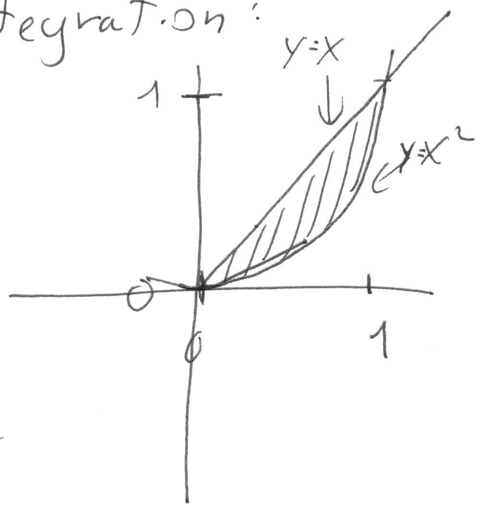
1. Calculate

$$\int_0^1 \int_{x^2}^{x} \frac{x}{y} \cos(y)\, dy\, dx\,.$$

The region of integration:



Switching order of integration:

$$= \int_0^1 \int_y^{\sqrt{y}} \frac{x}{y} \cos(y)\, dx\, dy =$$

$$= \int_0^1 \frac{\cos(y)}{y} \cdot \frac{x^2}{2} \Big|_{x=y}^{x=\sqrt{y}} dy = \int_0^1 \frac{\cos(y)}{y} \left(\frac{y}{2} - \frac{y^2}{2} \right) dy$$

$$= \int_0^1 \frac{\cos(y)}{2} - \frac{y\cos(y)}{2}\, dy = \frac{1}{2}\left(\sin(y)\Big|_0^1 - y\sin(y)\Big|_0^1 - \cos(y)\Big|_0^1 \right) =$$

$$= \frac{1}{2}\left(1 - \cos(1)\right)$$

2. Assume that the function $F(x, y)$ satisfies the equation $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$. Let A and B be some constants and let $G(s,t) = F(As + Bt, Bs - At)$.

For which A and B does G satisfy the equation $\frac{\partial^2 G}{\partial s^2} + \frac{\partial^2 G}{\partial t^2} = 0$?

$$\frac{\partial G}{\partial s} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$x = As + Bt$$
$$y = Bs - At$$

$$\frac{\partial G}{\partial s} = AF_x + BF_y$$

$$\frac{\partial G}{\partial t} = BF_x - AF_y$$

$$\frac{\partial^2 G}{\partial s^2} = A\frac{\partial F_x}{\partial s} + B\frac{\partial F_y}{\partial s} = A(AF_{xx} + BF_{xy}) + B(AF_{xy} + BF_y$$

$$= A^2 F_{xx} + 2ABF_{xy} + B^2 F_{yy}$$

$$\frac{\partial^2 G}{\partial t^2} = B\frac{\partial F_x}{\partial t} + A\frac{\partial F_y}{\partial t} = B(BF_{xx} - AF_{xy}) + A(BF_{xy} - AF_{yy}$$

$$= B^2 F_{xx} - 2ABF_{xy} + A^2 F_{yy}$$

$$\frac{\partial^2 G}{\partial s^2} + \frac{\partial^2 G}{\partial t^2} = (A^2 + B^2)(F_{xx} + F_{yy}) = 0 \quad \text{for all } A, B.$$

3. Find the absolute maximum and minimum of the function $F(x, y, z) = xy + z + z^2$ in the ball of radius 1 around the origin.

$F_x = y \quad F_y = x \quad F_z = 1 + 2z$

$y = 0 \quad x = 0 \quad 1 + 2z = 0 \Rightarrow (0, 0, -\frac{1}{2})$ is the only critical point.

For the boundary $G(x, y, z) = x^2 + y^2 + z^2 = 1$

$\nabla G = \langle 2x, 2y, 2z \rangle$

$\nabla F = \langle y, x, 1 + 2z \rangle$

Using Lagrange multipliers:

(1) $y = 2\lambda x$

(2) $x = 2\lambda y$

(3) $1 + 2z = 2\lambda z$

(4) $x^2 + y^2 + z^2 = 1$

(1)+(2) $\Rightarrow y = 4\lambda^2 y \Rightarrow y = 0$ or $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$

$y = 0 \overset{(2)}{\Rightarrow} x = 0 \overset{(4)}{\Rightarrow} z = \pm 1 \Rightarrow (0, 0, 1)\ (0, 0, -1)$ are candidates.

$\lambda = \frac{1}{2} \overset{(1)}{\Rightarrow} y = x$

$\overset{(3)}{\Rightarrow} 1 + 2z = z \Rightarrow z = -1 \overset{(4)}{\Rightarrow} y = x = 0 \quad (0, 0, -1)$ is a candidate,

$\lambda = -\frac{1}{2} \overset{(1)}{\Rightarrow} y = -x$

$\overset{3}{\Rightarrow} 1 + 2z = -z \Rightarrow z = -\frac{1}{3}$

④ $\Rightarrow \quad x^2 + (-x)^2 + \left(-\frac{1}{3}\right)^2 = 1$

$$2x^2 + \frac{1}{9} = 1$$

$$x = \pm \frac{2}{3}$$

$\left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$ and $\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ are candidates.

$$F\left(0, 0, -\frac{1}{2}\right) = -\frac{1}{4}$$

$$F(0, 0, 1) = 2$$

$$F(0, 0, -1) = 0$$

$$F\left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right) = \frac{2}{3} \cdot \left(-\frac{2}{3}\right) - \frac{1}{3} + \frac{1}{9} = \frac{-4-3+1}{9} = -\frac{2}{3}$$

$$F\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = -\frac{2}{3}$$

max is 2

min is $-\frac{2}{3}$

4. The temperature is given by $T(x, y, z) = xy + y^3 + x^2z^2$ and the humidity is given by $H(x, y, z) = e^{x^2+y^2+z^2}$. What is the rate of change of the humidity, at the point $(1, 0, 1)$, in the direction where the temperature increases most rapidly?

T increases quickest in the direction of the gradient.

$$\nabla T = \langle y + 2xz^2, x + 3y^2, 2x^2z \rangle$$

$$\nabla T(1,0,1) = \langle 2, 1, 2 \rangle$$

So $u = \dfrac{\langle 2,1,2 \rangle}{\sqrt{2^2+1^2+2^2}} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

$$D_u H = \nabla H \cdot u$$

$$\nabla H = \langle e^{x^2+y^2+z^2} \cdot 2x, e^{x^2+y^2+z^2} \cdot 2y, e^{x^2+y^2+z^2} \cdot 2z \rangle$$

$$\nabla H(1,0,1) = \langle 2e^2, 0, 2e^2 \rangle$$

$$D_u H(1,0,1) = \langle 2e^2, 0, 2e^2 \rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle = \frac{8}{3} e^2$$

5. Let $F(x, y) = e^x + e^y + (x + y)^2$ and let $g(x)$ satisfy the equation $F(x, g(x)) = 2$ and $g(0) = 0$ and g has a second order derivative at 0.

   (a) Find $g'(0)$.

   (b) Find $g''(0)$.

(a) $$g'(0) = -\frac{F_x(0,0)}{F_y(0,0)} = -\frac{e^0 + 2\cdot 0}{e^0 + 2\cdot 0} = -1$$

$$F_x = e^x + 2(x+y)$$

$$F_y = e^y + 2(x+y)$$

(b) $$F_x(x, g(x)) + F_y(x, g(x))\, g'(x) = 0$$

$$e^x + 2(x+g(x)) + \left(e^{g(x)} + 2(x+g(x))\right) g'(x) = 0$$

$$e^x + 2 + 2g'(x) + \left(e^{g(x)} + 2(x+g(x))\right) g''(x) + \left(e^{g(x)} \cdot g'(x) + 2 + 2g'(x)\right) g'(x$$

putting $x=0$ $g(0)=0$ $g'(0)=-1$ we get

$$1 + 2 - 2 + \left(e^0 + 2\cdot 0\right) g''(0) + \left(e^0 \cdot (-1) + 2 \cdot 2\right)(-1) = 0$$

$$1 + g''(0) + (-1)(-1) = 0$$

$$g''(0) = -2$$