

1. (25pts) The volume of the solid that lies under the function  $F(x, y) = (x + y)^2$  and above the square  $R = \{(x, y) \mid 0 \leq x \leq A, 0 \leq y \leq A\}$  is 1. Find  $A$ .

$$\int_0^A \int_0^A (x+y)^2 dy dx = 1$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$\int_0^A \int_0^A (x^2 + 2xy + y^2) dy dx = 1$$

$$\int_0^A \int_0^A x^2 dA + \iint_0^A 2xy dA + \int_0^A \int_0^A y^2 dA = 1$$

$$= \int_0^A x^2 dx \cdot \int_0^A 1 dy + \int_0^A 2x dx \cdot \int_0^A y dy + \int_0^A y^2 dy \cdot \int_0^A 1 dx = 1$$

$$\frac{1}{3}x^3 \Big|_{x=0}^{x=A} \cdot y \Big|_{y=0}^{y=A} + x^2 \Big|_{x=0}^{x=A} \cdot \frac{1}{2}y^2 \Big|_{y=0}^{y=A} + \frac{1}{3}y^3 \Big|_{y=0}^{y=A} \cdot x \Big|_{x=0}^{x=A} = 1$$

$$\frac{1}{3}A^3 \cdot A + A^2 \cdot \frac{1}{2}A^2 + \frac{1}{3}A^3 \cdot A = 1$$

$$\frac{1}{3}A^4 + \frac{1}{2}A^4 + \frac{1}{3}A^4 = 1$$

$$\frac{2A^4 + 3A^4 + 2A^4}{6} = 1$$

$$A^4 = \frac{6}{7}$$

$$A = \sqrt[4]{\frac{6}{7}}$$

2. (35pts) Find the absolute maximum and minimum of the function  $F(x, y, z) = xz + y + y^2$  in the ball of radius 1 around the origin. (be careful of square roots, they can be negative!)

ball of radius 1:  $x^2 + y^2 + z^2 \leq 1$ .

$$F(x, y, z) = xz + y + y^2 \quad G(x, y, z) = x^2 + y^2 + z^2 \leq 1$$

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle \\ = \langle z, 1+2y, x \rangle$$

critical pts, when  $\nabla F = 0$ ,

$$\begin{aligned} z &= 0, \\ 1+2y &= 0 \quad y = -\frac{1}{2}, \\ x &= 0 \end{aligned}$$

critical pt:  $(0, -\frac{1}{2}, 0)$

boundary, when  $G(x, y, z) = x^2 + y^2 + z^2 = 1$ .

$$\nabla G = \langle 2x, 2y, 2z \rangle$$

$$\nabla F = \lambda \nabla G \quad x^2 + y^2 + z^2 = 1 \quad \textcircled{1}$$

$$\begin{aligned} \lambda &= \frac{z}{2x} \quad \leftarrow \quad \textcircled{2} \quad z = 2x\lambda \\ \textcircled{2} \quad 1+2y &= 2y\lambda \\ \lambda = \frac{x}{2z} \quad \textcircled{3} \quad x &= 2z\lambda \end{aligned}$$

re-arranging  $\textcircled{2}$ ,

$$2y\lambda - 2y = 1$$

$$y(2\lambda - 2) = 1$$

$$y = \frac{1}{2\lambda - 2}$$

$$\therefore \cancel{\frac{1}{2\lambda - 2}} \quad \cancel{\frac{1}{2\lambda - 2}}$$

$$y = \frac{1}{2(\frac{z}{2x}) - 2} = \frac{1}{\frac{z}{x} - 2} = \frac{x}{z - 2x}$$

if  $z = x$ ,

$$y = \frac{1}{1-2} = -1$$

if  $z = -x$ ,

$$y = \frac{1}{-1-2} = -\frac{1}{3}$$

$$\textcircled{1} \div \textcircled{2}. \quad \frac{z}{x} = \frac{2x\lambda}{2y\lambda}$$

$$\cancel{\frac{z}{x}} = \frac{x}{z}$$

but what  $z^2 = x^2$

if  $x = 0$   $z = \pm x$ .  $\textcircled{4}$

~~nope~~

~~nope~~

$$\text{min max point} = \cancel{2x^2 + y^2 + z^2} \quad 1+2 = z^2(\lambda)$$

$$1+2 = z^2(\lambda)$$

$$1 = \frac{z^2}{9} + \frac{1}{z^2} \lambda^2$$

$$1 = 1$$

for  $y = \cancel{\frac{z}{2x}}$ , &  $z = x$   $(0, -1, 0)$

$$x^2 + x + x^2 = 1$$

$$2x^2 = 0$$

$$x = 0 \quad \therefore x = 0, z = 0, y = -1$$

$$(0, -1, 0)$$

$$\cancel{(0, -1, 0)}$$

$$-1 = -1 \checkmark$$

$$\checkmark$$

for  $y = \cancel{\frac{z}{2x}}$  &  $z = -x$ ,

$$x^2 + \frac{1}{9} + x^2 = 1$$

$$2x^2 = \frac{8}{9}$$

$$x^2 = \frac{4}{9}$$

$$x = \pm \frac{2}{3}, z = \mp \frac{2}{3}, y = -\frac{1}{3}$$

$$(\pm \frac{2}{3}, -\frac{1}{3}, \mp \frac{2}{3})$$

$$\cancel{(\pm \frac{2}{3}, -\frac{1}{3}, \mp \frac{2}{3})}$$

continue at back.

candidate points:  $(0, 1, 0)$   
 ~~$(0, -\frac{1}{2}, 0)$~~   
 $(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$   
 $(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$

critical point:  $(0, -\frac{1}{2}, 0)$

$$F(x, y, z) = xz + y + y^2$$

$$F(0, -\frac{1}{2}, 0) = 0 - \frac{1}{2} + \frac{1}{4} = \frac{1}{2}$$

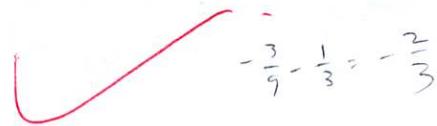
$$F(0, 1, 0) = 0 + 1 + 1 = 2 \leftarrow \text{abs max.}$$

$$F(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}) = \frac{2}{3}(-\frac{2}{3}) + (-\frac{1}{3}) + \frac{1}{9} = -\frac{4}{9} - \frac{1}{3} + \frac{1}{9} = -\frac{1}{3} + \frac{-3}{9} = -\frac{2}{3}$$

$$F(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}) = (-\frac{2}{3})(\frac{2}{3}) + (-\frac{1}{3}) + \frac{1}{9} = -\frac{4}{9} - \frac{1}{3} + \frac{1}{9} = -\frac{2}{3} \leftarrow \text{abs min.}$$

The absolute maximum of the function is  $F(0, 1, 0) = 2$

The absolute minimum of the function is  $F(\pm \frac{2}{3}, -\frac{1}{3}, \mp \frac{2}{3}) = -\frac{2}{3}$



$$-\frac{3}{9} - \frac{1}{3} = -\frac{2}{3}$$

3. Let  $f(r)$  be a function with second order derivative. Let  $z = F(x, y) = f(x^2 + y^2)$ .

(a) (10pts) Show that  $y \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial y}$ .

(b) (10pts) Show that if  $f$  satisfies  $r \frac{\partial^2 f}{\partial r^2} = -\frac{\partial f}{\partial r}$  then  $z$  satisfies  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

(c) (10pts) Let  $z = \log(x^2 + y^2)$  (where log is base  $e$ ). Show that  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

$$z = F(x, y) = f(x^2 + y^2) \quad \text{let } x^2 + y^2 = u$$

$$z = F(x, y) = f(u)$$

$$\cancel{\frac{\partial z}{\partial x}} = \cancel{\frac{\partial z}{\partial u}} \cdot \cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial z}{\partial y}} \cdot \cancel{\frac{\partial u}{\partial y}}$$

$$= \cancel{\frac{\partial z}{\partial u}} \cdot \cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial z}{\partial u}} \cdot \cancel{\frac{\partial u}{\partial y}}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot (2x) = f'(x^2 + y^2) \cdot 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = f'(u) \cdot (2y) = f'(x^2 + y^2) \cdot 2y$$

$$y \frac{\partial z}{\partial x} = 2xy f'(x^2 + y^2)$$

$$x \frac{\partial z}{\partial y} = 2xy f'(x^2 + y^2)$$

$$\therefore y \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial y}.$$

$$b) z = f(x^2 + y^2) \quad \text{let } x^2 + y^2 = r$$

$$\cancel{\frac{\partial z}{\partial x}} = \cancel{f'(u)} \cdot \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial z}{\partial r} \cdot 2x$$

$$\frac{d}{dx} \left( \frac{\partial z}{\partial x} \right) = \frac{d}{dx} \left( \frac{\partial z}{\partial r} \right) \cdot 2x + \frac{\partial z}{\partial r} \cdot 2$$

$$\frac{d}{dx} \left( \frac{\partial z}{\partial r} \right) = \frac{d}{dx} (f'(x^2 + y^2)) = f''(x^2 + y^2) \cdot 2x$$

$$\frac{d^2 z}{dx^2} = f''(x^2 + y^2) \cdot (4x^2) + 2 \cancel{\frac{\partial z}{\partial r}} f'(x^2 + y^2)$$

Similarly,

$$\frac{d}{dy} \left( \frac{\partial z}{\partial y} \right) = f''(x^2 + y^2) (4y^2) + 2f'(x^2 + y^2)$$

$$r \frac{d^2 f}{dr^2} = -\frac{df}{dr}$$

$$\frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} = 4x^2 f''(x^2 + y^2) + 2f'(x^2 + y^2) + 4y^2 f''(x^2 + y^2) + 2f'(x^2 + y^2)$$

$$\text{if } r = x^2 + y^2,$$

$$\textcircled{1} \quad (x^2 + y^2) f''(x^2 + y^2) = -f'(x^2 + y^2) \quad 4 = f''(x^2 + y^2) (4x^2 + 4y^2) + 4f'(x^2 + y^2)$$

Subbing in \textcircled{1},

$$f''(x^2 + y^2) (4x^2 + 4y^2) + 4(- (x^2 + y^2) f''(x^2 + y^2)) = 0 \quad \checkmark$$

$$\therefore \frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} = 0 \quad \text{if } r \frac{df}{dr} = -\frac{df}{dr}.$$

continue at back.

$$z = \log(x^2 + y^2) \quad \text{Show} \quad \frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} = 0.$$

$$\frac{dz}{dx} = \frac{1}{x^2 + y^2} \cdot 2x \quad \frac{d}{dx} \left( \frac{dz}{dx} \right) = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2}$$

$$\frac{dz}{dy} = \frac{1}{x^2 + y^2} \cdot 2y \quad \frac{d}{dy} \left( \frac{dz}{dy} \right) = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2}$$

$$\begin{aligned} \frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} &= \frac{2(x^2 + y^2) - 4x^2 + 2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} \\ &= \frac{\cancel{2x^2} + \cancel{2y^2} - \cancel{4x^2} + \cancel{2x^2} + \cancel{2y^2} - \cancel{4y^2}}{(x^2 + y^2)^2} = 0 \end{aligned}$$

$$\therefore \frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} = 0$$

$$x^2 + 2xy + y^2$$

4. Let  $F(x, y) = e^x + e^y + (x+y)^2$  and let  $g(x)$  satisfy the equation  $F(x, g(x)) = 2$  and  $g(0) = 0$  and  $g$  has a second order derivative at 0.

$$F(0, 0) = 2.$$

(a) (10pts) Find  $g'(0)$ .

(b) (10pts) Find  $g''(0)$ .

$$a) \frac{dF}{dx} = e^x + e^y \frac{dy}{dx} + 2x + 2y + 2x \frac{dy}{dx} + 2y \frac{dy}{dx}.$$

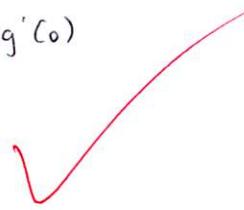
$$\Downarrow 0 = e^x + e^y g'(x) + 2x + 2y + 2xg'(x) + 2yg'(x) \quad ①$$

at  $(0, 0)$

$$= e^0 + e^0 g'(0) + 0 + 0 + 0 \cdot g'(0) + 0 \cdot g'(0)$$

$$= 1 + g'(0)$$

$$\boxed{g'(0) = -1}$$



$$\frac{dy}{dx} = g'(x) = -\frac{F_x}{F_y} = -\frac{e^x + 2x + 2y}{e^y + 2x + 2y}$$

$$\text{at } (0, 0) = -\frac{e^0}{e^0} = -1 \quad \checkmark$$

b) take  $\frac{d}{dx}$  of eq. ①.

$$0 = e^x + e^y g'(x) g'(x) + e^y g''(x) + 2 + 2g'(x) + 2g'(x) + 2xg''(x)$$

$$+ 2g'(x)g'(x) + 2y g''(x)$$

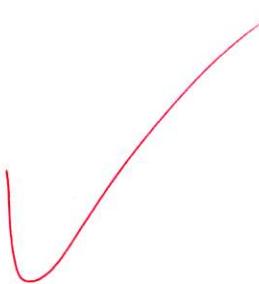
$$\text{at } (0, 0), \quad x=0, \quad y=0, \quad g'(0) = -1.$$

$$0 = e^0 + e^0 \cdot (-1)(-1) + e^0 g''(0) + \cancel{2} + \cancel{2(-1)} + 2(-1) + \cancel{2(0)g''(x)}$$

$$+ 2(-1)(-1) + \cancel{2(0)g''(x)}$$

$$= 1 + 1 + g''(0) - \cancel{2} + \cancel{2}$$

$$\boxed{g''(0) = -2}$$



Scratch sheet