MATH 253 - SEC 104 - W2011T1

1. Let $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$

Solution:

Using the chain rule (first time out of many) we get

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

Write $g(r,\theta) = \frac{\partial z}{\partial r} = f_x(r\cos\theta, r\sin\theta)\cos\theta + f_y(r\cos\theta, r\sin\theta)\sin\theta$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial g}{\partial r} = \frac{\partial f_x}{\partial r}\cos\theta + \frac{\partial f_y}{\partial r}\sin\theta$$

Since

$$\frac{\partial f_x}{\partial r} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r} = f_{xx}\cos\theta + f_{xy}\sin\theta$$

$$\frac{\partial f_y}{\partial r} = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r} = f_{xy}\cos\theta + f_{yy}\sin\theta$$

we get

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= (f_{xx}\cos\theta + f_{xy}\sin\theta)\cos\theta + (f_{xy}\cos\theta + f_{yy}\sin\theta)\sin\theta \\ &= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta\end{aligned}$$

Similarly,

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$$

To find $\frac{\partial^2 z}{\partial \theta^2}$ we must apply the product rule for derivatives to $f_x r\sin\theta$ and $f_y r\cos\theta$ and get

$$\frac{\partial^2 z}{\partial \theta^2} = -(\frac{\partial f_x}{\partial \theta} r\sin\theta + f_x r\cos\theta) + \frac{\partial f_y}{\partial \theta} r\cos\theta + f_y r(-\sin\theta)$$

and since

$$\frac{\partial f_x}{\partial \theta} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial \theta} = f_{xx} r(-\sin\theta) + f_{xy} r\cos\theta$$

$$\frac{\partial f_y}{\partial \theta} = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial \theta} = f_{xy} r(-\sin\theta) + f_{yy} r\cos\theta$$

we get

$$\begin{aligned}\frac{\partial^2 z}{\partial\theta^2} &= -((f_{xx}r(-\sin\theta) + f_{xy}r\cos\theta)r\sin\theta + f_x r\cos\theta) \\ &+(f_{xy}r(-\sin\theta) + f_{yy}r\cos\theta)r\cos\theta + f_y r(-\sin\theta) \\ &= f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\cos\theta\sin\theta + f_{yy}r^2\cos^2\theta - f_x r\cos\theta - f_y r\sin\theta\end{aligned}$$

All together now

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} =$$

$$= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta + \frac{1}{r}(f_x\cos\theta + f_y\sin\theta)$$

$$+f_{xx}\sin^2\theta - 2f_{xy}\cos\theta\sin\theta + f_{yy}\cos^2\theta - \frac{1}{r^2}f_x r\cos\theta - \frac{1}{r^2}f_y r\sin\theta$$

$$= f_{xx}(\cos^2\theta + \sin^2\theta) + f_{xx}(\sin^2\theta + \cos^2\theta) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

That wasn't too bad after all, was it?

2. Show that the function

$$u(x,y,t) = \frac{1}{t}e^{-\frac{x^2+y^2}{4t}}.$$

satisfies the *heat equation*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}.$$

Solution:

$$\frac{\partial u}{\partial x}(x,y,t) = \frac{-xe^{-\frac{x^2+y^2}{4t}}}{2t^2}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y,t) = \frac{-e^{-\frac{x^2+y^2}{4t}}}{2t^2} + \frac{x^2 e^{-\frac{x^2+y^2}{4t}}}{4t^3}$$

$$\frac{\partial u}{\partial y}(x,y,t) = \frac{-ye^{-\frac{x^2+y^2}{4t}}}{2t^2}$$

$$\frac{\partial^2 u}{\partial y^2}(x,y,t) = \frac{-e^{-\frac{x^2+y^2}{4t}}}{2t^2} + \frac{y^2 e^{-\frac{x^2+y^2}{4t}}}{4t^3}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) = \frac{-e^{-\frac{x^2+y^2}{4t}}}{t^2} + \frac{(x^2+y^2)e^{-\frac{x^2+y^2}{4t}}}{4t^3}$$

$$\frac{\partial u}{\partial t}(x,y,t) = -\frac{e^{-\frac{x^2+y^2}{4t}}}{t^2} + \frac{e^{-\frac{x^2+y^2}{4t}}}{t}\frac{(x^2+y^2)}{4t^2}$$

3. Let

$$F(x, y) = x \arctan(x^2 - y) \, .$$

(a) Assume g is differentiable such that $F(x, g(x)) = 0$ and $g(1) = 1$. Find $g'(1)$.

(b) Find a unit vector u, such that directional derivative of F at the point $(1, 1)$ in the direction u is 0.

Solution:

(a) By the implicit differentiation formula, if $y = g(x)$ we get

$$g'(x) = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{\arctan(x^2 - y) + \frac{2x^2}{1+(x^2-y)^2}}{\frac{-x}{1+(x^2-y)^2}} \, ,$$

so

$$g'(1) = \frac{2}{-1} \, .$$

(b) We have $\nabla F(1, 1) = \langle 2, -1 \rangle$ so we need some vector solving $v \cdot \langle 2, -1 \rangle = 0$. A possible solution is $v = \langle 1, 2 \rangle$, but this is not a unit vector, so to get a unit vector we take $u = \frac{v}{|v|} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$. The other possible solution is $\langle -1/\sqrt{5}, -2/\sqrt{5} \rangle$

4. Let $F(x, y, z) = x^2 + y^2 + z^2$.

(a) Write the gradient ∇F.

(b) Write the equation of a plane passing through (x_0, y_0, z_0) and orthogonal to $\nabla F(x_0, y_0, z_0)$.

(c) Find (x_0, y_0, z_0) which belong to the level surface $F(x_0, y_0, z_0) = 1$ for which the plane from (b) passes through the points $(1, 1, 1)$ and $(1, -2, 4)$.

Solution:

(a) $\nabla F(x, y, z) = \langle 2x, 2y, 2z \rangle$.

(b) $\nabla F(x_0, y_0, z_0) \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$

Equivalently, $2x_0(x - x_0) + 2y_0(y - y_0) + 2z_0(z - z_0) = 0$.

(c) We need to solve the system of equations:

$$2x_0(1 - x_0) + 2y_0(1 - y_0) + 2z_0(1 - z_0) = 0$$

$$2x_0(1 - x_0) + 2y_0(-2 - y_0) + 2z_0(4 - z_0) = 0$$

$$x_0^2 + y_0^2 + z_0^2 = 1$$

Subtracting the second from the third we get $6y_0 - 6z_0 = 0$, so $z_0 = y_0$.

The third equation is now $x_0^2 + 2y_0^2 = 1$, so $x_0 = \pm\sqrt{1 - 2y_0^2}$.

Putting this back in the first equation yields

$$\pm 2\sqrt{1 - 2y_0^2} - 2(1 - 2y_0^2) + 4y_0(1 - y_0) = 0$$

$$\pm\sqrt{1 - 2y_0^2} = 1 - 2y_0$$

Squaring

$$1 - 2y_0^2 = 1 - 4y_0 + 4y_0^2$$

$$4y_0 - 6y_0^2 = 0$$

So either $y_0 = 0$ and we get $z_0 = 0$ and $x_0 = 1$, or $y_0 = 2/3$ and we get $z_0 = 2/3$ and x_0 is either $+1/3$ or $-1/3$ and we can rule out $x_0 = +1/3$ by the first equation.

Notice that this is (once more) the same question you were asked to solve in the first 2 assignments.