1. (30pts) Let $f(x, y) = 2e^{x+y} + Axe^y + Bye^x$ be such that $(0, 0)$ is a critical point for f.

(a) (10pts) Find A and B.

(b) (10pts) Write the 2nd order Taylor Polynomial of f around $(0, 0)$.

(c) (10pts) Is $(0, 0)$ a local maximum for f? Explain.

a) $f_x = 2e^{x+y} + Ae^y + Bye^x$

$f_y = 2e^{x+y} + Axe^y + Be^x$

$(0,0)$ is a crit. pt. so $f_x(0,0) = f_y(0,0) = 0$

$\therefore\ 2e^{0+0} + Ae^0 + By(0)e^0 = 0$ $\qquad e^0 = 1$

$2e^{0+0} + A(0)e^0 + Be^0 = 0.$

$2 + A = 0$ $\qquad \boxed{A = -2 \quad B = -2.}$

$2 + B = 0$

b). $f_{xx} = 2e^{x+y} + Bye^x = 2e^{x+y} - 2ye^x$ $\qquad f_{xx}(0,0) = 2$

$f_{yy} = 2e^{x+y} + Axe^y = 2e^{x+y} - 2xe^y$ $\qquad f_{yy}(0,0) = 2$

$f_{xy} = 2e^{x+y} + Ae^y + Be^x = 2e^{x+y} - 2e^y - 2e^x$ $\qquad f_{xy}(0,0) = 2-2-2 = -2$

$$T_2(x_0,y_0) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f_{xx}(x_0,y_0)\frac{(x-x_0)^2}{2} + f_{yy}(x_0,y_0)\frac{(y-y_0)^2}{2} + f_{xy}(x_0,y_0)(x-x_0)(y-y_0).$$

$f(x,y) = 2e^{x+y} - 2xe^y - Bye^x$.

$f(0,0) = 2$ $\qquad f_x(0,0) = f_y(0,0) = 0$

$$\therefore T_2(0,0) = 2 + \frac{2(x-0)^2}{2} + \frac{2(y-0)^2}{2} - 2(x-0)(y-0).$$

$T_2(0,0) = 2 + x^2 + y^2 - 2xy$ $\qquad \boxed{T_2(x,y) = 2 + x^2 + y^2 - 2xy \text{ around } (0,0)}$

c). 2nd derivative test.

$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$

$D(0,0) = 2(2) - (-2)^2$

$= 4 - 4 = 0.$

inconclusive. ← By the 2 second derivative Test

However, $T_2(0,0) = \underbrace{(x-y)^2}_{\text{always positive.}} + 2.$

$\therefore (0,0)$ is NOT a local maximum for T_2. * T_2 has a local min but not a strict local min. any pt on the line $x = y$ will give the local min.

Therefore, $(0,0)$ is $\boxed{\text{not}}$ a local maximum for f.

50/50

2. (50pts) Let $f(x, y) = x^2y^2 - 2xy + 2x - x^2$.

(a) (25pts) Find all critical points of f and for each whether it is a local minimum, a local maximum or neither.

(b) (25pts) Find the absolute minimum and maximum of f in the domain $\{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 2\}$.

$f_x = 2xy^2 - 2y + 2 - 2x$

$f_y = 2x^2y - 2x$

$f_{xx} = 2y^2 - 2$ $\quad f_{yy} = 2x^2$ $\quad f_{xy} = 4xy - 2$

crit. pt when $f_x = 0$ and $f_y = 0$.

$2xy^2 - 2y + 2 - 2x = 0$ and $2x^2y - 2x = 0$

$x(2xy - 2) = 0$

$x = 0$ or $2xy = 2$

$xy = 1$

$x = \frac{1}{y}$.

① if $x = 0$,

$-2y + 2 = 0$

$y = 1$

$\therefore (0,1)$ is a crit. pt.

② if $x = \frac{1}{y}$

$2(\frac{1}{y})y^2 - 2y + 2 - 2(\frac{1}{y}) = 0$.

$\cancel{2y - 2y} + 2 - \frac{2}{y} = 0$

$2 = \frac{2}{y}$

$2y = 2$

$y = 1$ $\quad \therefore x = \frac{1}{1} = 1$

$\therefore (1,1)$ is a critical pt.

check:

$f_x(0,1) = 0 - 2 + 2 - 0 = 0$ ✓.

$f_y(0,1) = 0 - 0 = 0$ ✓.

$f_x(1,1) = 2(1)(1) - 2(1) + 2 - 2(1) = 0$ ✓.

$f_y(1,1) = 2(1)(1) - 2(1) = 0$ ✓.

Use second deriv. test $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)$

for pt $(0,1)$, $f_{xx}(0,1) = 2(1) - 2 = 0$.

$f_{yy}(0,1) = 2(0) = 0$

$f_{xy}(0,1) = 4(0)(1) - 2 = -2$.

$\therefore D = 0(0) - (-2)^2 = -4$

$\because D < 0$, $(0,1)$ is a saddle pt.

$\boxed{(0,1) \text{ is } \underline{\text{neither}} \text{ a local min or local max.}}$

For pt $(1,1)$:
$$f_{xx}(1,1) = 2(1) - 2 = 0$$
$$f_{yy}(1,1) = 2(1) = 2$$
$$f_{xy}(1,1) = 4(1,1) - 2 = 2.$$

$\therefore D = 0(2) - 2^2 = -4$

$\because D < 0$, $(1,1)$ is a saddle point.

$\boxed{(1,1) \text{ is } \underline{\text{neither}} \text{ a local max/min.}}$

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c) $f(x,y) = x^2y^2 - 2xy + 2x - x^2$.

"critical points: $(0,1)$, $(1,1)$.

Find "candidate" pts:

☐ look at line $x = 0$.

$g(y) = f(0,y) = 0$

$g'(y) = 0$.

☐ look at line $y = 0$

$g(x) = f(x,0) = 2x - x^2$

$g'(x) = 2 - 2x$.

set $g'(x) = 0$. $\quad 2 - 2x = 0$

$2 = 2x \quad x = 1$

$\therefore (1,0)$ is a candidate pt.

☐ look at line $x = 2$.

$g(y) = f(2,y) = 4y^2 - 4y + \cancel{4} - \cancel{4}$

$g'(y) = 8y - 4$.

set $g'(y) = 0 \quad 8y - 4 = 0$

$8y = 4 \quad y = \frac{1}{2}$

$\therefore (2, \frac{1}{2})$ is a candidate pt.

☐ look at line $y = 2$.

$g(x) = f(x,2) = 4x^2 - 4x + 2x - x^2$

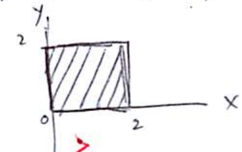
$g'(x) = 8x - 4 + 2 - 2x = 6x - 2$

set $g'(x) = 0$. $\quad 6x - 2 = 0$.

$x = \frac{1}{3}$.

$\therefore (\frac{1}{3}, 2)$ is a candidate pt.

Domain. $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$.



Find values:

$f(0,1) = 0$

$f(1,1) = 0$.

$f(1,0) = 0 - 0 + 2 - 1 = 1$

$f(2,\frac{1}{2}) = 4(\frac{1}{4}) - 2(2)(\frac{1}{2}) + 2(2) - (2)^2 = -1$

$f(\frac{1}{3},2) = \frac{1}{9}(4) - 2(\frac{1}{3})(2) + 2(\frac{1}{3}) - (\frac{1}{9}) = -\frac{1}{3}$

$\frac{4}{9} - \frac{4}{3} + \frac{2}{3} - \frac{1}{9} = \frac{1}{3} - \frac{4}{3} + \frac{2}{3}$

also check boundary pts.

$f(0,2) = 0$

$f(0,0) = 0$

$f(2,0) = 2(2) - (2)^2 = 0$.

$f(2,2) = 4(4) - 2(2)(2) + 2(2) - 2^2 =$

$16 - 8 + \cancel{4} - \cancel{4} = 8$.

Therefore, the absolute max of f is $f(2,2) = 8$

and the absolute min is $f(2,\frac{1}{2}) = -1$.

* in the Domain

3. (30pts) Let $f(x) = \sqrt{x}$.

(a) (10pts) Find the 2nd order Taylor Polynomial of f around $x_0 = 4$.

(b) (10pts) Give some bound on the absolute value of the 3rd order derivative of f in the interval $[2, 6]$.

(c) (10pts) Use part (a) to give an approximation for $\sqrt{2}$ and part (b) to give a bound on the error of your approximation.

a). Find $T_2(x_0)$

$$T_2(x_0) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$$

$x^{\frac{1}{2}}$ $\quad f(x) = \sqrt{x} \quad f(x_0) = \sqrt{4} = 2$

$\frac{1}{2}x^{-\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2\sqrt{x}} \quad f'(x_0) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$-\frac{1}{4}x^{-\frac{3}{2}}$ $\quad f''(x) = -\frac{1}{4}(x)^{-\frac{3}{2}} \quad f''(x_0) = f''(4) = -\frac{1}{4}\left(\frac{1}{(4)^{\frac{3}{2}}}\right) = -\frac{1}{4}\cdot\frac{1}{8} = -\frac{1}{32}$

$$T_2(x) = f(4) + f'(4)(x-4) + f''(4)\frac{(x-4)^2}{2}$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{32}\left(\frac{(x-4)^2}{2}\right)$$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\boxed{T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2}$$

b). $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$ ← decreasing

$f'''(2) = \frac{3}{8}(2^{-\frac{5}{2}}) = \frac{3}{8\sqrt{32}}$ ← bigger

$f'''(6) = \frac{3}{8}(6^{-\frac{5}{2}}) = \frac{3}{8\sqrt{6^5}}$ ← smaller.

$\therefore \boxed{|f'''(x)| \le \frac{3}{8\sqrt{32}}}$ in the interval $[2, 6]$

c). approximate $\quad \sqrt{2} = f(2) = \sqrt{2}$

$x = 2$.

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$T_2(2) = 2 + \frac{1}{4}(2-4) - \frac{1}{64}(2-4)^2$$

$$= 2 - \frac{1}{2} - \frac{1}{64}(4) = 2 - \frac{1}{2} - \frac{1}{16} = \frac{32}{16} - \frac{8}{16} - \frac{1}{16} \approx \frac{23}{16}$$

$\therefore \sqrt{2} \approx \frac{23}{16}$ $\quad \boxed{\sqrt{2} \approx \frac{23}{16}}$

Continue on next page

Scratch sheet

continuation of # 3.

$$|R_n^a(x_a)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

where $f'''(x_a) \leq M$.

interval: $2 \leq x \leq 6$.

$x_0 = 4$.

$2 - 4 \leq x - 4 \leq 6 - 4$

$-2 \leq x - 4 \leq 2$

$\therefore |x-4| \leq 2$

$$f'''(x_0) \leq \frac{3}{8\sqrt{32}} = M$$

$\therefore$

$$|R_2(x)| \leq \frac{3}{8\sqrt{32}} \cdot \frac{1}{(2+1)!}|2|^{2+1}$$

$$\frac{1}{8\sqrt{2}} = \frac{\sqrt{2}}{16}$$

$$\leq \frac{3}{8\sqrt{32}} \cdot \frac{1}{6} \cdot 8 = \frac{1}{2\sqrt{32}} = \frac{\sqrt{32}}{64} = \frac{4\sqrt{2}}{64} = \frac{\sqrt{2}}{16}$$

$$\therefore \boxed{|R_2(x)| \leq \frac{\sqrt{2}}{16}}$$ ~~for interval [2,6]~~

max. error for $x = 2$.