infty$, and hence there is a unique solution to (1): we analyze this solution in the next subsection, and now turn to show that indeed it approximates $I(\widetilde{\mathcal{G}}_K(t))$. This will follow from a general purpose theorem of [16] (Theorem 5.1), which we reformulate according to our needs, for the sake of simplicity (a simpler version of the theorem appears in [15]). If $\vec{y} = (y_1, \ldots, y_k)$, we use the notation $(x, c\vec{y})$ to describe the tuple $(x, cy_1, \ldots, cy_k)$.

**Theorem 3.1** ([16]). *Let $\vec{Y} = \vec{Y}(n) = (Y_1, \ldots, Y_l)$ denote $l \geq 1$ functions from graphs on $n$ vertices to the real interval $[-C_0 n, C_0 n]$, where $C_0 > 0$ is some constant, and let $\vec{f} = (f_1, \ldots, f_l)$ denote $l$ functions in $\mathbb{R}^{l+1} \to \mathbb{R}$. Let $\{H^T\}$ denote a graph process on $n$ vertices beginning with the edgeless graph. Let $\mathcal{D} \subset \mathbb{R}^{l+1}$ be a bounded connected open set such that $(0, \frac{\vec{Y}(H^0)}{n}) \in \mathcal{D}$, and let $T_{\mathcal{D}}(H)$ denote the minimal time $T$ such that $(\frac{T}{n}, \frac{\vec{Y}(H^T)}{n})$ no longer belongs to $\mathcal{D}$. Assume the following:*

1. *(Boundedness Hypothesis) For some function $\beta = \beta(n) \geq 1$, $|\vec{Y}(H^{T+1}) - \vec{Y}(H^T)|_\infty \leq \beta$ for every $T \leq T_{\mathcal{D}}(H)$.*

2. *(Trend Hypothesis) For some function $\lambda_1 = \lambda_1(n) = o(1)$, the following holds for every $T \leq T_{\mathcal{D}}(H)$: $\left|\mathbb{E}\left(\vec{Y}(H^{T+1}) - \vec{Y}(H^T)\right) - \vec{f}(\frac{T}{n}, \frac{\vec{Y}(H^T)}{n})\right|_\infty \leq \lambda_1$.*

3. *(Lipschitz Hypothesis) There exists a constant $L > 0$ such that the following holds for every $\vec{x}, \vec{y} \in \mathcal{D}$: $|\vec{f}(\vec{x}) - \vec{f}(\vec{y})|_\infty \leq L|\vec{x} - \vec{y}|_\infty$.*

*Then the following holds:*

1. *There exists a unique solution to the following system of $l$ differential equations:*

$$
\begin{cases}
\frac{du_1}{dx} = f_1(x, \vec{u}), & \ldots, \quad \frac{du_l}{dx} = f_l(x, \vec{u}) \\
\vec{u}(0) = \frac{\vec{Y}(H^0)}{n}
\end{cases},
$$

   *where $\vec{u}$ denotes $(u_1, \ldots, u_l)$. Let $\tilde{u}_1(x), \ldots, \tilde{u}_l(x)$ denote this solution.*

2. *Let $\lambda > \lambda_1$, $\lambda = o(1)$, and let $\sigma > 0$ be such that $(x, \vec{\tilde{u}}(x))$ is at least $C_1\lambda$ away from the boundary of $\mathcal{D}$ for every $x \in [0, \sigma]$, where $C_1 > 0$ is some constant. Then there exists a constant $C_2 > 0$, such that with probability $1 - O(\frac{\beta}{\lambda}\exp(-\frac{n\lambda^3}{\beta^3}))$, the following holds for every $0 \leq T \leq \sigma n$ and every $1 \leq i \leq l$: $|\frac{Y_i(H^T)}{n} - \tilde{u}_i(\frac{T}{n})| \leq C_2\lambda$.*

**Remark 3.2:** Note that for the purpose of proving Theorem 1.1 we will only need the special case $l = 1$ of Theorem 3.1 above. However, in Section 4, when analyzing the critical point for the emerging time of the giant component, we apply the above theorem for $l = 2$.

We claim that the following substitution completes the proof of Theorem 1.1:

$$
\begin{aligned}
Y(G) &= |\mathcal{C}_0(G)|, \\
f(x, y) &= 2\left(\frac{(1-y)K}{y^2 + (1-y^2)K} - 1\right).
\end{aligned}
$$

For the set $\mathcal{D}$ we choose a bounded connected open set containing the rectangle $[0, C] \times [0, 1]$. Notice that $0 \leq \frac{Y(G)}{n} \leq 1$ for every graph $G$ on $n$ vertices, hence $(\frac{T}{n}, \frac{Y(\widetilde{\mathcal{G}_K}^T)}{n})$ belongs to $\mathcal{D}$ for every $1 \leq T \leq Cn/2$ and $T_\mathcal{D}(\widetilde{\mathcal{G}_K})$ is at least $C$.

Indeed, $|Y(\widetilde{\mathcal{G}_K}^{T+1}) - Y(\widetilde{\mathcal{G}_K}^T)| \leq 2$, thus a choice of $\beta = 2$ confirms the Boundedness Hypothesis. Next, (16), (17) and (18) imply that

$$|\mathbb{E}\left(Y(\widetilde{\mathcal{G}_K}^{T+1}) - Y(\widetilde{\mathcal{G}_K}^T)\right) - f(\frac{T}{n}, \frac{Y(\widetilde{\mathcal{G}_K}^T)}{n})| = 2\mathrm{err_y}(\frac{Y(\widetilde{\mathcal{G}_K}^T)}{n}, K) = O(1/n) .$$

Thus, setting $\lambda_1 = \frac{1}{n}$ verifies the Trend Hypothesis. Finally, $f(x, y)$ is clearly $C^\infty$ (recall that $K > 0$) and hence satisfies the Lipschitz condition.

Proposition 3.3, proved in the next subsection by analyzing the differential equation (1), states that the solution $\tilde{u}(x)$ is bounded between $\exp(-x/Q)$ and $\exp(-Qx)$. Hence, for every $0 \leq x \leq C$ the solution $\tilde{u}(x)$ is at least a constant away from the boundary of $\mathcal{D}$, and we can easily choose $\sigma = C$ (regardless of our choice of $\lambda$).

Altogether, a choice of $\lambda = n^{-1/4}$ implies that with probability $1 - O(n^{1/4} \exp(-n^{1/4})) = 1 - o(\exp(-\frac{1}{2}n^{1/4})$, the quantity $Y(G)$ satisfies:

$$\frac{Y(\widetilde{\mathcal{G}_K}^T)}{n} = \tilde{u}(\frac{T}{n}) + O(n^{-1/4}) ,$$

for every $0 \leq T \leq Cn/2$, where $\tilde{u}$ is the (unique) solution to the equation $\frac{du}{dx} = f(x, u)$.

Scaling the time to units of $n/2$ edges, we obtain that, with probability $1 - o(\exp(-\frac{1}{2}n^{1/4})$, the following holds for every $0 \leq t \leq C$:

$$|I(\widetilde{\mathcal{G}_K}(t)) - y(t)| = O(n^{-1/4}) ,$$

where $y(x) = \tilde{u}(x/2)$. Hence, $y$ is the unique solution to the equation:

$$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} = \frac{(1-y)K}{y^2 + (1-y^2)K} - 1 ,$$

with the starting condition $y(0) = \tilde{u}(0) = 1$, completing the proof. ∎

## 3.2 The behavior of $y(t)$ when $K > 0$

The behavior of $y(t)$ along the biased process is crucial to the understanding of how the susceptibility grows, as we show in the next section. The behavior of the ratio of isolated vertices (and the corresponding function $y(t)$) was already stated for the case $K = 0$. We thus assume $K > 0$, and set $Q = \lceil \max\{K, \frac{1}{K}\} \rceil$. By Corollary 2.6, we obtain that for every $\varepsilon > 0$ and every $C > 0$, the following holds with high probability for every $t \in [0, C]$:

$$\mathrm{e}^{-Qt} - \varepsilon \leq I(\mathcal{G}_K(t)) \leq \mathrm{e}^{-t/Q} + \varepsilon . \tag{20}$$

This follows from the uniform continuity of the functions $\exp(-t/Q)$ and $\exp(-Qt)$, which describe $I(\mathcal{G}_1(t/Q))$ and $I(\mathcal{G}_1(Qt))$ respectively.

We next show that a stronger result than (20) can be easily derived directly from the differential analysis of $y(t)$, as stated by the next proposition:

**Proposition 3.3.** *The ODE* (1) *for $K > 0$ has the following properties:*

1. *Its solutions are strictly monotone increasing in $K$. That is, if $y_1(t), y_2(t)$ are the solutions that correspond to $K_1 < K_2$, then $y_1(t) < y_2(t)$ for all $t > 0$.*

2. *The solution $y(t)$ is strictly monotone decreasing in $t$, and satisfies:*

$$\exp\left(-\frac{t}{\min\{1, K\}}\right) \leq y(t) \leq \exp\left(-\frac{t}{\max\{1, K\}}\right), \tag{21}$$

*for all $t \geq 0$. Furthermore, the inequalities are strict whenever $K \neq 1$.*

*Proof.* We begin by proving an inequality analogous to (20), stating that whenever $K \neq 1$, $y(t)$ is strictly monotone decreasing in $t$ and satisfies:

$$\exp(-Qt) < y(t) < \exp(-t/Q) \tag{22}$$

for any $t > 0$, where $Q = \max\{\frac{1}{K}, K\}$. Inequality (21) will follow directly from (22) once we prove that $y$ is strictly monotone increasing in $K$, since $\exp(-t)$ is the solution to the ODE (1) for $K = 1$.

Suppose $K > 1$ and let $u(t) = \exp\{-t/Q\}$. First, we prove that $u$ satisfies the inequality

$$u' > \frac{(1 - u)K}{u^2 + (1 - u^2)K} - 1, \quad \text{for } t > 0. \tag{23}$$

Indeed, $u' = -\frac{1}{Q}\exp\{-t/Q\} = -\frac{u}{Q}$. Therefore, (23) holds iff

$$-\frac{u}{Q} > \frac{(1 - u)K}{u^2 + (1 - u^2)K} - 1, \quad \text{for } 0 < u < 1. \tag{24}$$

which, after some manipulations, is equivalent to the inequality

$$p_Q(u) \equiv (K - 1)u^2 - Q(K - 1)u + K(Q - 1) > 0, \quad \text{for } 0 < u < 1.$$

The quadratic polynomial $p_Q(u)$ satisfies $p_Q(u) \to +\infty$ as $u \to \pm\infty$, because $K > 1$. Furthermore, if $Q > 1$ then $p_Q(0) = K(Q - 1) > 0$ and $p_Q(1) = Q - 1 > 0$. Therefore, if $p_Q(u)$ does not have a root in the interval $(0, 1)$ then the inequality holds. In the special case of $Q = K$ we have $p_K(u) = (K - 1)\left(u^2 - Ku + K\right)$. The roots of $p_K(u)$ are $u_\pm = \frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - 4K}$. For $1 < K < 4$ there are no real roots. For $K = 4$ there is a double root at $u = 2$ and for $K > 4$ there are two distinct roots $u_\pm > 1$. Therefore, the inequality (23) holds for all values of $Q = K > 1$ as asserted. By the equivalent inequality (24) (recall that $0 < y < 1$ for $t > 0$), we obtain that $y'(t) < 0$ for every $t \geq 0$, and hence $y(t)$ is strictly monotone decreasing.

15

To complete the proof of the upper bound of (22), let $f(w) = \dfrac{(1-w)K}{w^2 + (1-w^2)K} - 1$. The differential equation (1) and inequality (23) imply that the functions $y(t)$ and $u(t)$ satisfy for $t > 0$

$$
\begin{aligned}
y' &= f(y), \\
u' &> f(u),
\end{aligned}
$$

together with the mutual initial condition $y(0) = u(0) = 1$. By standard analytical considerations, this implies that $u(t) > y(t)$ for all $t > 0$ (to see this, note that $u(t) > y(t)$ for $t \in [0, \varepsilon]$ and some small $\varepsilon > 0$, and that the smallest point $t^* > \varepsilon$ satisfying $u(t^*) = y(t^*)$ cannot satisfy $u'(t^*) - y'(t^*) > f(u(t^*)) - f(y(t^*)) = 0$, yielding a contradiction).

The lower bound in (22) and the case $0 < K < 1$ are similar.

To prove that $y$ is strictly monotone increasing in $K$, take $0 < K_1 < K_2$, and let $y_1$ and $y_2$ the solutions of equation (1) for $K_1$ and $K_2$ respectively. Considering $f$, defined as above, as a function of both $w$ and $K$, $f$ satisfies:

$$
\frac{\partial f}{\partial K}(w, K) = \frac{(1-w)w^2}{(w^2 + (1-w^2)K)^2} \; ,
$$

hence $\frac{\partial f}{\partial K}(w, K) > 0$ for every $0 < w < 1$. Recall that inequality (22) guarantees that $y(t) > 0$ for every $t$. Therefore, for any $t > 0$, $f(y(t), K)$ is monotone increasing in $K$, and in particular, for any $t > 0$ we have $y_1' = g(y_1)$ and $y_2' > g(y_2)$, where $g(y) = f(y, K)$. Thus, the above argument for proving inequality (22) completes the proof. ∎

We note that the bounds of (21) are much weaker than the estimations which can be obtained by examining the asymptotic behavior of the ODE (1), as we proceed to do in Section 5.

## 4 The susceptibility of the biased process

### 4.1 Proof of Proposition 1.2

We prove Proposition 1.2 by showing that there exists an $M > 0$ such that $0 < t_c(K) < M$ for every $K > 0$. The fact that $z(t)$ is continuous on $[0, t_c)$ follows from standard considerations in differential analysis. The case $K = 1$ is trivial, as in this case equation (2) takes the simple form

$$
z' = z^2 \; , \; z(0) = 1 \; ,
$$

hence its solution is $1/(1 - t)$ and $t_c(1) = 1$.

Take $K > 1$, and let $z(t)$ denote the (unique) solution to equation (2). Recall that by Proposition 3.3, $y(t)$, the solution to equation (1), satisfies:

$$
\mathrm{e}^{-t} < y(t) < \mathrm{e}^{-t/K}
$$

for every $t > 0$. Thus, $y(t) > 0$ for every $t \geq 0$, and we obtain that $\frac{K}{y^2 + (1 - y^2)K} > 1$ for every $t \geq 0$. Therefore:

$$z' > z^2 \; , \; z(0) = 1 \; ,$$

and by the method used in the proof of Proposition 3.3, we obtain that $z(t) \geq 1/(1 - t)$ for every $t \geq 0$. We deduce that $0 < t_c(K) \leq 1$ for every $K > 1$.

Let $0 < K < 1$, and again let $y(t)$ and $z(t)$ denote the solutions to equations (1) and (2) respectively. Recalling that $y(0) = 1$ and $y(t)$ decreases to 0 as $t \to \infty$, let $t^* > 0$ be such that $y(t^*) = \sqrt{K}$. It is easy to verify that the derivative of $f(y) = \frac{(1-y)K}{y^2 + (1-y^2)K} - 1$ is strictly negative for every $0 \leq y \leq 1$. Thus, (1) implies that for every $0 \leq t \leq t^*$,

$$y'(t) = f(y) \leq f(\sqrt{K}) = \frac{1 - \sqrt{K}}{2 - K} \; . \tag{25}$$

By the Mean Value Theorem, we obtain that:

$$t^* \leq \frac{1 - \sqrt{K}}{1 - \frac{1 - \sqrt{K}}{2 - K}} = \frac{1}{\frac{1}{1 - \sqrt{K}} - \frac{1}{2 - K}} \leq 2 \; , \tag{26}$$

where the last inequality is by the fact that the function $\frac{1}{1 - \sqrt{K}} - \frac{1}{2 - K}$ is monotone increasing from $\frac{1}{2}$ to $\infty$ as $K$ goes from 0 to 1. In addition, by (2), $z'(t) > 0$ for every $t \geq 0$ (this applies to every $K \geq 0$). Therefore:

$$z(t^*) > z(0) = 1 \; . \tag{27}$$

Hence:

$$c(t) = \frac{K}{y^2 + (1 - y^2)K} > \frac{1}{2} \tag{28}$$

for every $t \geq t^*$. By (27) and (28), defining $\hat{z}(t) = z(t + t^*)$ gives:

$$\hat{z}' > \frac{1}{2}\hat{z}^2 \; , \; \hat{z}(0) > 1 \; , \tag{29}$$

and by comparing $\hat{z}$ with the function $\frac{2}{2 - t}$ we deduce that $\hat{z}$ has a singularity point $\hat{t}_c \leq 2$. Altogether, $z(t)$ satisfies $t_c \leq \hat{t}_c + t^* \leq 4$, completing the proof. $\blacksquare$

**Remark 4.1:** Theorem 1.4 states that $t_c(K)$ is in fact monotone decreasing as a function of $K$ and thus bounded by $t_c(0) = \frac{3}{2}$. We note that Theorem 1.3, used in the proof of Theorem 1.4, requires a bound on $t_c(K)$.

## 4.2 The susceptibility at the subcritical phase

In this subsection we combine Theorem 2.1 with methods from [13] to prove part 1 of Theorem 1.3. The proof relies on the fact that a bounded susceptibility ensures a logarithmic upper bound for the components, and this in turn ensures that the solution to the differential equation (2) stays a good approximation for the susceptibility. Hence, the fact that $S(\mathcal{G}_K(t))$ is approximated by $z(t)$ and the fact that every component is of size $O(\ln n)$ ensure each other along the subcritical phase.

As we next show, the special case $K = 0$ again follows from the behavior of $S(\mathcal{G}_1)$. Indeed, (2) takes the following simple form when $K = 0$:

$$z' = 1 \ , \ z(0) = 1 \ ,$$

and hence its unique solution is $z(t) = 1 + z$. By definition, as $K = 0$, as long as there exists a pair of isolated vertices in the graph, such a pair forms the next edge. This amounts to trading two components of size 1 with one component of size 2, and hence at each such step the susceptibility, being the sum of squares of the component sizes, increases by $\frac{2}{n}$. Thus, at time $0 \le t \le 1$, the susceptibility equals $1 + \frac{2}{n}\lfloor tn/2 \rfloor$, hence:

$$|S(\mathcal{G}_K(t)) - (1+t)| \le \frac{2}{n} \ ,$$

where the $\frac{2}{n}$-term is the rounding error. As of this point, the process is equivalent to $\mathcal{G}_1$ on $\lfloor \frac{n+1}{2} \rfloor$ vertices, $\lfloor n/2 \rfloor$ of which represent components of size 2. Hence, the result on $S(G)$ from this point on is derived from the case $K = 1$ of Theorem 1.3. The solution for the differential equation (2) when $K$ equals 1 is $z = \frac{1}{1-t}$, hence, we obtain that the susceptibility is within $o(1)$ of the function $\hat{z} = \frac{2}{1-2(t-1)} = \frac{1}{3/2-t}$ for $t \ge 1$.

Throughout the remainder of the proof, assume therefore that $K > 0$. We need the following definition: a graph $G$ is said to have a $k, c$ *component tail* if, for every $s$, the probability that a uniformly chosen vertex has a component of size at least $s$ is at most $k \exp(-cs)$:

$$\Pr_{v \in V(G)}[|\mathcal{C}_v| \ge s] \le k e^{-cs} \ .$$

Notice that if a graph $G$ on $n$ vertices has this property then its largest component is of size at most $\max\{1, 1/c\} \ln n$, provided that $n$ is sufficiently large.

The following theorem ([13], Theorem 3.1) is crucial to the proof:

**Theorem 4.2** ([13]). *Let $\ell, k, c$ be positive real numbers. Let $G$ be a graph on $n$ vertices with a $k, c$ component tail. Let $H$ be a random graph with edge probability $p = \frac{\hat{t}}{n}$ on the same vertex set, where $\hat{t}$ is fixed. Set $G^+ = G \cup H$.*

1. *(Subcritical) Assume $S(G) \le \ell$ for all $n$. Let $\hat{t}\ell < 1$. Then there exist $k^+$, $c^+$ (dependent on $k, c, \ell, \hat{t}$ but not on $n$ nor $G$) such that $G^+$ almost surely has a $k^+,c^+$ component tail. In particular, all components have size $O(\ln n)$.*

2. *(Supercritical) Assume $S(G) > \ell$. Let $\hat{t}\ell > 1$. Then $G^+$ almost surely has a giant component. More precisely, there exists $\gamma > 0$ (dependent on $k, c, \ell, \hat{t}$ but not on $n$ nor $G$) such that $G^+$ has a component of size at least $\gamma n$.*

The following lemma proves part 1 of Theorem 1.3 (analogous to Theorem 4.2 of [13]):

**Lemma 4.3.** *Let $\varepsilon > 0$. For every $t \in [0, t_c - \varepsilon]$, where $t_c$ is as defined in Proposition 1.2, the following two conditions hold almost surely:*

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1. (*Susceptibility Approximation*) $|S(\mathcal{G}_K(t)) - z(t)| = o(1)$, *where* $z(t)$ *is the solution to the differential equation* (2).

2. (*Small Components*) $\mathcal{G}_K(t)$ *has a* $k, c$ *component tail, for some* $k = k(t), c = c(t)$.

*Proof.* Set $Q = \lceil \max\{\frac{1}{K}, K\} \rceil$, and let:

$$L = \lceil (z(t_c - \varepsilon) + \varepsilon) Q t_c \rceil , \ \ t_j = \frac{t_c - \varepsilon}{L} j \ (j = 0, \dots, L) . \tag{30}$$

Assume by induction that properties 1 and 2 of the lemma almost surely hold for $t \in [0, t_j]$ for some $0 \leq j < L$ (the case $j = 0$ is trivial). We show that with probability $1 - o(1)$ properties 1 and 2 hold for the interval $[t_j, t_{j+1}]$ as well, and the lemma follows from a union bound on the complement events.

First, consider the Small Components property. By the induction hypothesis and the monotonicity of $z(t)$ (recall that $z'(t) > 0$ for all $t \geq 0$, as mentioned in the proof of Proposition 1.2),

$$S(\mathcal{G}_K(t_j)) \leq z(t_j) + o(1) \leq z(t_L) + o(1) .$$

Let:

$$\ell = z(t_L) + \varepsilon , \ \hat{t} = Q \frac{t_L}{L} . \tag{31}$$

With high probability, $\mathcal{G}_K(t_j)$ has a $k, c$ component tail for some constants $k, c$ (dependent on $t_j$), $S(\mathcal{G}_K(t_j)) \leq \ell$ and $\hat{t} \ell \leq \frac{t_c - \varepsilon}{t_c} < 1$. Thus, Theorem 4.2 implies that $\mathcal{G}_1(\hat{t})|_{\mathcal{G}_K(t_j)}$ almost surely has a $k^+, c^+$ component tail for some constants $k^+, c^+$. By Theorem 2.1 (inequality (3)) we obtain that $\mathcal{G}_K(\hat{t}/Q)|_{\mathcal{G}_K(t_j)}$ almost surely has a $k^+, c^+$ component tail, hence $\mathcal{G}_K(t_{j+1})$ satisfies the Small Components property. Thus, in particular, with high probability all the components of $\mathcal{G}_K(t_{j+1})$ are of size $O(\ln n)$.

We next analyze the change in the susceptibility along the approximate biased process. Since $\mathcal{G}_K^T$ is equivalent to $\widetilde{\mathcal{G}_K}^{T'}$ for some $T' > T$ (where $T' - T$ is the number of redundant steps in $\widetilde{\mathcal{G}_K}$), the Small Components property which we proved above implies that, with high probability, the largest component of $\widetilde{\mathcal{G}_K}(t_{j+1})$ is of size at most $\alpha \ln n$ for some $\alpha > 0$. Assume therefore that this indeed is the case.

Let $G$ denote an instance of $\widetilde{\mathcal{G}_K}(t)$ on $n$ vertices for some $t < t_{j+1}$, and set $I = I(G)$ and $S = S(G)$. Recall that the total of the weights assigned to all ordered pairs is $(I^2 + K(1 - I^2))n^2$, and that in case that the chosen pair in the next step is a loop or an edge which already exists in $G$, the step is omitted. Let $G'$ denote the graph after the next step, and let $A_{C_1, C_2}$ denote the event in which the newly chosen pair joins the two components $C_1, C_2 \in \mathcal{C}(G)$. If $G'$ equals $G$ or if the new edge is an internal edge of some component $C \in \mathcal{C}(G)$ (the event $A_{C,C}$ occurred), then $S(G') = S(G)$. Otherwise, a new edge between $C_1, C_2 \in \mathcal{C}(G)$ implies:

$$S(G') = \frac{1}{n} \sum_{C \in \mathcal{C}(G')} |C|^2 = S + \frac{2}{n} |C_1||C_2| . \tag{32}$$

Hence:

$$\frac{\mathbb{E}(S(G') - S(G))}{2/n} = \sum_{C_1 \in \mathcal{C}} \sum_{\substack{C_2 \in \mathcal{C} \\ C_1 \neq C_2}} |C_1||C_2| \Pr[A_{C_1, C_2}] . \tag{33}$$

There are two cases to consider, according to which we divide the summands of the right hand side of (33) to two terms, $\Delta_1$ and $\Delta_2$. In the first case, a new edge $(u, v)$ is added between two distinct and formerly isolated vertices $u, v$. Since $|\mathcal{C}_u| = |\mathcal{C}_v| = 1$, the contribution of this case to the sum in (33) is its probability:

$$\Delta_1 = \sum_{u \neq v \in \mathcal{C}_0} \Pr[A_{\mathcal{C}_u, \mathcal{C}_v}] = \frac{I^2}{I^2 + (1 - I^2)K} - \Pr[\cup_{u \in \mathcal{C}_0} A_{\mathcal{C}_u, \mathcal{C}_u}] . \tag{34}$$

In remaining case, $A_{C_1, C_2}$ occurs where $C_1 \neq C_2$ and at least one of the components $C_1, C_2$ is not an isolated vertex. The contribution to the right hand side of (33) in this case is:

$$\Delta_2 = \sum_{\substack{C_1 \in \mathcal{C} \\ C_2 \in \mathcal{C}}} \frac{K|C_1|^2|C_2|^2}{(I^2 + (1 - I^2)K)n^2} - \frac{I^2 K}{I^2 + (1 - I^2)K} - \sum_{\substack{C \in \mathcal{C} \\ |C| > 1}} \frac{K|C|^4}{(I^2 + (1 - I^2)K)n^2}. \tag{35}$$

Define:

$$\begin{aligned} \mathrm{err_z}(I, K) &= \Pr[\cup_{u \in \mathcal{C}_0} A_{\mathcal{C}_u, \mathcal{C}_u}] + \sum_{\substack{C \in \mathcal{C} \\ |C| > 1}} \frac{K|C|^4}{(I^2 + (1 - I^2)K)n^2} = \\ &= \frac{K}{I^2 + (1 - I^2)K} \sum_{C \in \mathcal{C}} \frac{|C|^4}{n^2} + \frac{1 - K}{I^2 + (1 - I^2)K} \frac{I}{n} . \end{aligned} \tag{36}$$

Combining (34),(35) and (36), the following holds:

$$\Delta_1 + \Delta_2 = \frac{(1 - K)I^2}{I^2 + (1 - I^2)K} + \frac{K}{I^2 + (1 - I^2)K} \sum_{\substack{C_1 \in \mathcal{C} \\ C_2 \in \mathcal{C}}} \frac{|C_1|^2|C_2|^2}{n^2} - \mathrm{err_z}(I, K) ,$$

and therefore:

$$\frac{\mathbb{E}(S(G') - S(G))}{2/n} = \frac{(1 - K)I^2}{I^2 + (1 - I^2)K} + \frac{K}{I^2 + (1 - I^2)K}S^2(G) - \mathrm{err_z}(I, K) . \tag{37}$$

This explains the choice of the differential equation (2), provided that $\mathrm{err_z}$ is redundant; this is ensured by the fact that the largest component of $G$ is of size at most $\alpha \ln n$. To see this, notice that $G$ satisfies $\frac{1}{n} \sum_{C \in \mathcal{C}(G)} |C|^4 \leq \sum_{C \in \mathcal{C}(G)} \frac{|C|}{n}(\alpha \ln n)^3 = (\alpha \ln n)^3$. We obtain the following bound on $\mathrm{err_z}$:

$$\mathrm{err_z}(I, K) \leq \frac{K}{(I^2 + (1 - I^2)K} \frac{\alpha^3 \ln^3 n}{n} + \frac{1 - K}{I^2 + (1 - I^2)K} \frac{I}{n} = O(\frac{\ln^3 n}{n}) . \tag{38}$$

It is left to verify the conditions of Theorem 3.1. Although we may assume that the statements of Theorem 1.1 hold with the respect to the approximate biased process, and $|I(\widetilde{\mathcal{G}_K}(t)) - y(t)| = O(n^{-1/4})$ for every $0 \leq t \leq t_c$, where $y$ is the unique solution to (1), the $O(n^{-1/4})$ approximation

error will not be sufficient for proving that $S(G)$ is approximated by $z(G)$. The difficulty is in proving the Trend Hypothesis; $dS(G)/dt$ is approximated by $g_1(I)+g_2(I)S^2(G)$ for some continuous functions $g_1, g_2$, and using the $y$-approximation for $I(G)$ implies an $O(n^{-1/4})S^2(G)$ error. Therefore, we apply Theorem 3.1 on both $I(G)$ and $S(G)$ (re-proving the result on $I(G)$ for the interval $[0, t_{j+1}]$).

Keeping the notations of the theorem, define:

$$
\begin{aligned}
Y_1(G) &= nI(G) = |\mathcal{C}_0(G)| \ , \\
Y_2(G) &= nS(G) = \sum_{C \in \mathcal{C}(G)} |C|^2 \ , \\
f_1(x,y,z) &= 2(\frac{(1-y)K}{y^2 + (1-y^2)K} - 1) \ , \\
f_2(x,y,z) &= 2(\frac{K}{y^2 + (1-y^2)K}\left(z^2 - 1\right) + 1) \ .
\end{aligned}
$$

The set $\mathcal{D}$ will be a bounded connected open set containing the domain $[0, t_{j+1}] \times [0, 1] \times [0, \ell]$, where $\ell$ is the constant defined in (31). Clearly, $\mathcal{D}$ contains $(\frac{T}{n}, \frac{\vec{Y}(\widetilde{\mathcal{G}_K}^T)}{n})$ as long as $T \leq t_{j+1}n/2$ and $S(\widetilde{\mathcal{G}_K}^T) \leq \ell$, and in particular, for $T = 0$.

Recall that the first coordinate of $\vec{Y}$ satisfies the Boundedness Hypothesis for a choice of $\beta = 2$ and every $T \leq t_{j+1}n/2$. Since every component of $G$ is of size at most $\alpha \ln n$, (32) implies that $|Y_2(\widetilde{\mathcal{G}_K}^{T+1}) - Y_2(\widetilde{\mathcal{G}_K}^T)| \leq 2(\alpha \ln n)^2$, thus a choice of $\beta = 2\alpha^2 \ln n$ verifies the Boundedness Hypothesis for every $T \leq t_{j+1}n/2$.

The Trend Hypothesis was satisfied by $Y_1, f_1$ in the proof of Theorem 1.1 using a choice of $\lambda_1 = 1/n$ ($f_1$ is independent of $z$). By (37) and (38), we have:

$$
|\mathbb{E}\left(Y_2(\widetilde{\mathcal{G}_K}^{T+1}) - Y_2(\widetilde{\mathcal{G}_K}^T)\right) - f_2(\frac{T}{n}, \frac{\vec{Y}(\widetilde{\mathcal{G}_K}^T)}{n})| = 2\mathrm{err}_z(I, K) = O(\frac{\ln^3 n}{n}) \ .
$$

Therefore, a choice of $\lambda_1 = O(\frac{\ln^3 n}{n})$ verifies the Trend Hypothesis.

The Lipschitz condition is again satisfied by the fact that $f_i(x,y,z)$, $i = 1, 2$, are clearly $C^\infty$ for $K > 0$.

Notice that our definition of $\ell$, relying on Proposition 1.2, is such that the unique solution to the system of differential equations remains at least $\varepsilon$-far from the boundary of the domain $\mathcal{D}$ for every $0 \leq x \leq t_{j+1}$, hence a choice of $\sigma = t_{j+1}$ is valid.

Altogether, a choice of $\lambda = n^{-1/4}$ implies that with probability $1 - o(\exp(-\frac{1}{2}n^{1/4}))$ there exists some constant $C_2 > 0$ such that the following holds for every $0 \leq T \leq t_{j+1}n/2$:

$$
\left| \frac{\vec{Y}(\widetilde{\mathcal{G}_K}^T)}{n} - (\tilde{y}(\frac{T}{n}), \tilde{z}(\frac{T}{n})) \right|_\infty \leq C_2 n^{-1/4} \ ,
$$

where $(\tilde{y}, \tilde{z})$ is the (unique) solution to the equation system:

$$\begin{cases} \frac{dy}{dx} = f_1(x, y, z), \ \frac{dz}{dx} = f_2(x, y, z) \\ y(0) = z(0) = 1 \end{cases} .$$

Scaling the time to units of $n/2$ edges, we obtain that, with probability $1 - o(\exp(-\frac{1}{2}n^{1/4}))$, the following holds for every $0 \leq t \leq t_{j+1}$:

$$|S(\widetilde{\mathcal{G}_K}(t)) - z^*(t)| = O(n^{-1/4}) ,$$

where $z^*(x) = \tilde{z}(x/2)$. Hence, $z^*$ is the unique solution to the equation:

$$\frac{dz^*}{dx} = \frac{1}{2}\frac{d\hat{z}}{dx} = \frac{K}{\tilde{y}^2 + (1 - \tilde{y}^2)K}z^2 + \frac{(1 - K)\tilde{y}^2}{\tilde{y}^2 + (1 - \tilde{y}^2)K} ,$$

with the initial condition $z^*(0) = \tilde{z}(0) = 1$, completing the proof of the lemma. ∎

## 4.3 The susceptibility at the supercritical phase

**Proof of Theorem 1.3 part 2**. Let $Q = \lceil\max\{K, \frac{1}{K}\}\rceil$, and assume without loss of generality that $\varepsilon < t_c$. By Proposition 1.2 there exists some $0 < t^- < t_c$ satisfying $z(t^-) = \frac{Q+2}{\varepsilon}$. Lemma 4.3 implies that, with high probability,

$$S(\mathcal{G}_K(t^-)) \geq \frac{Q + 2}{\varepsilon}(1 - o(1)) > \frac{Q + 1}{\varepsilon} ,$$

and in addition, $\mathcal{G}_K(t^-)$ has a $k, c$ component tail. Fix $H = \mathcal{G}_K(t^-)$. Setting $\ell = \frac{Q+1}{\varepsilon}$ and $\hat{t} = \frac{\varepsilon}{Q}$, Theorem 4.2 implies that $\mathcal{G}_1(\frac{\varepsilon}{Q})|_H$ almost surely has a giant component. Hence, by Theorem 2.1 (inequality (4)), $\mathcal{G}_K(\varepsilon)|_H = \mathcal{G}_K(t^- + \varepsilon)$ almost surely has a giant component, and in particular, $\mathcal{G}_K(t_c + \varepsilon)$ has a giant component with high probability.

∎

## 4.4 Computer simulations and numerical results

We have implemented an efficient simulator of the model $\mathcal{G}_K$, which runs in a time complexity of $O(n \log n)$, and uses a data structure of size $O(n)$. Maintaining the sets of isolated and non-isolated vertices allows an $O(1)$ cost for randomizing the next edge at each step whenever one of its endpoints is isolated; the cost of selecting an edge between two non-isolated vertices is $O(\log n)$, where the set of existing edges is kept in a balanced tree, which is an efficient data-structure. Finally, maintaining the connected components in linked lists according to the Weighted-Union Heuristic (see, e.g., [8] p. 445) provides an average cost of $O(\log n)$ for uniting components.

We conducted a series of simulations with $n = 10^6$, and values of $K$ ranging from 0 to 10 in steps of 0.1. In order to assess the critical time $t_c(K)$, we considered the minimal time at which
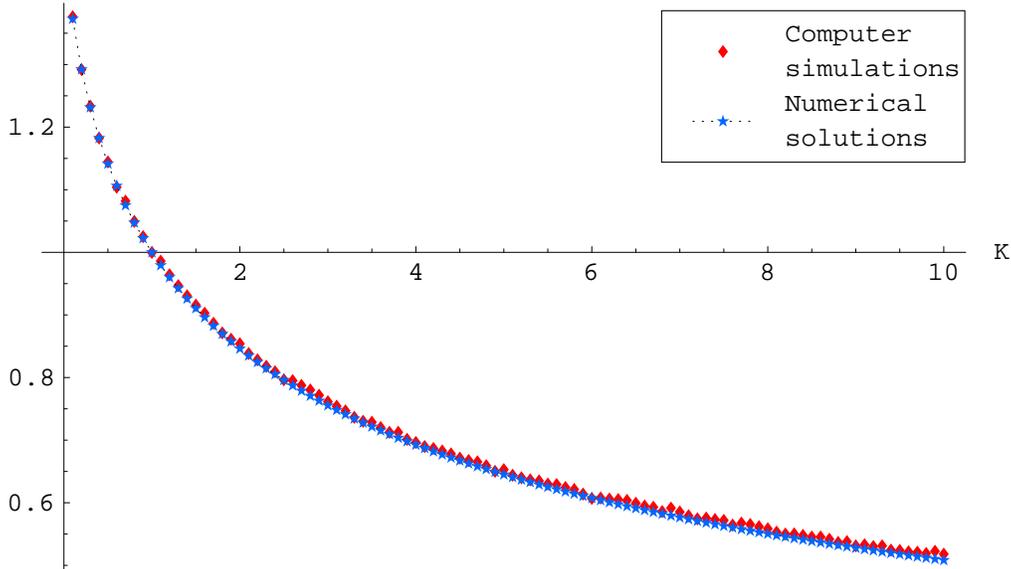
Figure 2: Comparison of the numerical results for $t_c(K)$, and estimations of $t_c(K)$ according to computer simulations of the model.

the largest component of $\mathcal{G}_K$ was of size $\alpha n$, where $\alpha = 0.01$. These estimations of $t_c(K)$ were supported by tests of higher values of $n$, such as $2.5 \cdot 10^6$ and $5 \cdot 10^6$ vertices. We note that the constant $\alpha$ decreases with $K$ (independently of $n$). Therefore, automatically assessing $t_c(K)$ for large values of $K$ requires a different method, such as a threshold for the derivative of the size of the largest component, or for the derivative of the susceptibility.

The simulated value of $t_c(K)$ in each setting was averaged over 10 tests conducted as mentioned above. Figure 2 shows an excellent agreement between these values and the numerical approximations of $t_c(K)$, obtained by numerical solutions of the ODEs (1) and (2) by Mathematica.

# 5 Analytical properties of the critical time

In this section we prove Theorem 1.4 regarding the analytical properties of the critical time $t_c(K)$.

## 5.1 Asymptotical Behavior of $t_c(K)$ for $K \gg 1$

**Proof of Theorem 1.4**. The proof begins with the analysis of $t_c(K)$ as $K \to \infty$, whose leading order asymptotics is constructed by the method of matched asymptotics.

The differential equation (1) for $y(x)$

$$y' = \frac{(1-y)K}{y^2 + (1-y^2)K} - 1,$$ (39)

with its initial condition $y(0) = 1$, suggests the transformation $u = 1 - y$, so $u \ll 1$ near the origin, and satisfies

$$-u' = \frac{Ku}{(1-u)^2 + Ku(2-u)} - 1,$$

with the initial condition $u(0) = 0$. Hereafter $\approx$ means equality of leading order terms. For $u \ll 1$ we have

$$u' \approx \frac{1 + Ku}{1 + 2Ku},$$ (40)

whose solution is

$$2Ku - \ln(1 + Ku) \approx Kx.$$ (41)

For $K^{-1} \ll u \ll 1$ the logarithmic term is smaller compared to the linear term, hence

$$u(x) \approx \frac{x}{2}, \quad \text{for } K^{-1} \ll x \ll K^{-\varepsilon},$$ (42)

where $\varepsilon > 0$ is needed to ensure $u \ll 1$.

We rewrite the differential equation (2) for $z(x)$ as

$$z' = c(x)(z^2 - 1) + 1,$$ (43)

where

$$c(x) = \frac{K}{y^2 + (1-y^2)K}.$$ (44)

We recognize (43) as a Riccati equation (see, e.g., [11]). However, it has no closed-form solution for a general function $c(x)$, and we resort to asymptotic methods.

We obtain an asymptotic approximation of $z(x)$ by the method of matched asymptotics. As shown below, we first construct an asymptotic solution that is valid for $K^{-1} \ll x \ll K^{-2/3}$, which is then matched to a different asymptotic solution that is valid for $x \gg K^{-3/4}$ up to the blowup point.

First, we use a linear approximation of (43) near $x = 0$, by substituting $v = z - 1$ that satisfies

$$v' = c(v^2 + 2v) + 1,$$ (45)

with the initial condition $v(0) = 0$, so $v$ is small near the origin. Therefore, as long as the nonlinear term satisfies $cv^2 \ll 1$, it can be viewed as a perturbation to the linear equation

$$v' - 2cv = 1 + O(cv^2).$$ (46)

Multiplying (46) by its integrating factor we obtain

$$v(x) \approx \exp\left\{ \int 2c(x)\,dx \right\} \int \exp\left\{ -\int 2c(x')\,dx' \right\} dx,$$ (47)

Equations (40) and (44) imply

$$c \approx \frac{Ku'}{1 + Ku}. \tag{48}$$

Therefore,

$$\int c(x)\, dx \approx \int \frac{Ku'}{1 + Ku}\, dx = \ln(1 + Ku). \tag{49}$$

Hence, by equation (47) we have

$$v \approx (1 + Ku)^2 \int \frac{dx}{(1 + Ku)^2}. \tag{50}$$

We calculate the integral in equation (50) by implying equation (40)

$$
\begin{aligned}
\int \frac{dx}{(1 + Ku)^2} &= \int \frac{u'}{(1 + Ku)^2} \frac{1 + 2Ku}{1 + Ku}\, dx = \int \left[ \frac{2}{(1 + Ku)^2} - \frac{1}{(1 + Ku)^3} \right] du \\
&= \frac{1}{K} \left[ -\frac{2}{1 + Ku} + \frac{1}{2(1 + Ku)^2} + \frac{3}{2} \right]
\end{aligned}
$$

Thus,

$$v \approx \frac{1}{2} u(2 + 3Ku),$$

which combined with equation (42) for $x \gg K^{-1}$ gives

$$v(x) \approx \frac{3}{8} K x^2. \tag{51}$$

Equations (44) and (42) imply that $c(x)$ is to leading order

$$c(x) \approx \frac{K}{1 + 2Ku} \approx \frac{1}{x}, \quad \text{for } K^{-1} \ll x \ll K^{-\varepsilon}. \tag{52}$$

Recall that the linear approximation for $v(x)$ is valid whenever $cv^2 \ll 1$. Equations (51) and (52) give $cv^2 \approx K^2 x^3$, hence the linear approximation (46) holds for $x \ll K^{-2/3}$. Thus, the approximation (51) holds for $K^{-1} \ll x \ll K^{-2/3}$.

Second, we choose an intermediate point $x_1$ such that $K^{-1} \ll x_1 \ll K^{-2/3}$, e.g., $x_1 = K^{-3/4}$, for which $c(x_1) = O\left(K^{3/4}\right)$, $v(x_1) = O\left(K^{-1/2}\right)$, and $z(x_1)^2 - 1 = O\left(K^{-1/2}\right)$. Therefore, $c(x_1)\left(z(x_1)^2 - 1\right) = O\left(K^{1/4}\right)$ and we may approximate equation (43) by neglecting the constant term $1 \ll K^{1/4}$

$$z' \approx c(x)(z^2 - 1), \tag{53}$$

with the initial condition

$$z_1 \equiv z(x_1) = 1 + v(x_1) \approx 1 + \frac{3}{8} K x_1^2. \tag{54}$$

The blowup point $t_c$ satisfies

$$\int_{z_1}^{\infty} \frac{dz}{z^2 - 1} = \int_{x_1}^{t_c} c(x)\, dx. \tag{55}$$

The left hand side of (55) is

$$\int_{z(x_1)}^{\infty} \frac{dz}{z^2 - 1} = \frac{1}{2} \ln\left( \frac{z_1 + 1}{z_1 - 1} \right),$$

whereas the right hand side of (55) is calculated by equation (49) and the approximation (42)

$$\int_{x_1}^{t_c} c(x)\, dx \approx \ln\left(\frac{1 + Ku_c}{1 + Ku_1}\right) \approx \ln\left(\frac{1 + \frac{1}{2}Kt_c}{1 + \frac{1}{2}Kx_1}\right).$$

Therefore,

$$\frac{z_1 + 1}{z_1 - 1} \approx \left(\frac{1 + \frac{1}{2}Kt_c}{1 + \frac{1}{2}Kx_1}\right)^2 \approx \frac{t_c^2}{x_1^2} \tag{56}$$

Substituting (54) and rearranging we obtain

$$t_c^2 \approx x_1^2 \frac{2 + \frac{3}{8}Kx_1^2}{\frac{3}{8}Kx_1^2} \approx \frac{16}{3K}. \tag{57}$$

Note that the result is independent of the choice of the intermediate point $x_1$, as expected. Hence

$$t_c \approx \frac{4}{\sqrt{3K}}, \quad \text{for } K \gg 1, \tag{58}$$

is the leading order asymptotics of the blow up point. Throughout the asymptotic analysis we used merely the leading order terms in their validity regimes. Therefore, the higher order terms are guaranteed to be asymptotically smaller than the leading order term, that is,

$$t_c(K) = \frac{4}{\sqrt{3K}}\left(1 + o(1)\right), \quad \text{for } K \gg 1. \tag{59}$$

We constructed numerical solutions of the ODEs (1),(2) for various values of $K$ using Mathematica. Figure 3 shows the comparison of the numerical approximation for $t_c(K)$ and the asymptotic result $4/\sqrt{3K}$.

## 5.2 Strict monotone decreasing behavior of $t_c(K)$

The continuity of the critical point $t_c(K)$ follows from the general continuous dependence of ODEs on their parameters, which is in our case a single parameter $K$. Thus, to complete the proof we must show that $t_c(K)$ is strictly monotone decreasing in $K$.

We first show that $c(x)$ is strict monotone increasing in $K$, i.e., for any $0 < K_1 < K_2$, the corresponding functions $c_1, c_2$ (where $y$ is replaced by the corresponding solutions to (1)) satisfy $c_1(x) < c_2(x)$ for every $x \geq 0$.

Recall that by the properties of $y$, as stated in Proposition 3.3, $c(0) = K$ and for every $K \neq 0, 1$, $c(x)$ either strictly increases or strictly decreases to 1 as $x \to \infty$ (depending on whether $K > 1$ or $K < 1$). Hence, if $K_1 \leq 1 < K_2$ or $K_1 < 1 \leq K_2$, clearly $c_1(x) < c_2(x)$ for every $x \geq 0$. Furthermore, if $1 < K_1 < K_2$, the strict monotonicity of $y$ in $K$ implies that for every $x > 0$:

$$c_1 = \frac{1}{y_1^2(\frac{1-K_1}{K_1}) + 1} < \frac{1}{y_2^2(\frac{1-K_2}{K_2}) + 1} = c_2 \ .$$
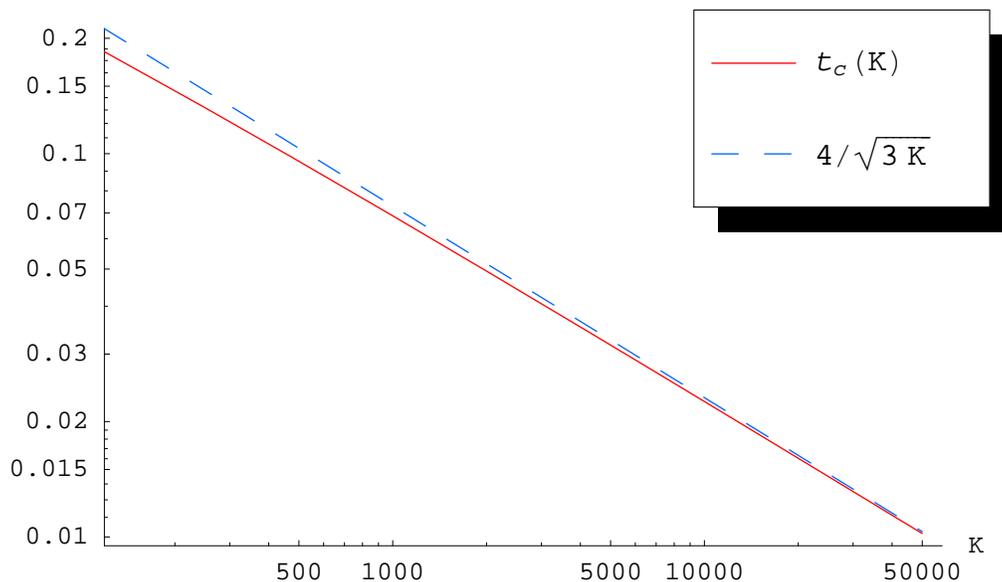
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Figure 3: Comparison of the numerical results for $t_c(K)$ obtained by numerical solutions of the ODEs (1)-(2), and the asymptotic approximation of Theorem 1.4. Logarithmic scale was used in both axes.

Assume therefore that $0 < K_1 < K_2 < 1$. Chain rule differentiation of (44), together with (1), gives:

$$c'(x) = -2c^2 \frac{1-K}{K} y \left( (1-y)c + 1 \right) . \tag{60}$$

Rearranging equation (44), we get:

$$y = \left( \frac{(1-c)K}{c(1-K)} \right)^{1/2} . \tag{61}$$

Combining equations (60) and (61), we obtain an equality of the form $c'(x) = g(c, K)$, where $g(c, K)$ satisfies:

$$\frac{\partial g}{\partial K}(c, K) = \frac{c^2(1+c)}{K} \sqrt{\frac{(1-c)K}{c(1-K)}} > 0 , \tag{62}$$

for all $0 < K < 1$ and $x \geq 0$. By the argument stated in the proof of Proposition 3.3, this implies that $c$ is indeed strictly monotone increasing in $K$.

Returning to equation (43), since $z' = c(z^2 - 1) + 1$ and $c$ is strictly monotone increasing in $K$, we conclude that $z'(t)$ is monotone increasing in $K$ for any $t > 0$ (recall that, as stated in the proof of Proposition 1.2, $z'(t) > 0$ for any $t \geq 0$ and hence $z(t) > 1$ for any $t > 0$). Hence, the above consideration implies that $z$ is strictly monotone increasing in $K$. Therefore, $t_c(K)$ is monotone decreasing in $K$, that is,

$$t_c(K_1) \geq t_c(K_2), \quad \text{for } K_1 < K_2. \tag{63}$$

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Yet, the fact that $z$ is strictly monotone increasing in $K$ does not guarantee by itself that $t_c(K)$ is *strictly* monotone decreasing in $K$.

To this end, we consider solutions $z_1, z_2$ and $c_1, c_2$ corresponding to $K_1 < K_2$ and prove that $t_c(K_1) > t_c(K_2)$. The functions $c_1$ and $c_2$ are continuous on $[0, \infty)$ and satisfy $c_1 < c_2$, therefore $\exists \varepsilon > 0$ such that

$$c_2(x) - c_1(x) > \varepsilon, \quad \forall x \in [0, t_c(K_1)]. \tag{64}$$

The function (whose choice is made clear below)

$$A(z) = \frac{\ln \dfrac{z^2}{z^2 - 1}}{\ln \dfrac{z+1}{z}}$$

satisfies $\lim_{z \to \infty} A(z) = 0$. Since $z_1(x) \to \infty$ as $x \to t_c(K_1)$, there exists a point $0 < t^* < t_c(K_1)$, such that

$$z_1(t^*) \geq \max\{1, K_1^{-1}\} \quad \text{and} \quad A(z_1(t^*)) < \frac{2\varepsilon}{\max\{1, K_1\}}. \tag{65}$$

Let $z_1^*(x)$ be the solution of the differential equation

$$z' = c_1(x)z^2 + c_1(x)z, \quad z(t^*) = z_1(t^*). \tag{66}$$

Recall that $z_1(x)$ is the solution of

$$z' = c_1(x)z^2 + 1 - c_1(x), \quad z(t^*) = z_1(t^*). \tag{67}$$

Comparing (66) and (67) we note that $c_1(x)z_1^*(x) > 1 - c_1(x)$ for all $x \geq t^*$, because $z_1^*(x)$ is monotone increasing in $x$, $c_1(x) \geq \min\{1, K_1\}$, and the choice of $t^*$ in equation (65), indicated by

$$c_1(x)z_1^*(x) \geq c_1(x)z_1^*(t^*) \geq \min\{1, K_1\} \max\{1, K_1^{-1}\} = 1 > 1 - c_1(x).$$

Hence, $z_1^*(x) > z_1(x)$ for all $x > t^*$ and

$$t_c(K_1) \geq t_c^*(K_1), \tag{68}$$

where $t_c^*(K_1)$ is the blowup point of $z_1^*(x)$. Similar considerations show that the blowup point $t_c^*(K_2)$ of the solution, $z_2^*$, to the differential equation

$$z' = c_2(x)(z^2 - 1), \quad z(t^*) = z_2(t^*), \tag{69}$$

(remark: if $t^* \geq t_c(K_2)$ then $t_c(K_1) > t_c(K_2)$ and the proof is completed, so we assume $t^* < t_c(K_2)$) satisfies

$$t_c^*(K_2) \geq t_c(K_2), \tag{70}$$

because the positive term 1 was omitted in (69) compared to (43). The advantage of the differential equations (66) and (69) over the original equation (43) is that their blowup points satisfy closed

form relations. Namely,

$$\ln\left(\frac{z_1^*(t^*)+1}{z_1^*(t^*)}\right) = \int_{t^*}^{t_c^*(K_1)} c_1(x)\,dx, \tag{71}$$

$$\frac{1}{2}\ln\left(\frac{z_2^*(t^*)+1}{z_2^*(t^*)-1}\right) = \int_{t^*}^{t_c^*(K_2)} c_2(x)\,dx. \tag{72}$$

Assume to the contrary that $t_c(K_1) = t_c(K_2)$. Then, by equations (68) and (70) it follows that

$$t_c^*(K_2) \geq t_c^*(K_1). \tag{73}$$

Combining equations (71),(72),(73),(64) and the positivity of $c_2(x)$, we obtain

$$\frac{1}{2}\ln\left(\frac{z_2^*(t^*)+1}{z_2^*(t^*)-1}\right) \geq \int_{t^*}^{t_c^*(K_1)} c_2(x)\,dx > \varepsilon(t_c^*(K_1)-t^*) + \int_{t^*}^{t_c^*(K_1)} c_1(x)\,dx$$
$$\tag{74}$$
$$= \ln\left(\frac{z_1^*(t^*)+1}{z_1^*(t^*)}\right) + \varepsilon(t_c^*(K_1)-t^*).$$

Since $\ln\frac{z+1}{z-1}$ is a monotonic decreasing function of $z$, and $z_1^*(t^*) < z_2^*(t^*)$, it follows that

$$\ln\left(\frac{z_1^*(t^*)+1}{z_1^*(t^*)-1}\right) - 2\ln\left(\frac{z_1^*(t^*)+1}{z_1^*(t^*)}\right) > 2\varepsilon(t_c^*(K_1)-t^*), \tag{75}$$

or equivalently

$$\ln\left(\frac{z_1^*(t^*)^2}{z_1^*(t^*)^2-1}\right) > 2\varepsilon(t_c^*(K_1)-t^*). \tag{76}$$

Employing equation (71) with $c_1(x) \leq \max\{1, K_1\}$, we find

$$t_c^*(K_1) - t^* \geq \frac{1}{\max\{1, K_1\}}\ln\left(\frac{z_1^*(t^*)+1}{z_1^*(t^*)}\right). \tag{77}$$

Equations (75) and (77) give

$$A(z_1^*(t^*)) = \frac{\ln\left(\dfrac{z_1^*(t^*)^2}{z_1^*(t^*)^2-1}\right)}{\ln\left(\dfrac{z_1^*(t^*)+1}{z_1^*(t^*)}\right)} > \frac{2\varepsilon}{\max\{1, K_1\}}, \tag{78}$$

which contradicts (65). Therefore, $t_c(K_1) > t_c(K_2)$. ∎

# 6 Concluding remarks and open problems

As mentioned in [13], there is a well known relation between the "double jump" phenomenon in $\mathcal{G}_1$ and the critical threshold in percolation. Indeed, the susceptibility corresponds to the size of the component containing the origin, and the appearing of giant component is analogous to the appearing of an infinite cluster. The subcritical and supercritical stages in the $\mathcal{G}_1$ correspond to

those in percolation. However, it is unclear how to define a percolation process which is analogous to the biased process.

We conclude by describing several natural ways to extend the family of biased processes and their analysis.

Consider processes where the probability of adding an edge at each step is biased according to the sizes of the components of its endpoints. For instance, it is possible to assign the weight $K_{i,j}$ to a missing edge $(u, v)$, such that $|\mathcal{C}_u| = i, |\mathcal{C}_u| = j$ for $i, j < C$, where $C > 0$ is some large constant, and some weight $K_C$ otherwise. It seems plausible that the above methods can be applied in this scenario as-well.

The biased process $\mathcal{G}_K$ was defined as a graph process on the complete graph on $n$ vertices, $K_n$; in other words, at each step we considered the entire set of missing edges, and the process ended when obtaining $K_n$. It is possible to define an analogous process on a different underlying graph $H = (V, E)$, where the edge pool at each step contains only edges belonging to $E$, and the process ends once $H$ is obtained. For instance, it is possible to consider the biased process on $Q_n$, the $n$-th dimensional cube, or $L_2(n)$, the $n \times n$ grid. In these cases, it may be interesting to study the critical time for satisfying other monotone increasing properties, such as left-right crossing in $L_2(n)$. It seems plausible that this crossing probability is monotone decreasing in $K$.

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# References

[1] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, Springer, New York (1999).

[2] B. Bollobás, *Random graphs*, volume 73 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second edition (2001).

[3] T. Bohman and A. Frieze, Avoiding a giant component, Random Structures and Algorithms, 19 (2001), 75-85.

[4] T. Bohman, A. Frieze and N. Wormald, Avoidance of a giant component in half the edge set of a random graph, Random Structures and Algorithms, 25 (2004), 432-449.

[5] T. Bohman and J.H. Kim, A phase transition for avoiding a giant component, Random Structures and Algorithms, 28 (2006), 195-214.

[6] T. Bohman and D. Kravitz, Creating a giant component, Combinatorics, Probability and Computing , to appear.

[7] A. Flaxman, D. Gamarnik and G. Sorkin, Embracing the giant component, Random Structures and Algorithms, 27 (2005), 277-289.

[8] T. H. Cormen, C.E. Leiserson, and R.L. Rivest, *Introduction to Algorithms*, The MIT Press/McGraw-Hill (1990).

[9] P. Erdős and A. Rényi, On the evolution of random graphs, Publ. math. Inst. Hungar. Acad. Sci., 5 (1960), 17-61.

[10] L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland, Amsterdam (1986).

[11] G.M. Murphy, *Ordinary Differential Equations and Their Solutions*, D. van Nostrand Comp., Inc. Princeton, New Jersey (1960).

[12] C.J. Preston, A generalization of the FKG inequalities. Communications in Mathematical Physics, 36 (1974), 233-241.

[13] J. Spencer and N. Wormald, Birth control for giants, Combinatorica, 27 (2007), 587-628.

[14] V. Strassen, The existence of probability measures with given marginals, Annals of Mathematical Statistics, 36 (1965), 423-439.

[15] N.C. Wormald, Analysis of greedy algorithms on graphs with bounded degrees, Discrete Mathematics, 273 (2003), 235-260.

[16] N.C. Wormald, The differential equation method for random graph processes and greedy algorithms, Lectures on Approximation and Randomized Algorithms, M. Karoński and H.J. Prömel (eds), PWN, Warsaw (1999), 73-155.