MATH 200 - SEC 104 - 2012W1

Midterm no. 4

1-1:50pm, Nov 16, 2012

Name (Last, First):_____________Oui_____________

Student I.D. Number:_________________________

Signature: _________________________

INSTRUCTIONS: This is a closed-book exam. You may not use any books, notes, papers, calculators, or other aids. Do all work on the sheets provided. There is an extra sheet on the back for scratch work. If you need an extra sheet, raise your hand and one will be provided. If you need more space for your solution, use the back of the sheets and leave an arrow for the grader. Please draw a box around your final answer.

There are 3 questions. Explain all your answers. Good Luck!

1. (15pts) Find the absolute minimum and maximum values, and all points where they occur, of the function $F(x, y, z) = x^2 + yz$ over the ellipsoid $(x-1)^2 + y^2 + 4z^2 = 1$.

$\nabla F = \langle 2x, z, y \rangle$ $\quad\quad$ $\nabla G = \langle 2x-2, 2y, 8z \rangle$

Using Lagrange multipliers we need to solve the system

$$\begin{cases} ① \ 2x = \lambda(2x-2) \\ ② \ z = 2\lambda y \\ ③ \ y = 8\lambda z \\ ④ \ (x-1)^2 + y^2 + 4z^2 = 1 \end{cases}$$

② + ③ $\Rightarrow$ $z = 16\lambda^2 z \Rightarrow z = 0$ or $\lambda = \frac{1}{4}$ or $\lambda = -\frac{1}{4}$

- If $z = 0$ then by ③ $y = 0$ and then by ④ $(x-1)^2 = 1$ so $x = 0$ or $x = 2$

$(0,0,0)$ and $(2,0,0)$ are candidates.

- If $\lambda = \frac{1}{4}$ then ① $\Rightarrow 2x = \frac{x-1}{2} \Rightarrow 4x = x-1 \Rightarrow$ $\Rightarrow x = -\frac{1}{3}$. Then ④ cannot be satisfied since $(-\frac{1}{3}-1)^2 > 1$. No candidates here.

- If $\lambda = -\frac{1}{4}$ then ① $\Rightarrow 2x = -\frac{x-1}{2} \Rightarrow 4x = -x+1 \Rightarrow$ $\Rightarrow x = \frac{1}{5}$.

③ $\Rightarrow y = -2z$.

Using ④ we get $\left(\frac{1}{5}-1\right)^2 + 4z^2 + 4z^2 = 1 \Rightarrow$

$$\frac{16}{25} + 8z^2 = 1 \Rightarrow \quad 8z^2 = \frac{9}{25} \Rightarrow 2\sqrt{2}\,z = \pm\frac{3}{5}$$

So $z = \frac{3}{10\sqrt{2}}$ or $z = -\frac{3}{10\sqrt{2}}$.

Candidates are

$\left(\frac{1}{5}, -\frac{3}{5\sqrt{2}}, \frac{3}{10\sqrt{2}}\right)$ and $\left(\frac{1}{5}, \frac{3}{5\sqrt{2}}, -\frac{3}{10\sqrt{2}}\right)$.

$F(0,0,0) = 0$

$F(2,0,0) = 4 \Leftarrow$ abs max

$$F\left(\frac{1}{5}, -\frac{3}{5\sqrt{2}}, \frac{3}{10\sqrt{2}}\right) = \frac{1}{25} - \frac{3}{5\sqrt{2}} \cdot \frac{3}{10\sqrt{2}} = \frac{4-9}{100} = -\frac{1}{20}$$

$$F\left(\frac{1}{5}, \frac{3}{5\sqrt{2}}, -\frac{3}{10\sqrt{2}}\right) = -\frac{1}{20} \Leftarrow \text{abs min}$$

2. (10pts) Calculate

$$\int\int_D \frac{1}{(x+y)^2}\,dA.$$

Where $D = \{(x, y) \mid 1 \le x \le 2\ ,\ 0 \le y \le 1\}$. Simplify your answer.

$$\int_1^2\int_0^1 \frac{1}{(x+y)^2}\,dy\,dx = \int_1^2 -\frac{1}{(x+y)}\Big|_{y=0}^{y=1}\,dx =$$

$$= \int_1^2\left(-\frac{1}{x+1}+\frac{1}{x}\right)dx = -\ln(x+1)+\ln x\Big|_{x=1}^{x=2} =$$

$$= (-\ln 3+\ln 2)-(-\ln 2+\ln 1) =$$

$$= 2\ln 2-\ln 3 = \ln\frac{4}{3}$$

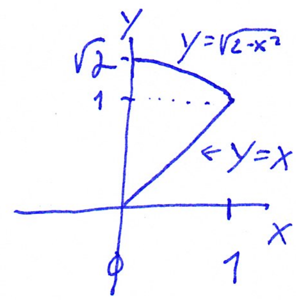
3. (15pts) Consider the iterated integral

$$I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{y} \, dy \, dx$$

(a) Write I as an iterated integral or sum of iterated integrals in the reverse order ($dx\,dy$).

(b) Write I as an iterated integral in polar coordinates.

(c) Evaluate I. Simplify your answer.



(a)

$$\int_0^1 \int_0^y \frac{x}{y} \, dx \, dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{y} \, dx \, dy$$

(b)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \frac{r\cos\theta}{r\sin\theta} \, r \, dr \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r \frac{\cos\theta}{\sin\theta} \, dr \, d\theta$$

(c) Using (b) we get

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} \, d\theta \cdot \int_0^{\sqrt{2}} r \, dr = \ln(\sin\theta)\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cdot \frac{r^2}{2}\Big|_{r=\phi}^{r=\sqrt{2}}$$

$$= \left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) \cdot (1-0) = \frac{\ln 2}{2}$$

Alternatively, using (a)

$$\int_0^1 \frac{x^2}{2\cancel{y}}\Bigg|_{x=0}^{x=y} dy + \int_1^{\sqrt{2}} \frac{x^2}{2y}\Bigg|_{x=0}^{x=\sqrt{2-y^2}} dy =$$

$$\int_0^1 \frac{y}{2}\,dy + \int_1^{\sqrt{2}} \frac{2-y^2}{2y}\,dy = \int_0^1 \frac{y}{2}dy + \int_1^{\sqrt{2}} \frac{1}{y} - \frac{y}{2}\,dy$$

$$= \frac{y^2}{4}\Bigg|_{y=0}^{y=1} + \left(\ln y - \frac{y^2}{4}\right)\Bigg|_{y=1}^{y=\sqrt{2}} = \frac{1}{4} + \left(\ln\sqrt{2} - \frac{1}{2}\right) - \left(\ln 1 - \frac{1}{4}\right)$$

$$= \ln\sqrt{2} = \frac{\ln 2}{2}$$