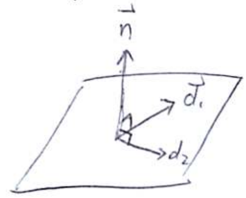
1. (20pts) Find an equation describing the plane containing the line $(0, 1, 3) + t(1, 1, 1)$ and the point $(0, 0, 0)$.

Plane contains points : $P(0,1,3)$ and $Q(0,0,0)$

also has a direction vector $(1,1,1) = \vec{d}_1$

another direction vector of the plane is.

$\vec{QP} = (0,1,3) = \vec{d}_2$



the normal of the plane is $\vec{d}_1 \times \vec{d}_2$

$(1,1,1) \times (0,1,3) = (2,-3,1)$.

1	1	1	1	1	1	
1	1	3	0	1	3	
		2	-3	1		

The cartesian equation of a plane is

$Ax + By + Cz + D = 0$

where $\vec{n} = (A, B, C)$

so,

$2x - 3y + z + D = 0$. ①

To find D, sub in a pt. (x,y,z) on the plane onto eq. ①.

$P(0,1,3)$.

$2(0) - 3(1) + 1(3) + D = 0$

$-3 + 3 + D = 0$

$D = 0$.

$2x - 3y + z = 0$.

check if pt. $Q(0,0,0)$ is on the plane:

$2(0) - 3(0) + 0 = 0$ ✓.

$$\boxed{2x - 3y + z = 0}$$

20/20

2. Let $\vec{v} = (1, 2, 5)$ and $\vec{u} = (3, 2, 0)$.

(a) (10pts) Find the angle between u and v [express it as arccos(...)].

(b) (10pts) Find a **unit** vector perpendicular to both v and u.

$|u| = \sqrt{9+4+0} = \sqrt{13}$

$|v| = \sqrt{1+4+25} = \sqrt{30}$

a) $u \cdot v = |u||v| \cos\theta$

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{(3,2,0)\cdot(1,2,5)}{\sqrt{13}\cdot\sqrt{30}}$$

$$\cos\theta = \frac{3+4+0}{\sqrt{390}} = \frac{7}{\sqrt{390}}$$

$$\boxed{\theta = \arccos\left(\frac{7}{\sqrt{390}}\right)}$$

b). a vector perp. to both v & u is the cross-product of v & u.

$(1,2,5) \times (3,2,0) = (-10, 15, -4)$

$$\begin{matrix} 2 & 5 & 1 & 2 \\ 2 & 0 & 3 & 2 \\ -10 & 15 & -4 & \end{matrix}$$

check if they are $\perp$.

$(1,2,5)\cdot(-10,15,-4) = -10+30-20 = 0$ ✓.

$\because a \cdot b = 0$ iff $a \perp b$.

$(3,2,0)\cdot(-10,15,-4) = -30+30+0 = 0$ ✓.

$(-10, 15, -4)$.

$|(-10,15,-4)| = \sqrt{100+225+16}$

$= \sqrt{341}$

unit vector has magnitude 1.

$\rightarrow$ unit vector $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$

$\therefore$ unit vector: $\boxed{\left(-\frac{10}{\sqrt{341}}, \frac{15}{\sqrt{341}}, -\frac{4}{\sqrt{341}}\right)}$.

check: $\left|\left(-\frac{10}{\sqrt{341}}, \frac{15}{\sqrt{341}}, -\frac{4}{\sqrt{341}}\right)\right| = \sqrt{\left(-\frac{10}{\sqrt{341}}\right)^2 + \left(\frac{15}{\sqrt{341}}\right)^2 + \left(\frac{-4}{\sqrt{341}}\right)^2} =$

$\sqrt{\frac{100}{341} + \frac{225}{341} + \frac{16}{341}} = \sqrt{\frac{341}{341}} = 1$ ✓.

100
225
341

15
15
75
15
225

20/20

3. Let $a = (1, 1, 0)$ and $b = (2, 2, 2)$.

(a) (10pts) Find a vector or parametric equation for the line going through a and b.

(b) (10pts) Find the point where this line intersect the plane P whose equation is $x + y + z = 4$.

(c) (5pts) Does the plane P intersect the line **segment** between a and b?

25/25

a). line through a & b has direction vector:

$\vec{ab} = (2,2,2) - (1,1,0) = (1,1,2)$

$\therefore$ $\boxed{\vec{r} = (1,1,0) + t(1,1,2). \quad t \in \mathbb{R}.}$

$x = 1+t$

$y = 1+t$

$z = 2t.$

b). ~~plane~~ $x+y+z=4$. plug in values from parametric eq. of the line into this plane to find intersection:

$(1+t) + (1+t) + (2t) = 4$

$1+t+1+t+2t = 4$

$4t = 2$

$t = \frac{1}{2}.$

So the pt. where they intersect is.

$x = 1+\frac{1}{2} = \frac{3}{2}$

$y = 1+\frac{1}{2} = \frac{3}{2}$

$z = 2(\frac{1}{2}) = 1$

$\boxed{(\frac{3}{2}, \frac{3}{2}, 1)}$

c). line segment b/w a & b. $\Rightarrow$ $(1,1,0) + t(1,1,2)$, $0 \le t \le 1$.

~~as found previously, the planes.~~

the eq. of the line segment from a to b is similar to the eq. of the line found in (a), except for the restriction $0 \le t \le 1$. from (b) it is found that the plane intersects the line at $t = \frac{1}{2}$.

$\because$ this is within $0 \le t \le 1$, the plane intersects the line segment between a & b. $\boxed{\text{Yes.}}$

4. ~~(40pts)~~ Match the surfaces described by the following equations with their drawings. Justify your answers.

(a) (15pts) $x^2 - 2y^2 - 5z = 0$ $\boxed{H}$ $x^2 - 2y^2 = 5z.$ ~~paraboloid hyperbolic~~

(b) (15pts) $z = -\log(x^2 + 5y^2)$ $\boxed{A}$

(c) (15pts) $z^2 - 2x^2 - 3 = 0$ $\boxed{D}$ $z^2 - 2x^2 = 3.$ ~~hyperboloid~~

a). $x^2 - 2y^2 - 5z = 0.$

$$\frac{x^2}{10} - \frac{y^2}{5} - \frac{z}{2} = 0 \qquad \frac{x^2}{10} - \frac{y^2}{5} = \frac{z}{2}.$$

parallel to $z=0$ plane, at $z=k$,
the level curve is a hyperbola

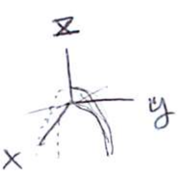
$$\left(\frac{x^2}{10} - \frac{y^2}{5} = \frac{k}{2}\right). \leftarrow \text{eq. of a hyperbola}$$

$$\frac{x^2}{\frac{10k}{2}} - \frac{y^2}{\frac{5k}{2}} = 1 \quad \text{with semi-axes. } \pm\sqrt{\frac{10k}{2}} \text{ and } \pm\sqrt{\frac{5k}{2}}.$$

parallel to $x=0$ plane, at $x=k$. level curve: parabola.

$$\left(\frac{k^2}{10} - \frac{y^2}{5}\right) = \frac{z}{2}. \leftarrow \text{equation of a parabola.}$$

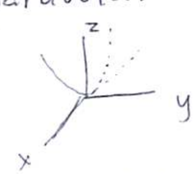
$$z = \frac{k^2}{5} - \frac{2y^2}{5} \quad \text{upside down parabola. } \because p < 0.$$



Similarly, $\parallel$ to $y=0$ plane, at $y=k$.

$$\frac{x^2}{10} - \frac{k^2}{5} = \frac{z}{2} \quad \text{level curve: parabola.}$$

$$z = \frac{x^2}{5} - \frac{2k^2}{5}$$



Therefore. a) has to be a <u>hyperbolic paraboloid</u>, which is graph $\boxed{H}$

b). $z = -\log(x^2+5y^2)$

$-z = \log(x^2+5y^2).$

$10^{-z} = x^2+5y^2$

at diff. values of z:

at $z=0$.

$1 = x^2+5y^2 \leftarrow$ ellipse.

at $z=-1$

$10 = x^2+5y^2 \leftarrow$ ellipse

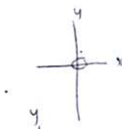
at $z=1$,

$\frac{1}{10} = x^2+5y^2$

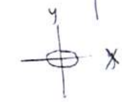
ellipse.

Scratch sheet

$z = 1 \quad \frac{1}{10} = x^2 + 5y^2 \rightarrow 1 = 10x^2 + 50y^2 \rightarrow$ semi-axes: $\pm\sqrt{\frac{1}{10}}$ & $\pm\sqrt{\frac{1}{50}}$.

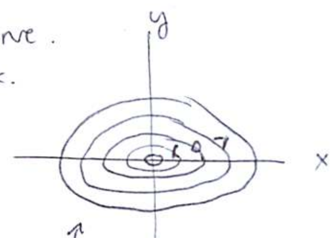


$z = 0 \quad 1 = x^2 + 5y^2 \rightarrow$ semi-axes: $\pm 1 \quad \pm\sqrt{\frac{1}{5}}$



$z = -1 \quad 10 = x^2 + 5y^2 \rightarrow 1 = \frac{x^2}{10} + \frac{y^2}{2}$. $\rightarrow$ semi-axes $\pm\sqrt{10}$, $\pm\sqrt{2}$

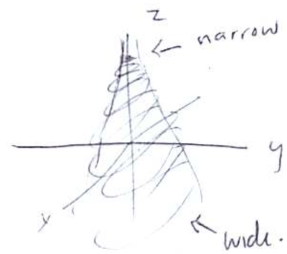
level curve. for z=k.



not to scale!

as $z \rightarrow \infty$, ellipse becomes smaller

as $z \rightarrow -\infty$, ellipse becomes bigger



so the graph is $\boxed{A.}$

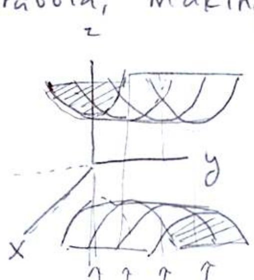
c). $z^2 - 2x^2 - 3 = 0$

$z^2 - 2x^2 = 3$. in 2d, this is the equation of a hyperbola

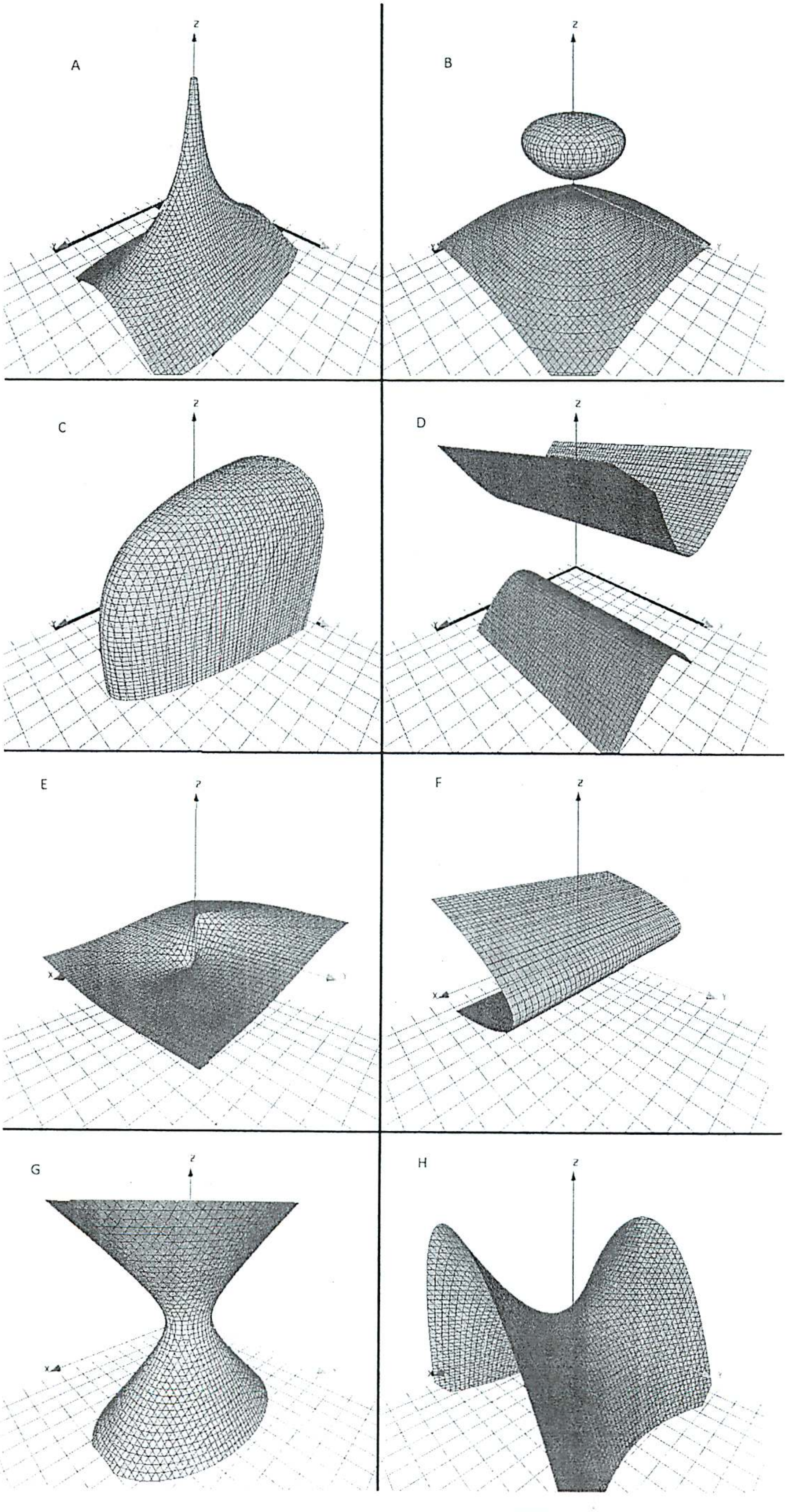
↑ does not contain y, so at diff y-values, it should still have the same curve, w/c is the parabola, making it a cylinder, parallel to y-axis.

hyperbolic cylinder.

for $y = k$, level curve will always be $z^2 - 2x^3 = 3$



so the graph is $\boxed{D.}$


A
z
B
z
C
z
D
z
E
z
F
z
G
z
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z