Scoring Guideline
MATH 200/253 Sec 922
Summer 2012
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If you find that there is a problem with the marking scheme or find that your quiz was marked incorrectly or unfairly, come and see me or your professor. You can find me at the Math Learning Center at LSK basement at 2:00pm to 3:00pm on Tuesday and Thrusday.

1. Consider

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+3y^4}$$

Either find the value of this limit or show that the limit does not exist.

Solution

Instead of testing various possible paths, let's just take the generic path $y = mx$

Let $f(x,y) = \dfrac{xy^3}{x^4+3y^4}$, then $f(x,mx) = \dfrac{x(mx)^3}{x^4+3(mx)^4} = \dfrac{m^3x^4}{x^4+3m^3x^4} = \dfrac{m^3}{1+3m^3}$

So clearly for different values of m, the value of the limit is different everywhere and hence the limit DNE.

5 marks for showing the limit DNE

5 marks for stating the DNE and justifying it

At most **4 marks** was given if student shows the limit exists with justification

Remark 1: A lot of people tried to use the Squeeze Theorem to show that the limit does exists at (0,0), and they also misused the theorem.

For instance, the "correct" (because the limit actually DNE) technique is:

Proof (though not really)

Note that $x^4 + 3y^4 > x^4$

$$0 < x < x^4 + 3y^4 \iff 0 < \frac{x}{x^4+3y^4} < 1 \tag{1}$$

$$\iff 0 < \frac{x|y^3|}{x^4+3y^4} < |y^3| \tag{2}$$

$$\iff \lim_{(x,y)\to(0,0)} 0 < \lim_{(x,y)\to(0,0)} \frac{x|y^3|}{x^4+3y^4} < \lim_{(x,y)\to(0,0)} |y^3| \tag{3}$$

$$\iff 0 < \lim_{(x,y)\to(0,0)} \frac{x|y^3|}{x^4+3y^4} < 0 \tag{4}$$

See the difference in (3) and (4)? Of course, I stress again that the proof above is actually incorrect, but the technique is similiar to what most people wanted to employ.

Remark 2: Carrying the mistake from **Remark 1**, many people also thought that by showing $\lim_{(x,y)\to(0,0)} \left|\dfrac{xy^3}{x^4+3y^4}\right|$ exists implies $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^4+3y^4}$ also has a limit. This is NOT always true, that's like saying $-1 = 1$.

2. Let $f(x, y) = x^4 + 4x^3y + Ax^2y^2 + 4xy^3 + 5$. Suppose that f satisfies the differential equation $f_{xx} = f_{xy}$. What must A be?

Solution

Find f_{xx} and f_{xy}

$f_{xx} = 12x^2 + 24xy + 2Ay^2$

$f_{xy} = 12x^2 + 4Axy + 12y^2$

3 marks each for finding the second-order partials

Set $f_{xx} = f_{xy}$ and solve

$$\begin{aligned} f_{xx} &= f_{xy} \\ 12x^2 + 24xy + 2Ay^2 &= 12x^2 + 4Axy + 12y^2 \\ 24xy + 2Ay^2 &= 4Axy + 12y^2 \end{aligned}$$

2 marks for substituting the second order partials and simplifying correctly

Match the coefficients

$$\begin{aligned} 2Ay^2 = 12y^2 &\iff 2A = 12 \iff A = 6 \\ 24xy = 4Axy &\iff 24 = 4A \iff A = 6 \end{aligned}$$

Therefore A = 6

2 marks for solving A

3. Let $f(x, y) = \sin(x^2 + y)$. Give an approxmiation for the value of $f(1.03, -1.08)$

Solution

I think the question should have been worded better. The question should have asked specifcally for a *linear* approximation because technically you can improve the approximation by adding more terms (and the question doesn't ask to what accuracy, i.e. when to stop adding terms!).

To approximate $f(1.03, -1.08)$, you need to pick points *near* $(1.03, -1.08)$. The optimal choice here would be $(1, -1)$

Find $f_x(1, -1)$, $f_y(1, -1)$, and $f(1, -1)$

$f_x = 2x\cos(x^2 + y)$,

$f_y = \cos(x^2 + y)$

$f_x(1, -1) = 2(1)\cos(0) = 2$

$f_y(1, -1) = \cos(0) = 1$

$f(1, -1) = \sin(1 - 1) = \sin(0) = 0$

1 mark each for finding the first-ordered partials

1 mark each for finding the first-ordered partials at (1,-1)

1 mark for finding $f(1, -1)$

The formula for the linear approximation is

$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

The desired plane is

$$\begin{aligned} L(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &= f(1, -1) + f_x(1, -1)(x - 1) + f_y(1, -1)(y + 1) \\ &= 0 + 2(x - 1) + (y + 1) \end{aligned}$$

3 marks for finding the tangent plane

The linear approximation to $f(1.03, -1.08)$ is

$$\begin{aligned} L(1.03, -1.08) &= 2(1.03 - 1) + (-1.08 + 1) \\ &= 2(0.03) - 0.08 \\ &= -0.02 \end{aligned}$$

2 marks for finding $L(1.03, -1.08)$

Remark: The actual value of $f(1.03, -1.08) \approx -0.019$. So our approximation wasn't too bad.

At least **2 marks** was awarded for recognizing to approximate near (1,-1)

4. Consider the equation $2x^2 - y^2 + 2z^2 + 1 = 0$. Which of the following is a picture of the surface defined by this equation? Explain your answer

Solution

The correct answer is **B**

For these type of questions, always use the method of elimation.

There are many ways to go about this problem and the solution presented here is only one of many methods. So chances are, your method is probably different from this solution.

Notice that the origin is not a solution to this equation. This immediately eliminates **F** and **E**.

When $z = k$, the traces are hyperbolas of the form

$$
\begin{aligned}
2x^2 - y^2 + 2k^2 + 1 = 0 &\iff 2x^2 - y^2 = -1 - 2k^2 && (5)\\
&\iff y^2 - 2x^2 = 1 + 2k^2 && (6)\\
&\iff y^2 - \frac{x^2}{\frac{1}{2}} = 1 + 2k^2 && (7)\\
&\iff \frac{y^2}{1+2k^2} - \frac{x^2}{\frac{1}{2}(1+2k^2)} = 1 && (8)\\
&\iff \frac{y^2}{1+2k^2} - \frac{x^2}{(\frac{1}{2}+k^2)} = 1 && (9)\\
&\iff \frac{y^2}{\sqrt{(1+2k^2)^2}} - \frac{x^2}{\sqrt{(\frac{1}{2}+k^2)^2}} = 1 && (10)
\end{aligned}
$$

Perhaps (5) and (6) wasn't necessary, but you could now see for any choice of k, the focal axis will *always* be the y-axis since $1 + 2k^2 \geq \frac{1}{2}(1 + 2k^2)$

Together with the arguements made from before, we can eliminate **A**, **C**, **D**, **F**, **G**, and **H**.

So the only choice left is **B**

Remark: People were generally weak on identifying simple quadratic curves. For instance, a lot of people got themselves mixed up the difference between a circle and an ellipse. In the end, I decided it was best not to deduct marks for being sloppy unless they were the sole justification and if the supporting arguments were too weak.

2 marks stating choice **B**

8 marks for justification

At most **8 marks** was also given if justification seems to *lead very closely* to the correct answer