

MATH 200 - SEC 201 - 2010W

Assignment no. 2

Due: 9am, Feb 11, 2011

1. Let

$$f(x, y) = \frac{xy^3 - x^3y}{x^2 + y^2}.$$

(a) What is the domain of definition of f ?

(b) Calculate f_x, f_y, f_{xy}, f_{yx} and verify that $f_{xy} = f_{yx}$.

(c) Calculate $\lim_{x \rightarrow 0} f_{xy}(x, 0)$ and $\lim_{y \rightarrow 0} f_{xy}(0, y)$. Does $\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x, y)$ exist?

Solution:

(a) $(x, y) \neq (0, 0)$

$$(b) f_x(x, y) = \frac{y^5 - x^4y - 4x^2y^3}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{-x^5 + xy^4 + 4x^3y^2}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = f_{yx}(x, y) = \frac{-x^6 - 9x^4y^2 + 9x^2y^4 + y^6}{(x^2 + y^2)^3}$$

(c) For any $x \neq 0$, $f_{xy}(x, 0) = \frac{-x^6}{x^6} = -1$ so $\lim_{x \rightarrow 0} f_{xy}(x, 0) = -1$. Similarly, $\lim_{y \rightarrow 0} f_{xy}(0, y) = \lim_{y \rightarrow 0} \frac{y^6}{y^6} = 1$. Since $1 \neq -1$ we see that $\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x, y)$ does not exist.

2. Let

$$f(x, y) = x \arctan(x^2 - y).$$

- (a) What is the domain of definition of f ?
- (b) Calculate f_x and f_y and find the only point where the tangent plane is horizontal.
- (c) Find the second order Taylor approximation at the point from (b). Can you say whether this point is a local minimum, a local maximum or neither?

Solution:

(a) f is defined on the entire plane.

(b) $f_x(x, y) = \arctan(x^2 - y) + \frac{2x^2}{1+(x^2-y)^2}.$

$$f_y(x, y) = \frac{-x}{1+(x^2-y)^2}.$$

The tangent plane is horizontal when both $f_x(x, y)$ and $f_y(x, y)$ are zero, i.e. when (x, y) is a critical point for f . The equation $\frac{-x}{1+(x^2-y)^2} = 0$ yield that $x = 0$. Putting this back into $\arctan(x^2 - y) + \frac{2x^2}{1+(x^2-y)^2} = 0$ we get $\arctan(-y) = 0$ and so $y = 0$ too. So our point is $(0, 0)$.

(c) $f_{xx}(x, y) = \frac{2x^5+4x^3y-6x-6xy^2}{(1+(x^2-y)^2)^2}$ so $f_{xx}(0, 0) = 0.$

$$f_{xy}(x, y) = \frac{1-3x^4+2x^2y+y^2}{(1+(x^2-y)^2)^2} \text{ so } f_{xy}(0, 0) = 1.$$

$$f_{yy}(x, y) = \frac{2x^3-2xy}{(1+(x^2-y)^2)^2} \text{ so } f_{yy}(0, 0) = 0.$$

$$f(0, 0) = 0$$

Hence, the 2nd order Taylor polynomial of f around $(0, 0)$ is $T_2(x, y) = xy$. This function describes a hyperbolic paraboloid and so $(0, 0)$ is a saddle point for f .

3. Let

$$f(x, y) = \sqrt{1 - x^2 - y^2}.$$

- (a) What is the domain of definition of f ?
- (b) Write the equation of the tangent plane at the point $(a, b, f(a, b))$ in terms of a and b .
- (c) Find a and b such that this tangent plane passes through the points $(1, 1, 1)$ and $(1, -2, 4)$.

Solution:

(a) $x^2 + y^2 \leq 1$ - the disc of radius 1 around the origin.

(b) $f_x(x, y) = \frac{-x}{\sqrt{1-x^2-y^2}}$

$f_y(x, y) = \frac{-y}{\sqrt{1-x^2-y^2}}$

The tangent plane at $(a, b, f(a, b))$ is described by

$$z = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(x - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(y - b).$$

(c) We want to solve the following equations for a and b :

$$1 = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(1 - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(1 - b).$$

$$4 = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(1 - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(-2 - b).$$

Multiplying by $\sqrt{1 - a^2 - b^2}$ and simplifying yields

$$\begin{aligned}\sqrt{1 - a^2 - b^2} &= 1 - a - b \\ 4\sqrt{1 - a^2 - b^2} &= 1 - a + 2b\end{aligned}\tag{1}$$

This implies that

$$4(1 - a - b) = 1 - a + 2b$$

so

$$a = 1 - 2b.$$

Putting this back into equation 1 gives

$$\sqrt{1 - (1 - 2b)^2 - b^2} = 1 - (1 - 2b) - b.$$

Squaring yields

$$1 - (1 - 4b + 4b^2) - b^2 = b^2$$

so

$$4b - 6b^2 = 0$$

which means that either $b = 0$ (and $a = 1$) or $b = \frac{2}{3}$ (and $a = 1 - 2\frac{2}{3} = -\frac{1}{3}$).

Note that the first solution $(a, b) = (0, 1)$ is on the boundary of our domain and the partial derivatives are not defined there. The second solution $(a, b) = (\frac{2}{3}, -\frac{1}{3})$ is the only solution.