TAYLOR'S POLYNOMIALS IN 1 OR MORE VARIABLES

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1. Single Variable

Definition 1.1. The n-th order Taylor's Polynomial of f around x_0 is

$$T_n(x) =$$

$$f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2}+f'''(x_0)\frac{(x-x_0)^3}{6}+\cdots+f^{(n)}\frac{(x-x_0)^n}{n!}.$$

For a fixed x_0, T_n is a polynomial of degree at most n, whose coefficients depend on x_0 and f.

Example 1.2. $T_1(x) = f(x_0) + f'(x_0)(x - x_0)$ is the line tangent to f at $(x_0, f(x_0))$.

Example 1.3. If $f(x) = \log(x)$ (where log is the natural logarithm - base e) then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. The 2nd order Taylor Polynomial around 1 is $T_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$.

This T_n has the property that its value and first n derivatives at the point x_0 coincide with the value of f and its first n derivatives at x_0, causing T_n to behave very similarly to f near x_0. To see this, notice that $g(x) = \frac{(x-x_0)^k}{k!}$ has the following properties:

$$g^{(\ell)}(x) = \begin{cases} \frac{(x-x_0)^{k-\ell}}{(k-\ell)!} & \ell < k \\ 1 & \ell = k \\ 0 & \ell > k \end{cases}$$

In particular,

$$g^{(\ell)}(x_0) = \begin{cases} 1 & \ell = k \\ 0 & \ell \neq k \end{cases}$$

So, the k-th term in the definition of $T_n(x)$ contribute exactly $f^{(k)}(x_0)$ to the k-th derivative of T_n at x_0 and nothing to the other derivatives.

Other examples and more explanations can be found in the book (section 12.10 and 12.11).

Definition 1.4. The *remainder* or *error* of T_n is defined to be $R_n(x) = f(x) - T_n(x)$, so we can write $f(x) = T_n(x) + R_n(x)$.

Notice that the remainder at x_0 is 0. Actually, something much stronger is true:

Theorem 1.5. *If $f^{(n)}(x_0)$ exists, and T_n is the n-th order Taylor Polynomial around x_0, then*

$$\lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} = 0\,.$$

Proof. Apply l'Hopital's rule n times. ∎

So the remainder goes to 0 faster than $(x-x_0)^n$. There are ways to bound the remainder at specific points (Taylor's inequality, section 12.10). We will not use this, but here's an example.

Example 1.6. We use the second order Taylor Polynomial of $\log(x)$ around 1, which we calculated in example 1.3 to approximate the value of $\log(5/4)$. We get $T_2(5/4) = 1/4 - 1/32 = 0.21875$. The true value of $\log(5/4)$ is about 0.2231 so the error here is about $R_2(5/4) \approx 0.0044$.

Exercise 1.7. *Find the 5-th order Taylor polynomial of* $\sin(x)$ *around 0 and use it to estimate* $\sin(1)$.

Exercise 1.8. *Find the 4-th order Taylor polynomial of $\sqrt{x}$ around 4 and use it to estimate $\sqrt{3}$.*

Exercise 1.9. *Find the 4-th order Taylor polynomial of e^x around 0.*

Exercise 1.10. *Find the 4-th order Taylor polynomial of e^x around 1.*

2. Multiple Variables

Like in single variable, the n-th order Taylor's polynomial of $f(x,y)$ around (x_0,y_0) is the unique polynomial of degree at most n, such that all of its partial derivatives up to order n at (x_0,y_0) are the same as those of f at (x_0,y_0).

The 0-th order Taylor's polynomial of f around (x_0,y_0) is

$$T_0(x,y) = f(x_0,y_0)\,.$$

The 1-st order Taylor's polynomial of f around (x_0,y_0) is

$$T_1(x) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\,.$$

This is exactly the function describing the tangent plane at $(x_0, y_0, f(x_0, y_0))$. In other words, $z = T_1(x)$ is the equation of this tangent plane.

The 2nd order Taylor's Polynomial of f around (x_0, y_0) is

$$T_2(x) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) +$$
$$+ \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2.$$

Recall that for "reasonable" functions Clairaut's theorem apply and we get that $f_{xy} = f_{yx}$ explaining why only one of them appear in the formula (for functions where $f_{xy} \neq f_{yx}$ we will generally not be interested in T_2 since it won't be a good approximation).

In general, The n-th order Taylor's Polynomial of f around (x_0, y_0) is

$$T_n(x, y) = \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} \frac{\partial^k \partial^\ell f}{\partial x^k \partial y^\ell}(x_0, y_0) \frac{(x - x_0)^k (y - y_0)^\ell}{k!\ell!}$$

For more than 2 variables the formula is similar. For example, in 3 variables, the terms would be of the form

$$\frac{\partial^k \partial^\ell \partial^m f}{\partial x^k \partial y^\ell \partial z^m}(x_0, y_0, z_0) \frac{(x - x_0)^k (y - y_0)^\ell (z - z_0)^m}{k!\ell!m!}$$

and we would sum over all indices such that $k + \ell + m \leq n$.

Example 2.1. Let $f(x, y) = \log(x + y^2 - 1)$.

Then $f_x(x, y) = 1/(x + y^2 - 1)$, so $f_x(1, 1) = 1$.

$f_y(x, y) = 2y/(x + y^2 - 1)$ so $f_y(1, 1) = 2$.

$f_{xx}(x, y) = -1/(x + y^2 - 1)^2$, so $f_{xx}(1, 1) = -1$.

$f_{xy}(x, y) = -2y/(x + y^2 - 1)^2$, so $f_{xy}(1, 1) = -2$.

Finally, $f_{yy}(x, y) = 2/(x + y^2 - 1) - 4y^2/(x + y^2 - 1)^2$, so $f_{yy}(1, 1) = -2$.

Therefore, the second order Taylor's polynomial around $(1, 1)$ is

$$T_2(x, y) = 0 + (x - 1) + 2(y - 1) - \frac{1}{2}(x - 1)^2 - 2(x - 1)(y - 1) - (y - 1)^2.$$

Example 2.2. Let $f(x, y, z) = e^{xyz}$. Then calculating all derivatives up to 3rd order, we find that the only nonzero one at $(0, 0, 0)$ is $f_{xyz}(0, 0, 0) = 1$. Therefore, $T_3(x, y, z) = 1 + xyz$.

Once more, we define the remainder $R_n(x, y) = f(x, y) - T_n(x, y)$. We also have a theorem bounding the rate that the remainder tends to 0 at (x_0, y_0).

Theorem 2.3. *If all n-th order partial derivatives of f at (x_0, y_0) exist and are continuous near (x_0, y_0) and T_n is the n-th order Taylor Polynomial around (x_0, y_0), then*

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R_n(x,y)}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)^n} = 0\,.$$

In words, the error is small compared to the distance between (x, y) and (x_0, y_0) to the n-th power. This finally allows us to rigorously define the tangent plane:

Definition 2.4. A linear function $L(x, y) = Ax+By+C$ is the tangent plane to $f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0\,.$$

We will focus mostly on the case of 2nd order Taylor Polynomial of a 2 variables function.

Exercise 2.5. *Find the 1st, 2nd and 3rd order Taylor Polynomial of $f(x, y) = x^3 + xy^2$ around $(1,2)$. Is $T_3(x, y)$ identical to $f(x, y)$? Why?*