Scoring Guideline
MATH 200/253 Sec 922
Summer 2012
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Instructor: Dr. Ori-Gurel

If you have concerns about the quiz, come see me at the MLC at LSK100 basement. You can find me between 2:00pm to 3:00pm there. If you really can't make it, see the professor.

After your final exam, all grade issues (this includes past quizzes and midterms) should go to your professor, thanks.

I had a wonderful time conversing Mathematics with some of you and I hope you will use that knowledge to smoke the final!

1. Suppose that charge is distributed according to the following rule: The charge density on a point of distance r from the origin $\frac{1}{r}$. What is the total amount of charge contained in the annulus $D = \left\{(x, y) \in \mathbb{R}^2; 1 \leq x^2 + y^2 \leq 4\right\}$?

Solution

Let the charge density be $\sigma = \frac{1}{r}$, then $dQ = \sigma dA$

$$\begin{aligned}
Q &= \iint_D \sigma dA \\
&= \int_0^{2\pi} \int_1^2 \frac{1}{r} r dr d\theta \\
&= \int_0^{2\pi} \int_1^2 dr d\theta \\
&= \int_0^{2\pi} d\theta \\
&= 2\pi
\end{aligned}$$

5 marks for correct setup

4 marks for correct integration

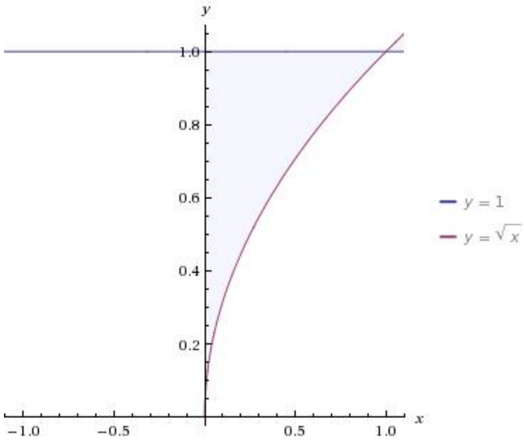
1 mark for correct answer

2. Calculuate

$$\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)dydx$$

Solution

A plot of the region of integration.



The region of integration is $R = \left\{(x,y) \in \mathbb{R}^2; \sqrt{x} \leq y \leq 1, 0 \leq x \leq 1\right\}$

2 marks for correct plot of the region

Making horizontal slices, the region can be also written as $R = \left\{(x,y) \in \mathbb{R}^2; 0 \leq x \leq y^2, 0 \leq y \leq 1\right\}$

$$\int_0^1 \int_{\sqrt{x}}^1 \sin(y^3)dydx = \int_0^1 \int_0^{y^2} \sin(y^3)dxdy \tag{1}$$
$$= \int_0^1 x\sin(y^3)\Big|_0^{y^2} dy \tag{2}$$
$$= \int_0^1 y^2\sin(y^3)dy \tag{3}$$
$$= -\frac{\cos(y^3)}{3}\Big|_0^1 \tag{4}$$
$$= \frac{1-\cos(1)}{3} \tag{5}$$
$$\tag{6}$$

5 marks for correct setup

2 marks for correct integration and substituting the bounds

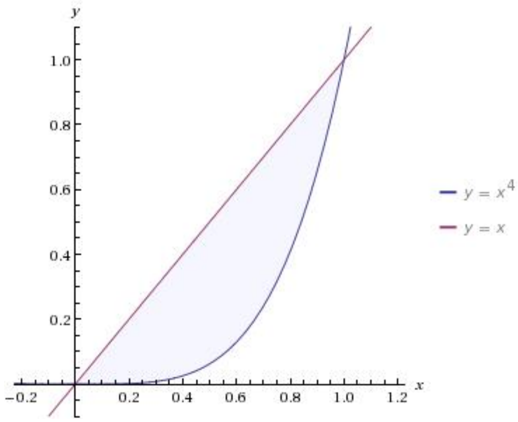
1 mark for correct final answer

Note: This goes for all the problems. Plotting the region allows me to follow your understanding of the problem instead of just randomly plugging in things into the integral and hoping whatever comes out is correct.

3. Let D be the region bounded between the curve $y = x^4$ and the line $y = x$

(a) Find the centroid of D

Solution



2 marks for correct plot of the region

Find the mass, i.e. the area of the region.

$$m = \iint_D dA = \int_0^1 \int_{x^4}^{x} dydx = \frac{3}{10}$$

2 marks for finding the mass

Thus,

$$\begin{aligned}\bar{x} &= \frac{1}{m} \iint_D x dA \\ &= \frac{10}{3} \int_0^1 \int_{x^4}^{x} x dydx \\ &= \frac{10}{3} \int_0^1 x(x - x^4) dx \\ &= \frac{10}{3} \int_0^1 x^2 - x^5 dx \\ &= \frac{5}{9}\end{aligned}$$

2 marks for correct setup

1 mark for correct $\bar{x}$

$$\begin{aligned}\bar{y} &= \frac{1}{m}\iint_D y\,dA \\ &= \frac{10}{3}\int_0^1\int_{x^4}^{x} y\,dy\,dx \\ &= \frac{10}{3}\int_0^1 x^2 - x^8\,dx \\ &= \frac{10}{27}\end{aligned}$$

2 marks for correct setup

1 mark for correct $\bar{y}$

Therefore, the centroid of D is $(\bar{x}, \bar{y}) = \left(\frac{5}{9}, \frac{10}{27}\right)$

Remark: Centroid means the density function $\sigma = 1$

(b) Find the average value of the function $f(x,y) = \sqrt{x^2+y^2}$ over D

Solution

$f_{ave} = \frac{1}{A(D)} \iint_D f(x,y)dA$

1 mark for the formula of average value

Change to polar coordinates.

$$y = x \iff r\sin\theta = r\cos\theta \iff \tan\theta = 1 \iff \theta = \frac{\pi}{4}$$

$$y = x^4 \iff r\sin\theta = r^4\cos^4\theta \text{ (for } r \neq 0) \implies \sin\theta = r^3\cos^4\theta \iff r^3 = \frac{\sin\theta}{\cos^4\theta} \implies r = \left(\frac{\sin\theta}{\cos^4\theta}\right)^{1/3}$$

2 marks each for correctly converting to polar coordinates

The region swept by the polar curve $r = \left(\frac{\sin\theta}{\cos^4\theta}\right)^{1/3}$ runs from $\theta = 0$ to $\theta = \frac{\pi}{4}$

Hence the average value of the function is

$$\begin{aligned}
f_{ave} &= \frac{10}{3}\int_0^{\frac{\pi}{4}}\int_0^{\left(\frac{\sin\theta}{\cos^4\theta}\right)^{1/3}} r^2 dr d\theta \\
&= \frac{10}{9}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta} d\theta \\
&= \frac{10}{27}\left[\frac{1}{\cos^3\theta}\right]_0^{\frac{\pi}{4}} \\
&= \frac{10}{27}\left[\left(\frac{2}{\sqrt{2}}\right)^3 - 1\right] \\
&= \frac{10}{27}\left[\frac{4}{\sqrt{2}} - 1\right] \\
\text{Or} & \\
&= \frac{10}{27}\left[2\sqrt{2} - 1\right]
\end{aligned}$$

3 marks for correct setup

2 marks for correct answer