MATH 253 - SEC 104 - W2011T1

Midterm no. 1

2:00-3:20pm, Oct 18, 2011

Name (Last, First):_______________Ori_______________

Student I.D. Number:______________________________________

Signature: ____

INSTRUCTIONS: This is a closed-book exam. You may not use any books, notes, papers, calculators, or other aids. Do all work on the sheets provided. There is an extra sheet on the back for scratch work. If you need an extra sheet, raise your hand and one will be provided. If you need more space for your solution, use the back of the sheets and leave an arrow for the grader. Please draw a box around your final answer.

There are 5 questions, each worth 12 points. Explain all your answers. Good Luck!

1. Let $a = (0,1,2)$ and $b = (3,3,3)$.

   (a) Find a vector or parametric equation for the line going through a and b.

   (b) Find the point where this line intersect plane P given by the equation $x+y+z=1$.

   (c) Does plane P intersect the line **segment** between a and b? Explain.

(a) $u = \vec{oa} + t \cdot \vec{ab}$

$u = \langle 0,1,2 \rangle + t\langle 3,2,1 \rangle$

(b) $3t + 1 + 2t + 2 + t = 1$

$3 + 6t = 1$

$6t = -2$

$t = -\frac{1}{3} \quad \Rightarrow \quad u = \langle -1, \frac{1}{3}, 1\frac{2}{3} \rangle$

(c) No. The line segment is all the points with $0 \leq t \leq 1$ and here $t = -\frac{1}{3}$

2. Let $f(x,y) = x^2 + \ln(x+y)$.

(a) Write the equation of the tangent plane at $(0, e, 1)$.

(b) Find a point (a, b) such that the tangent plane through $(a, b, f(a,b))$ is parallel to the plane $7x + y - 3z = 0$.

(a) $f_x(x,y) = 2x + \frac{1}{x+y}$ $\qquad f(0,e) = 1$

$f_y(x,y) = \frac{1}{x+y}$

$f_x(0,e) = \frac{1}{e}$

$f_y(0,e) = \frac{1}{e}$

$$\boxed{z - 1 = \frac{1}{e}(x-0) + \frac{1}{e}(y-e)}$$

(b) The tangent plane through $(a, b, f(a,b))$ is

$$z - a^2 + \ln(a+b) = \left(2a + \frac{1}{a+b}\right)(x-a) + \frac{1}{a+b}(y-b)$$

Its normal is $\langle 2a + \frac{1}{a+b}, \frac{1}{a+b}, -1\rangle$

We want it to be in the same direction as $\langle 7, 1, -3\rangle$ which has the same direction as $\langle \frac{7}{3}, \frac{1}{3}, -1\rangle$. Now the z-coordinate is the same so the other must also be the same

$\frac{7}{3} = 2a + \frac{1}{a+b}$

$\frac{1}{3} = \frac{1}{a+b}$

$\Rightarrow$ $2 = 2a$

$a = 1$

$\frac{1}{3} = \frac{1}{1+b}$

$\Rightarrow$ $b = 2$

so

$\boxed{(1, 2)}$

3. Let $f(x, y) = e^y(y^2 - x^2 + xy)$. Find all critical points and for each critical point find whether it is a local minimum, a local maximum or neither (if the methods we use are inconclusive, just write so).

$$f_x = e^y(-2x + y) = 0$$

$$f_y = e^y(y^2 - x^2 + xy + 2y + x) = 0$$

e^y never 0 so $y - 2x = 0 \Rightarrow y = 2x$

$$4x^2 - x^2 + 2x^2 + 4x + x = 0$$

$$5x^2 + 5x = 0$$

$x = 0$ or $x = -1$

$y = 0$ $\quad y = -2$

$(0,0)$ $\quad (-1,-2)$

$$f_{xx} = e^y(-2)$$

$$f_{xy} = e^y(-2x + y + 1)$$

$$f_{yy} = e^y(y^2 - x^2 + xy + 2y + x + 2y + x + 2) = e^y(y^2 - x^2 + xy + 4y + 2x + 2)$$

at $(0,0)$

$D = (-2)\cdot 2 - 1^2 = -5 < 0$ so $(0,0)$ is neither local max or min.

at $(-1,-2)$

$$D = e^{-2}(-2)e^{-2}(-3) - (e^{-2}\cdot 1)^2 = e^{-4}\cdot 5 > 0$$

$f_{xx}(-1,-2) = e^{-2}(-2) < 0$ so $(-1,-2)$ is local max.

4. Let $f(x, y) = \sin(x^2 + y)$.

(a) Use linear approximation around $(1, -1)$ to approximate the value of $f(1.1, -1.1)$.

(b) Use the second order Taylor's polynomial around $(1, -1)$ to approximate the value of $f(1.1, -1.1)$.

(a) $f(1,-1) = 0$

$f_x = \cos(x^2+y) \cdot 2x \qquad f_x(1,-1) = 2$

$f_y = \cos(x^2+y) \qquad f_y(1,-1) = 1$

$z - 0 = 2(x-1) + (y+1)$

$z - 0 = 2(0.1) + (-0.1) = 0.1$

Approximation is 0.1

(b) $f_{xx} = -\sin(x^2+y)4x^2 + 2\cos(x^2+y)$

$f_{xy} = -\sin(x^2+y)2x$

$f_{yy} = -\sin(x^2+y)$

at $(1,-1)$ $\quad f_{xx} = 2 \quad f_{xy} = 0 \quad f_{yy} = 0$

$T_2(x,y) = 2(x-1) + (y+1) + (x-1)^2$

$T_2(1.1, -1.1) = 0.2 - 0.1 + (0.1)^2 = 0.11$

5. Match the following functions with pictures of their graphs. Explain your answers.

(a) $f(x, y) = -\ln(x^2 + 5y^2)$

(b) $f(x, y) = x^2 e^{-y}$

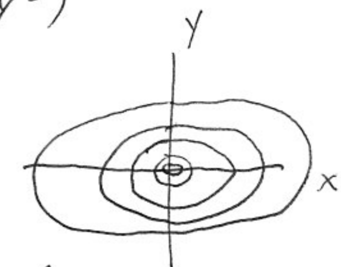
(c) $f(x, y) = y \sin(3x)$

(a) The level curves $k = -\ln(x^2+5y^2)$

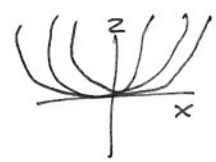
$\Updownarrow$

$e^{-k} = x^2+5y^2$

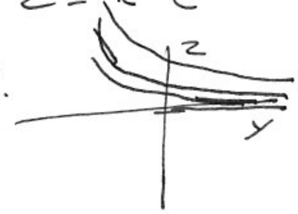
are ellipses around the origin, which are smaller as k increases. This fits picture A.



(b) Intersections with $y=k$ are $z = x^2 e^{-k}$ that is parabolas which are wider as k increases.



Intersection with $x=k$ are $z = k^2 e^{-y}$ exponential functions decreasing in y.



This fits picture F

(c) Intersections with $y=k$ give $z = k\sin(3x)$ sin curves with increasing height as $|k|$ increases.

Intersections with $x=k$ give lines with periodically changing slopes.

$z = y\sin(3k)$

This fits picture D.