1. (25pts) The volume of the solid that lies under the function $F(x, y) = (x + y)^2$ and above the square $R = \{(x, y) \mid 0 \le x \le A, 0 \le y \le A\}$ is 1. Find A.

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$$\int_0^A \int_0^A (x+y)^2 \, dy \, dx = 1$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$\int_0^A \int_0^A (x^2 + 2xy + y^2) \, dy \, dx = 1$$

$$\int_0^A \int_0^A x^2 \, dA + \iint_0^A 2xy \, dA + \int_0^A \int_0^A y^2 \, dA = 1$$

$$= \int_0^A x^2 dx \cdot \int_0^A 1 \, dy + \int_0^A 2x \, dx \cdot \int_0^A y \, dy + \int_0^A y^2 dy \cdot \int_0^A 1 \, dx = 1$$

$$\frac{1}{3}x^3 \Big|_{x=0}^{x=A} \cdot y \Big|_{y=0}^{y=A} + x^2 \Big|_{x=0}^{x=A} \cdot \frac{1}{2}y^2 \Big|_{y=0}^{y=A} + \frac{1}{3}y^3 \Big|_{y=0}^{y=A} \cdot x \Big|_{x=0}^{x=A} = 1$$

$$\frac{1}{3}A^3 \cdot A + A^2 \cdot \frac{1}{2}A^2 + \frac{1}{3}A^3 \cdot A = 1$$

$$\frac{1}{3}A^4 + \frac{1}{2}A^4 + \frac{1}{3}A^4 = 1$$

$$\frac{2A^4 + 3A^4 + 2A^4}{6} = 1$$

$$A^4 = \frac{6}{7} \qquad A = \pm \sqrt[4]{\frac{6}{7}}$$

$$A = \sqrt[4]{\frac{6}{7}} \quad \because A > 0.$$

2. (35pts) Find the absolute maximum and minimum of the function $F(x, y, z) = xz + y + y^2$ in the ball of radius 1 around the origin. (be careful of square roots, they can be negative!)

ball of radius 1: $x^2 + y^2 + z^2 \le 1$.

$F(x,y,z) = xz + y + y^2$ $\quad$ $G(x,y,z) = x^2 + y^2 + z^2 \le 1$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle$

$= \langle z, 1+2y, x \rangle$

critical pts, when $\nabla F = 0$, $\quad$ critical pt: $(0, -\frac{1}{2}, 0)$

$z = 0$,

$1 + 2y = 0 \quad y = -\frac{1}{2}$.

$x = 0$

boundary, when $G(x,y,z) = x^2 + y^2 + z^2 = 1$.

$\nabla G = \langle 2x, 2y, 2z \rangle$

$\nabla F = \lambda \nabla G$ $\quad$ $x^2 + y^2 + z^2 = 1$ ④

$\lambda = \frac{z}{2x} \leftarrow$ ① $z = 2x\lambda$ $\quad$ ①÷③. $\frac{z}{x} = \frac{2\lambda x}{2\lambda z}$

② $1 + 2y = 2y\lambda$ $\quad$ when $x \neq 0$

$\lambda = \frac{x}{2z}$ $\quad$ ③ $x = 2z\lambda$ $\quad$ $\frac{z}{x} = \frac{x}{z}$

but what if $x = 0$ $\quad$ $z^2 = x^2$

re-arranging ②, $\quad$ $z = \pm x$. ⑤

$2y\lambda - 2y = 1$

$y(2\lambda - 2) = 1$

$y = \frac{1}{2\lambda - 2}$

$y = \frac{1}{2\left(\frac{z}{2x}\right) - 2} = \frac{1}{\frac{z}{x} - 2} = \frac{x}{z - 2x}$

if $z = x$,

$y = \frac{1}{1-2} = -1$

if $z = -x$,

$y = \frac{1}{-1-2} = -\frac{1}{3}$

for $y = -1$, & $z = x$ $\quad$ $(0, -1, 0)$

$x^2 + 1 + x^2 = 1$

$2x^2 = 0$

$x = 0$ $\quad$ ∴ $x = 0, z = 0, y = -1$

for $y = -\frac{1}{3}$ & $z = -x$, $\quad$ $(\pm\frac{2}{3}, -\frac{1}{3}, \mp\frac{2}{3})$

$x^2 + \frac{1}{9} + x^2 = 1$

$2x^2 = \frac{8}{9}$ $\quad$ $x^2 = \frac{4}{9}$ $\quad$ $x = \pm\frac{2}{3}$, $z = \mp\frac{2}{3}$, $y = -\frac{1}{3}$

continue at back.

candidate points: $(0, 1, 0)$

$\cancel{(\;\;)}$

$\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

$\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

critical point: $\left(0, -\frac{1}{2}, 0\right)$

$F(x,y,z) = xz + y + y^2$

$F\left(0, -\frac{1}{2}, 0\right) = 0 - \frac{1}{2} + \frac{1}{4} = \frac{1}{2}$

$-\frac{3}{9} - \frac{1}{3} = -\frac{2}{3}$

$F(0, 1, 0) = 0 + 1 + 1 = 2$ ⟵ abs max.

$F\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) = \frac{2}{3}\left(-\frac{2}{3}\right) + \left(-\frac{1}{3}\right) + \frac{1}{9} = -\frac{4}{9} - \frac{1}{3} + \frac{1}{9} = -\frac{1}{3} + \frac{-3}{9} = -\frac{2}{3}$

$F\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{1}{3}\right) + \frac{1}{9} = -\frac{4}{9} - \frac{1}{3} + \frac{1}{9} = -\frac{2}{3}$ ⟵ abs min.

The absolute maximum of the function is $F(0, 1, 0) = 2$

The absolute minimum of the function is $F\left(\pm\frac{2}{3}, -\frac{1}{3}, \mp\frac{2}{3}\right) = -\frac{2}{3}$

3. Let $f(r)$ be a function with second order derivative. Let $z = F(x,y) = f(x^2+y^2)$.

(a) (10pts) Show that $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$.

(b) (10pts) Show that if f satisfies $r\frac{\partial^2 f}{\partial r^2} = -\frac{\partial f}{\partial r}$ then z satisfies $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

(c) (10pts) Let $z = \log(x^2+y^2)$ (where log is base e). Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

$z = F(x,y) = f(x^2+y^2)$ let $x^2+y^2 = u$

$z = F(x,y) = f(u)$

$\frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx} = f'(u)\cdot(2x) = f'(x^2+y^2)\cdot 2x$

$\frac{dz}{dy} = \frac{dz}{du}\cdot\frac{du}{dy} = f'(u)\cdot(2y) = f'(x^2+y^2)\cdot 2y$

$y\frac{dz}{dx} = 2xy\, f'(x^2+y^2)$

$x\frac{dz}{dy} = 2xy\, f'(x^2+y^2)$

$\therefore\ y\frac{dz}{dx} = x\frac{dz}{dy}$.

b). $z = f(x^2+y^2)$ let $x^2+y^2 = r$

$\frac{dz}{dx} = \frac{dz}{dr}\cdot\frac{dr}{dx} = \frac{dz}{dr}\cdot 2x$

$\frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dz}{dr}\right)\cdot 2x + \frac{dz}{dr}\cdot 2$

$\frac{d}{dx}\left(\frac{dz}{dr}\right) = \frac{d}{dx}\left(f'(x^2+y^2)\right) = f''(x^2+y^2)\cdot 2x$

$\frac{d^2z}{dx^2} = f''(x^2+y^2)\cdot(4x^2) + 2 f'(x^2+y^2)$

Similarly,

$\frac{d}{dy}\left(\frac{dz}{dy}\right) = f''(x^2+y^2)(4y^2) + 2f'(x^2+y^2)$

$r\frac{d^2f}{dr^2} = -\frac{df}{dr}$

if $r = x^2+y^2$,

① $(x^2+y^2)f''(x^2+y^2) = -f'(x^2+y^2)$

$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 4x^2 f''(x^2+y^2) + 2f'(x^2+y^2) + 4y^2 f''(x^2+y^2) + 2f'(x^2+y^2)$

$= f''(x^2+y^2)(4x^2+4y^2) + 4f'(x^2+y^2)$

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subbing in ①,

$f''(x^2+y^2)(4x^2+4y^2) + 4(-(x^2+y^2)f''(x^2+y^2)) = 0$ ✓.

$\therefore\ \frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0$ if $r\frac{d^2f}{dr^2} = -\frac{df}{dr}$.

continue at back.

$z = \log(x^2 + y^2)$ show $\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0$.

$$\frac{dz}{dx} = \frac{1}{x^2+y^2} \cdot 2x \qquad \frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2}$$

$$\frac{dz}{dy} = \frac{1}{x^2+y^2} \cdot 2y \qquad \frac{d}{dy}\left(\frac{dz}{dy}\right) = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2}$$

$\cancel{x^4+2x^2y^2+y^4}$

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = \frac{2(x^2+y^2) - 4x^2 + 2(x^2+y^2) - 4y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^2} + \cancel{2y^2} - \cancel{4x^2} + \cancel{2x^2} + \cancel{2y^2} - \cancel{4y^2}}{(x^2+y^2)^2} = 0 \quad ✓.$$

$$\therefore \frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0$$

$x^2 + 2xy + y^2$

4. Let $F(x, y) = e^x + e^y + (x+y)^2$ and let $g(x)$ satisfy the equation $F(x, g(x)) = 2$ and $g(0) = 0$ and g has a second order derivative at 0.

$F(0,0) = 2.$

(a) (10pts) Find $g'(0)$.

(b) (10pts) Find $g''(0)$.

a) $\frac{dF}{dx} = e^x + e^y \frac{dy}{dx} + 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx}.$

$\Downarrow$

$0 = e^x + e^y g'(x) + 2x + 2y + 2xg'(x) + 2y\,g'(x)$ ①

at $(0,0)$

$= e^0 + e^0 g'(0) + 0 + 0 + 0\cdot g'(0) + 0\cdot g'(0)$

$= 1 + g'(0)$

$\boxed{g'(0) = -1}$

$\frac{dy}{dx} = g'(x) = -\frac{F_x}{F_y} = -\frac{e^x + 2x + 2y}{e^y + 2\cdot x + 2y}$

at $(0,0)$ $= -\frac{e^0}{e^0} = -1$ ✓

b) take $\frac{d}{dx}$ of eq. ①.

$0 = e^x + e^y g'(x) g'(x) + e^y g''(x) + 2 + 2g'(x) + 2g'(x) + 2x\,g''(x)$

$+ 2g'(x)\,g'(x) + 2y\,g''(x)$

at $(0,0)$, $x=0$, $y=0$, $g'(0) = -1$.

$0 = e^0 + e^0\cdot(-1)(-1) + e^0 g''(0) + \cancel{2} + 2(\cancel{-1}) + 2(-1) + \cancel{2(0)g''(x)}$

$+ 2(-1)(-1) + \cancel{2(0)g''(x)}$

$= 1 + 1 + g''(0) - \cancel{2} + \cancel{2}$

$\boxed{g''(0) = -2}$

Scratch sheet